Sequences:

Definition: A **sequence** is a function whose domain is the set of natural numbers or a subset of the natural numbers. We usually use the symbol a_n to represent a sequence, where n is a natural number and a_n is the value of the function on n.

A sequence may be **finite** or **infinite**.

If a sequence is finite, we sometimes write $\{a_1, a_2, a_3, a_4, \ldots a_n\}$ to represent the sequence, If a sequence is infinite, we write $\{a_1, a_2, a_3, \ldots\}$ or $\{a_i\}_{i=1}^{\infty}$.

The notation $\{a_i\}$ implies that we have a sequence whose first term is a_1, the second term is a_2, the third term is a_3...etc. The **index** i starts from 1 (or any other positive integer) and increases by 1 each time to represent each subsequent term in the sequence.

Intuitively, a sequence is just an ordered list of (possibly infinitely many) numbers. Each number in a sequence is a **term** of the sequence. We usually use the letter i as the **index**, and a_i is the **i-th term** of the sequence.

A sequence can be represented by a formula expressed as an expression in i or in n. If a sequence has a pattern we can also write the first few terms of the sequence and assume that the pattern continues and let the reader figure out the values of the subsequent values A sequence can also be defined **recursively**, where value of each subsequence is defined by one or more of the previous terms. We can also describe a sequence verbally if there's no obvious formula or pattern that we can use to express the sequence.

Example: Consider the sequence $\{a_i = -6\}_{i=1}^{\infty}$. Starting with $i = 1$, since $a_i = -6$, so $a_1 = -6$ is the first term of the sequence. If $i = 2$, then $a_2 = -6$. If $i = 3$, then $a_3 = -6$. The value of a_i is always the same value, so we have the sequence of constant terms: $\{a_i = -6\}_{i=1}^{\infty} = \{-6, -6, -6, -6, \ldots\}$

Example: Consider the sequence $\{a_i = i\}_{i=1}^{\infty}$. Starting with $i = 1$, since $a_i = i$, so $a_1 = 1$ is the first term of the sequence. If $i = 2$, then $a_2 = 2$. If $i = 3$, then $a_3 = 3$. Continue in this fashion, we obtain the sequence of positive integers: $\{a_i\}_{i=1}^{\infty} = \{1, 2, 3, 4, \ldots\}$

Example: Consider the sequence $\left\{a_i = i^2 - 3\right\}_{i=1}^{\infty}$. Starting with $i = 1$, since $a_i = i^2 - 3$, so $a_1 = 1^2 - 3 = -2$ is the first term of the sequence. If $i = 2$, then $a_2 = 2^2 - 3 = 1$ is the second term. If $i = 3$, then $a_3 = 3^2 - 3 = 6$ is the third term. If $i = 4$, then

$a_4 = 4^2 - 3 = 13$ is the fourth term. Continue in this fashion, we obtain the following sequence: $\{a_i\}_{i=1}^{\infty} = \{-2, 1, 6, 13, 22, 33, 46, \ldots\}$

Example: Consider the sequence $\left\{a_i = \frac{i}{i+1}\right\}_{i=1}^{\infty}$. Starting with $i = 1$, since $a_i = \frac{i}{i+1}$, so $a_1 = \frac{1}{1+1} = \frac{1}{2}$ is the first term of the sequence. If $i = 2$, then $a_2 = \frac{2}{2+1} = \frac{2}{3}$ is the second term. If $i = 3$, then $a_3 = \frac{3}{3+1} = \frac{3}{4}$ is the third term. If $i = 4$, then $a_4 = \frac{4}{4+1} = \frac{4}{5}$ is the fourth term. Continue in this fashion, we obtain the following sequence: $\{a_i\}_{i=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\right\}$

Example: Consider the sequence: $\{2, 4, 6, 8, \ldots\}$. Assuming the pattern continues, this is the sequence of positive even integers. We can also represent this sequence using a formula: $\{a_i = 2i\}_{i=1}^{\infty}$ or $\{2i\}_{i=1}^{\infty}$

Example: $\left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6} \ldots\right\}$ Assuming the pattern continues, this is a sequence whose terms alternate in sign. It can be expressed by $a_i = \frac{(-1)^{i+1}}{i}$

Example: Consider the sequence defined by: $a_i =$ the i-th prime number.

This is the sequence $\{2, 3, 5, 7, 11, \ldots\}$. This sequence cannot be expressed as an expression in i, but is well-defined.

Example: The sequence $\{1, 1, 1, \ldots\}$ is a sequence defined by $a_i = 1$.

Example: The sequence $\{i^3\}_{i=1}^{\infty}$ is the sequence of positive perfect cubes,

$\{a_i\} = \{1, 8, 27, 64, 125, \ldots\}$

Example: The sequence $\frac{1}{i^2+1}$ is the sequence: $\left\{\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17} \ldots\right\}$

Example: Find an expression in i for the sequence $\left\{\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \ldots\right\}$

Ans: For each fraction, the denominator is the square of a number one greater than the numerator, so we may use: $a_i = \frac{i}{(i+1)^2}$

Notice that we may also use: $\left\{a_i = \frac{i-1}{i^2}\right\}_{i=2}^{\infty}$. For any sequence, i does not have to start at 1.

Example: Find an expression in i for the sequence $\left\{2, -\frac{3}{2}, \frac{4}{3}, -\frac{5}{4}, \frac{6}{5}, \dots\right\}$

Ans: This is a sequence where the denominator is one less than the numerator. To make the terms alternate in sign, we use a power of -1:

$a_i = (-1)^{i+1}\left(\frac{i+1}{i}\right)$, $i \geq 1$

Sigma Notation

Suppose we have a sequence $\{a_i\} = \{a_1, a_2, a_3, \dots\}$, often times we want to add some or all of the terms of the sequence to find the sum. Instead of writing $a_1 + a_2 + a_3 + \cdots + a_i + \cdots$ every time, we use the **sigma notation**, Σ, to represent the sum of all these terms:

Definition: $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$

Example: Consider: $\sum_{i=1}^{8} i$. This means that $a_i = i$, we start with $i = 1$, so $a_1 = 1$, then increases i by 1 each time until we get to 8, we have:

$$\sum_{i=1}^{8} i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36.$$

Example: Consider: $\sum_{i=1}^{11} i^2$. This means that $a_i = i^2$, we start with $i = 1$, so $a_1 = 1^2 = 1$, then increases i by 1, we get $a_2 = 2^2 = 4$, then increases i by 1 again, so $a_3 = 3^3 = 9$. Continue in this pattern until $i = 11$, we have:

$$\sum_{i=1}^{11} i^2 = 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 + 121 = 506$$

Example: Consider: $\sum_{i=5}^{12}(2i + 1)$. This means that $a_i = (2i + 1)$, but this time to get the sum, we start with $i = 5$, so $a_5 = 2(5) + 1 = 11$, then increasment i by 1, we have $a_6 = 2(6) + 1 = 13$. Continue in this fashion until $i = 12$, we have:

$$\sum_{i=5}^{12}(2i + 1) = 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 = 144$$

Example: Consider: $\sum_{i=1}^{9} 4$. This means that $a_i = 4$ for all i. In other words, $a_1 = 4$,

$a_2 = 4$,...etc., so $\sum_{i=1}^{9} 4 = 4+4+4+4+4+4+4+4+4 = 36$

Some formula involving sigma that would be useful to know:

If c is a constant,

$\sum_{i=1}^{n} c = nc$ (we are adding the same constant, c, for n many times, the result is n times c)

$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$ (this is just the distributive property)

$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$ (communtative property of addition)

$$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} i = \frac{(n)(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{(n)(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{(n)(n+1)}{2}\right]^2$$