Derivatives

Using limits, we can define the slope of a tangent line to a function.

When given a function $f(x)$, and given a point $P (x_0, f(x_0))$ on $f$, if we want to find the slope of the tangent line to $f$ at $P$, we can do this by picking a nearby point $Q (x_0 + h, f(x_0 + h))$ ($Q$ is $h$ units away from $P$, $h$ is small) then find the slope of the secant line containing $PQ$, the slope of this secant line, from algebra, is known to be:

$$\text{slope of secant line} = m_{sec} \frac{f(x_0 + h) - f(x_0)}{h}$$

Our intuition tells us that, if $h$ is small, the slope of this secant line should be a good approximation of the slope of the tangent line. So we define the slope of the tangent line to be the limit of the slope of secant lines as $h$ approaches 0:

Definition:

$$m_{tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

is the slope of the tangent line to $f$ at the given point $(x_0, f(x_0))$. 
If instead of using a constant \( x_0 \) in the above formula, we replace \( x_0 \) with the variable \( x \), the resulting limit (if it exists) will be an expression in terms of \( x \). We can treat this expression in terms of \( x \) as another function of \( x \). This very useful function, denoted by \( f'(x) \), is called the derivative function of \( f \).

![Graph of a function and its derivative]

Definition: Let \( f(x) \) be a function of \( x \), the derivative function of \( f \) at \( x \) is given by:

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

If the limit exists, \( f \) is said to be differentiable at \( x \), otherwise \( f \) is non-differentiable at \( x \).

If \( y = f(x) \) is a function of \( x \), then we also use the notation \( \frac{dy}{dx} \) to represent the derivative of \( f \). The notation is read ”D \( y \) D \( x \)”. Do not read it as "D \( y \) over D \( x \)”, as the differentials \( dy \), \( dx \) are not numbers.

It is important that we know what \( \frac{dy}{dx} \) means. \( \frac{dy}{dx} \) is the derivative of \( y \) with respect to \( x \). In other words, it means the rate at which \( y \) changes when \( x \) changes. Since rate of change is a relative quantity, it is important that we distinguish the change of which quantity with respect to which other quantity.

For example, \( \frac{dy}{dx} \) is the change of \( y \) with respect to \( x \). On the other hand, \( \frac{dy}{dz} \) is the rate of change of \( y \) with respect to \( z \). These two quantities might be very different even though they are both derivatives of \( y \).

Let’s look at an example which illustrates this point.

Suppose you are riding in a train that is going at 40 miles per hour. Inside the train, your friend is walking toward the front of the train at 5 mile per hour. The
train passes a station with an observer outside who is standing still.

According to you, your friend is moving at 5 miles per hour; however, according to the observer standing on the station, your friend is moving at 45 miles per hour. If we let \( y \) represents your friend, \( x \) represents you, and \( z \) represents the observer on the station, then \( \frac{dy}{dx} = 5 \), but \( \frac{dy}{dz} = 45 \). \( \frac{dy}{dx} \) is how your friend (\( y \)) is changing \textit{with respect to} you, \( x \). \( \frac{dy}{dz} \) is how your friend is changing \textit{with respect to} the observer, \( z \). This example shows that even though we may be referring to just the \textit{derivative} of \( y \), it does make a difference as to what is the derivative of \( y \) \textit{with respect to}. 

Most of the time, if \( y \) is given explicitly as a function of another variable then when we talk about the \textit{derivative} of \( y \) it is implicitly understood that we are talking about the derivative of \( y \) with respect to the given variable. So if we are given that \( y = t^3 + 2 \), and when we ask for \textit{derivative} of \( y \), it is understood that we meant \( \frac{dy}{dt} \).

We have seen two notations, the \textit{prime} (') notation and the \( \frac{dy}{dx} \) notation for the derivative. Given a function \( f(x) \), we use \( f'(x) \) to represent the derivative of \( f \) \textit{with respect to} \( x \). We may also use \( y' \) to mean the derivative of \( y \) or \( f' \) to mean the derivative of \( f \). Use extra caution if you use the prime notation without explicitly indicating which variable is being differentiated with respect to.

You should understand the meaning of the two notation. Suppose 
\[ y = f(x) = x^3, \]
then \( f'(x) \) and \( \frac{dy}{dx} \) mean the same thing, the derivative of the \( x^3 \) function, with respect to \( x \). Later you will learn that the derivative of the \( x^3 \) function is \( 3x^2 \). Therefore, if you are given the function \( f(x) = x^3 \), and you are asked for the derivative of \( f \) (implicitly or explicitly, with respect to \( x \)), you may write:
\[ f'(x) = 3x^2 \]
If we use \( y = x^3 \), you may write
\[ \frac{dy}{dx} = 3x^2 \]

Sometimes we can explicitly use the prime or the \( \frac{d}{dx} \) notation:
\[ (x^3)' \text{ means the same thing as the derivative of the } x^3 \text{ function (implicitly with respect to } x). \text{ So you may write:} \]
\[ (x^3)' = 3x^2 \]
to mean that the derivative of \( x^3 \) (implicitly with respect to \( x \)) is equal to \( 3x^2 \).
With the $\frac{d}{dx}$ notation, we write:

\[ \frac{d}{dx} (x^3) \quad \text{or} \quad \frac{d}{dx} (x^3) \]

to mean the derivative of the $x^3$ function (explicitly with respect to $x$). You can write:

\[ \frac{d}{dx} (x^3) = 3x^2 \quad \text{or} \quad \frac{d}{dx} (x^3) = 3x^2 \]

to mean the derivative of $x^3$ (explicitly with respect to $x$) is equal to $3x^2$. 
Example:

Find the derivative of \( f(x) = \sqrt{x} \):

Using the definition we have:

\[
  f'(x) = \lim_{h \to 0} \frac{f(h + x) - f(x)}{h}
\]

\[
  = \lim_{h \to 0} \frac{\sqrt{h + x} - \sqrt{x}}{h}
\]

\[
  = \lim_{h \to 0} \frac{\sqrt{h + x} - \sqrt{x}}{h} \cdot \frac{\sqrt{h + x} + \sqrt{x}}{\sqrt{h + x} + \sqrt{x}}
\]

\[
  = \lim_{h \to 0} \frac{(\sqrt{h + x} - \sqrt{x})(\sqrt{h + x} + \sqrt{x})}{h(\sqrt{h + x} + \sqrt{x})}
\]

\[
  = \lim_{h \to 0} \frac{(h + x) - x}{h(\sqrt{h + x} + \sqrt{x})}
\]

\[
  = \lim_{h \to 0} \frac{h}{h(\sqrt{h + x} + \sqrt{x})}
\]

\[
  = \lim_{h \to 0} \frac{1}{\sqrt{h + x} + \sqrt{x}}
\]

At this stage, notice that when we substitute \( h = 0 \) into the limit, we get an expression in terms of \( x \).

\[
  \lim_{h \to 0} \frac{1}{\sqrt{h + x} + \sqrt{x}}
\]

\[
  = \frac{1}{\sqrt{0 + x} + \sqrt{x}}
\]

\[
  = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
\]

So, \( f'(x) = \frac{1}{2\sqrt{x}} \)
What is the derivative?

Since the derivative is defined as the limit which finds the slope of the tangent line to a function, the derivative of a function \( f \) at \( x \) is the instantaneous rate of change of the function at \( x \).

For example, if \( s(t) \) represents the displacement of a particle at any time \( t \), then \( s'(t) \) represents the velocity of the particle at any moment in time \( t \). If \( y = f(x) \) is a function of \( x \), then \( f'(x) \) represents how \( y \) changes when \( x \) changes. If \( f'(x) \) is positive at a given point, then at that point \( y \) increases as \( x \) increases; if the derivative is negative at a given point, then at that point \( y \) decreases as \( x \) increases.

When is a function not differentiable?

The graph of a differentiable function does not have any sharp corners. It also does not have any points with vertical slope.

Since the derivative represents velocity, imagine that before \( t = 1 \) you are driving toward one direction with a velocity of, say, 30 miles per hour. Then once you reached \( t = 1 \), without any stopping or slowing down, you suddenly are driving at 30 miles per hour to the opposite direction. Physically this is impossible, and on the graph of the function you will have a sharp corner, and that’s where the function is not differentiable.

How are differentiability related to continuity? It turns out that differentiability is stronger than continuity. What this means is that:

If \( f \) is differentiable at \( a \) then \( f \) is continuous at \( a \).

In other words, a differentiable function is necessarily continuous.

The converse of the above statement is not true. i.e. a continuous function may not necessarily be differentiable.

The relationship between differentiability, continuity, and having a limit is this: a function \( f \) can be continuous but not differentiable, and it can have a limit but not continuous. In summary, being continuous is stronger than having a limit, and being differentiable is stronger than being continuous. This means that if a function \( f \) is differentiable at \( a \), then it must be continuous at \( a \), and if \( f \) is continuous at \( a \), then it must have a limit at \( a \).

Another point to note is that, differentiability, continuity, and existence of a limit are all what we call local properties of a function. What this means is that a function may be differentiable at one point, but fail to be differentiable at a different point; similarly a function may be continuous or have a limit at one point, but not be continuous or have a limit at another point.
Example:

In the above graph of a function \( f \),

\( f \) is continuous (therefore has a limit) at \( x = -4 \) but is not differentiable at \( x = -4 \) (sharp corner).

\( f \) is discontinuous but has a limit at \( x = -2 \) (What is this limit?). Since \( f \) is discontinuous at \( x = -2 \), it cannot be differentiable there, as differentiability is stronger than continuity.

\( f \) has a left and a right hand limit at \( x = 0 \), but does not have a limit, \( f \) is discontinuous, and not differentiable at \( x = 0 \).

\( f \) has a left and a right hand limit at \( x = 2 \), but does not have a limit, \( f \) is discontinuous, and not differentiable at \( x = 2 \).

\( f \) is continuous at \( x = 3 \) but is not differentiable at \( x = 3 \) (vertical tangent).

\( f \) has a left hand limit of 0 at \( x = 4 \), the right hand limit of \( f \) at \( x = 4 \) is \(-\infty\), \( f \) does not have a limit at \( x = 4 \).
E.g. \( f(x) = |x| \) is **non-differentiable** at \( x = 0 \). To find the derivative of \( f \) at 0, we need to use the definition:

\[
f'(0) = \lim_{h \to 0} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}
\]

From previous examples we already knew that this limit does not exist, since \( \lim_{h \to 0^-} \frac{|h|}{h} = -1 \) while \( \lim_{h \to 0^+} \frac{|h|}{h} = 1 \). If we look at the graph of the \(|x|\) function we see that there’s a sharp corner at \( x = 0 \).

![Graph of \(|x|\)](image)

E.g. \( f(x) = \sqrt{x} \) is **non-differentiable** at \( x = 0 \). We will see how we can evaluate the derivative of \( f \) in the next section. At this moment, let’s look at the graph of \( f(x) = \sqrt{x} \) and observe that the tangent line to \( f \) at 0 is a vertical line, and therefore its slope is undefined.

![Graph of \(\sqrt{x}\)](image)
Differentiation Formulas:

We have seen how to find the derivative of a function using the definition. While this is fine and still gives us what we want, it is too tedious and time-consuming. We want to develop formulas to help us find derivatives more easily:

**Constant Rule:**

If $f(x) = c$, where $c$ is a constant, then $f'(x) = 0$. In other words, the derivative of a constant is zero.

E.g. if $y = 2$, then $\frac{dy}{dx} = 0$.

The following formulas are less obvious, you should try to memorize them as they will be used time and time again in any calculus class.

**Derivative of $x$ to a power:**

**Power Rule:** If $f(x) = x^n$, where $n$ is any real number, then $f'(x) = nx^{n-1}$. If we use the prime notation, understanding that we are differentiating with respect to $x$, we have:

$(x^n)' = nx^{n-1}$

E.g. $y = x^7$, then $\frac{dy}{dx} = 7x^6$. In applying the power rule, $n$ can be any real number. Consider $f(x) = x^{\frac{2}{3}}$, then

$f'(x) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$

$f(x) = x^{\sqrt{3}}$, then

$f'(x) = \sqrt{3}x^{\sqrt{3}-1}$

$f(x) = x^\pi$, then

$f'(x) = \pi x^{\pi-1}$
**Constant Multiple Rule:**

If \( c \) is a constant, \( g(x) = cf(x) \), and \( f \) is differentiable, then \( g \) is differentiable and \( g'(x) = cf'(x) \). i.e. the derivative of a constant times a function is the constant times the derivative of the function.

\[
(cf(x))' = cf'(x)
\]

E.g. \( y = 3x^5 \), then \( \frac{dy}{dx} = 3(x^5)' = 3(5x^4) = 15x^4 \).

**Sum and Difference Rule** Suppose \( h(x) = f(x) + g(x) \), and \( f \) and \( g \) are both differentiable, then \( h \) is also differentiable and \( h'(x) = f'(x) + g'(x) \). i.e. The derivative of the sum is the sum of the derivatives.

\[
[f(x) + g(x)]' = f'(x) + g'(x)
\]

Suppose \( h(x) = f(x) - g(x) \), and \( f \) and \( g \) are both differentiable, then \( h \) is also differentiable and \( h'(x) = f'(x) - g'(x) \). i.e. The derivative of the difference is the difference of the derivatives.

\[
[f(x) - g(x)]' = f'(x) - g'(x)
\]

E.g.: We may combine the rules to find complicated derivatives such as this:

\( f(x) = \sqrt{x} + x^3 - 4x^2 + 2x - 5 \).

To find \( f'(x) \), let us rewrite the radical in a more convenient form:

\( f(x) = x^{1/2} + x^3 - 4x^2 + 2x - 5 \)

Using the sum and different rule, we know that

\[
\frac{dy}{dx} = \left( x^{1/2} \right)' + (x^3)' - (4x^2)' + (2x)' - (5)'
\]

\[
= \frac{1}{2}x^{(1/2)-1} + 3x^2 - 4(x^2)' + 2(x)' - 0
\]

\[
= \frac{1}{2}x^{-1/2} + 3x^2 - 4(2x) + 2
\]

\[
= \frac{1}{2\sqrt{x}} + 3x^2 - 8x + 2
\]
**Product Rule** Suppose \( h(x) = f(x) \cdot g(x) \) and \( f \) and \( g \) are both differentiable, then \( h \) is differentiable, and \( h'(x) = g(x) \cdot f'(x) + f(x) \cdot g'(x) \). i.e.

\[
[f(x) \cdot g(x)]' = g(x)f'(x) + f(x)g'(x)
\]

E.g. Find the derivative of

\[
y = (x^4 + 2x^2 - x)(2x^7 - 9x^4 - 3x + 4)
\]

Notice that \( y \) is a product of two functions, \((x^4+2x^2-x)\) and \((2x^7-9x^4-3x+4)\).

To find \( \frac{dy}{dx} \) we use the product rule:

\[
\frac{dy}{dx} = (x^4 + 2x^2 - x)(2x^7 - 9x^4 - 3x + 4)' + (2x^7 - 9x^4 - 3x + 4)(x^4 + 2x^2 - x)'
\]
\[
= (x^4 + 2x^2 - x)(14x^6 - 36x^3 - 3) + (2x^7 - 9x^4 - 3x + 4)(4x^3 + 4x - 1)
\]

**Warning:** It is important to note that

\[
[f(x) \cdot g(x)]' \neq f'(x) \cdot g'(x)
\]

In other words, The derivative of a product is not the product of the derivatives.

**Quotient Rule:** Suppose \( h(x) = \frac{f(x)}{g(x)} \), and \( f \) and \( g \) are differentiable, then \( h \) is differentiable where \( g(x) \neq 0 \) and \( h'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2} \). i.e.

\[
\left[ \frac{f(x)}{g(x)} \right]' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}
\]

**Warning:**

\[
\left[ \frac{f(x)}{g(x)} \right]' \neq \frac{f'(x)}{g'(x)}
\]

In other words, the derivative of a quotient is not the quotient of the derivatives.

Another important note is that in the numerator of the quotient rule, we are taking the difference of the two terms, so the order of which derivative to take first matters.
Example:

Let \( y = \frac{x^3 - 4x^2 + 1}{2x^2 - x} \) Find \( \frac{dy}{dx} \)

Since this is a quotient, we use the quotient rule:

\[
\frac{dy}{dx} = \frac{(2x^2 - x)[(x^3 - 4x^2 + 1)]' - (x^3 - 4x^2 + 1)[2x^2 - x]'}{(2x^2 - x)^2}
\]

\[
= \frac{(2x^2 - x)(3x^2 - 8x) - (x^3 - 4x^2 + 1)(4x - 1)}{(2x^2 - x)^2}
\]

E.g.

Let \( y = \frac{\sqrt{x} - 3x^4(x^2 + 1)}{(2x + 1)} \) Find \( \frac{dy}{dx} \)

This is a function which is a quotient, so in order to take its derivative, we must use the quotient rule:

\[
\frac{dy}{dx} = \frac{[2x + 1][((\sqrt{x} - 3x^4)(x^2 + 1))' - [\sqrt{x} - 3x^4](x^2 + 1)][2x + 1]'}{(2x + 1)^2}
\]

However, the numerator is itself a product, so to differentiate the numerator, we must use the product rule:

\[
[(\sqrt{x} - 3x^4)(x^2 + 1)]'
= (x^2 + 1)\left(\frac{1}{2\sqrt{x}} - 12x^3\right) + (\sqrt{x} - 3x^4)(2x)
\]

So the derivative of \( y \) becomes:

\[
\frac{dy}{dx} = \frac{[2x + 1][x^2 + 1]\left(\frac{1}{2\sqrt{x}} - 12x^3\right) + (\sqrt{x} - 3x^4)(2x)] - [\sqrt{x} - 3x^4](x^2 + 1)]2}{(2x + 1)^2}
\]
If we use the *prime* (′) notation, we can summarize the differentiation rules in compact form:

\[
\begin{align*}
(c)' &= 0 \\
(x^n)' &= nx^{n-1} \\
(cf)' &= cf' \\
(f + g)' &= f' + g' \\
(f - g)' &= f' - g' \\
(fg)' &= gf' + fg' \\
\left(\frac{f}{g}\right)' &= \frac{gf' - fg'}{g^2}
\end{align*}
\]
All the differentiation formulas introduced so far allows us to find derivatives of power, sum, difference, products and quotients. However, none of the differentiation formula allows us to find composition of functions. For example, suppose we want to find the derivative of $y = \sqrt{x^3 + x + 2}$, how are we going to do that?

In order to differentiate the above function, or any function that are the result of composition of two or more functions, we must use:

**Chain Rule:** Suppose $f$ and $g$ are differentiable functions and $h = f \circ g$ is the composite function defined by $h(x) = f(g(x))$, then $h$ is differentiable and

$$h'(x) = f'(g(x))(g'(x))$$

If we use the $d$ notation, and let $y = f(u)$ and $u = g(x)$, then we have:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

To many people, the chain rule is the most confusing differentiation rule and it needs to be explained in more detail. The chain rule says that the derivative of the composition of two functions is equal to the derivative of the first (the outside) function evaluated at the derivative of the second (the inside) function times the derivative of the second (the inside) function.

Example: find the derivative of $y = (x^4 - 5x^3 + 2x - 1)^8$.

Ans: Notice that $y$ is really a composition of two functions. If we were to differentiate $y$ (with respect to $x$), we need to use the chain rule. If we want to use the chain rule formally, here’s how it is done:

Let $f(x) = x^8$, $g(x) = x^4 - 5x^3 + 2x - 1$, then $h(x) = (x^4 - 5x^3 + 2x - 1)^8 = f(g(x))$ is the function we want to differentiate. The chain rule says that we should differentiate the first (outside) function, $f$, then evaluate it on the second function, $g$. i.e $h'(x) = f'(g(x))g'(x)$

Since $f'(x) = 8x^7$, and $g(x) = x^4 - 5x^3 + 2x - 1$, so we have

$$f'(g(x)) = 8(x^4 - 5x^3 + 2x - 1)^7$$

Now, $g'(x) = 4x^3 - 15x^2 + 2$, so according to the chain rule,

$$h'(x) = f'(g(x))g'(x) = 8(x^4 - 5x^3 + 2x - 1)^7(4x^3 - 15x^2 + 2)$$
Another way to apply the chain rule, using \( \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \)

We want to differentiate \( y = (x^4 - 5x^3 + 2x - 1)^8 \)

Let \( u = x^4 - 5x^3 + 2x - 1 \) (\( u \) is the inside), then \( y = u^8 \). We want to find \( \frac{dy}{dx} \).

According to the chain rule,
\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

Since \( \frac{dy}{du} = 8u^7 \), and \( \frac{du}{dx} = 4x^3 - 15x^2 + 2 \), so
\[
\frac{dy}{dx} = (8u^7)(4x^3 - 15x^2 + 2) = 8(x^4 - 5x^3 + 2x - 1)^7(4x^3 - 15x^2 + 2)
\]

The above method of applying the chain rule is the proper way to use the chain rule. However, most of the time people find it more convenient (and less confusing) to think of the chain rule as **derivative of the outside** followed by **derivative of the inside**. For the above example, \( y = (x^4 - 5x^3 + 2x - 1)^8 \), the **outside** function is the **eighth power** function, \( ()^8 \); the **inside** function is \( x^4 - 5x^3 + 2x - 1 \), so
\[
\frac{dy}{dx} = \frac{d}{dx}(\text{outside}) \cdot \frac{d}{dx}(\text{inside})
\]
\[
= 8(x^4 - 5x^3 + 2x - 1)^7(4x^3 - 15x^2 + 2)
\]

**Warning:** It is important to note that, when you differentiate the **outside**, the **inside** is unchanged, then you differentiate the inside, and multiply the two. **Never try to differentiate the outside and the inside at the same time.**

In the above example, \( y = (x^4 - 5x^3 + 2x - 1)^8 \),
\[
\frac{dy}{dx} \neq 8(4x^3 - 15x^2 + 2)^7
\]
Let \( y = \sqrt{x^3 + x + 2} \) Find \( \frac{dy}{dx} \)

This is once again a composition of two functions. The outside function is the square root \( \sqrt{\cdot} \) function, and the inside is the \( x^3 + x + 2 \) function. The derivative of the outside is \( \frac{1}{2\sqrt{\cdot}} \), and the derivative of the inside is \( 3x^2 + 1 \).

\[
\frac{dy}{dx} = \frac{1}{2\sqrt{x^3 + x + 2}} (3x^2 + 1) = \frac{3x^2 + 1}{2\sqrt{x^3 + x + 2}}
\]

Once again, when you differentiate the outside, do not change the inside. You differentiate the outside with the inside unchanged, then differentiate the inside and multiply the two together.

If you want to differentiate composition of more than two functions, you will need to apply the chain rule multiple times.

E.g.

Let \( f(x) = \sqrt{(3x^2 + 5)^4 + 1} \) Find \( f'(x) \)

Notice that \( f \) is a composition of three functions. The outer-most square root \( \sqrt{\cdot} \) function, the middle fourth power plus one \( [(\cdot)^4 + 1] \) function, and the inner-most \( 3x^2 + 5 \) function. To differentiate this function, we can treat the middle fourth power plus one \( (\cdot)^4 + 1 \) function and the inner-most \( 3x^2 + 5 \) function together as one single function, apply the chain rule once; then apply the chain rule once again to differentiate the composition of the middle function with the inner-most function.

\[
f'(x) = \frac{1}{2\sqrt{(3x^2 + 5)^4 + 1}} [(3x^2 + 5)^4 + 1]' \]

To differentiate the \( (3x^2 + 5)^4 + 1 \) function, we have to apply the chain rule one more time:

\[
[(3x^2 + 5)^4 + 1]' = [4(3x^2 + 5)^3][6x]
\]

so

\[
f'(x) = \frac{1}{2\sqrt{(3x^2 + 5)^4 + 1}} [(3x^2 + 5)^4 + 1]' = \frac{1}{2\sqrt{(3x^2 + 5)^4 + 1}} [4(3x^2 + 5)^3][6x] = \frac{24x(3x^2 + 5)^3}{2\sqrt{(3x^2 + 5)^4 + 1}} = \frac{12x(3x^2 + 5)^3}{\sqrt{(3x^2 + 5)^4 + 1}}
\]
Let’s do a problem that incorporates all the rules that we’ve learned so far:

\[
\text{Let } f(x) = \frac{(3x + 1)^8(x - 1)}{x^2 - 4x} \quad \text{Find } f'(x)
\]

This function is a quotient, so the first thing we want to do is to apply the quotient rule:

\[
f'(x) = \frac{(x^2 - 4x)[(3x + 1)^8(x - 1)]' - [(3x + 1)^8(x - 1)][x^2 - 4x]'}{(x^2 - 4x)^2}
\]

Since the numerator is a product, we must use the product rule to differentiate the numerator.

\[
[(3x + 1)^8(x - 1)]' = (x - 1)[(3x + 1)^8]' + [(3x + 1)^8](x - 1)'
\]

To differentiate the first factor, which involves an outside \([()^8]\) and an inside \(3x + 1\), we must apply the chain rule, this gives us:

\[
[(3x + 1)^8(x - 1)]' = (x - 1)[8(3x + 1)^7(3)] + (3x + 1)^8(1)
\]

\[
= 24(x - 1)(3x + 1)^7 + (3x + 1)^8
\]

The derivative of the function \(f(x) = \frac{(3x + 1)^8(x - 1)}{x^2 - 4x}\) can now be completed:

\[
f'(x) = \frac{(x^2 - 4x)[(3x + 1)^8(x - 1)]' - [(3x + 1)^8(x - 1)][x^2 - 4x]'}{(x^2 - 4x)^2}
\]

\[
= \frac{(x^2 - 4x)[24(x - 1)(3x + 1)^7 + (3x + 1)^8] - [(3x + 1)^8(x - 1)][2x - 4]}{(x^2 - 4x)^2}
\]
Differentiation of Exponential and Log Functions:

It turns out that the derivative of the natural exponential function \( f(x) = e^x \) is itself, that is:

If \( f(x) = e^x \), then \( f'(x) = e^x \)

Furthermore, the derivative of the natural logarithm is the reciprocal function, that is:

If \( f(x) = \ln x \), then \( f'(x) = \frac{1}{x} \)

Differentiation of trigonometric functions

Let \( f(x) = \sin x \)  Find \( f'(x) \)

Before we try to find the derivative of \( \sin x \), let’s first look at how the sine function behaves:

![Graph of sine function]

Notice that sine is a periodic function, so we should expect that its derivative to be periodic. In addition, notice that sine has horizontal tangents at the points \( \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \), so we should expect that the derivative of sine to be equal to 0 at these points. While we cannot prove anything with just this observation, it seems that cosine is a function that would satisfy the conditions mentioned above. Let us see how we can use the definition of derivative to prove this is the case:

We use the definition of derivative:

\[
 f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}
\]

We use the sum of angle formula to expand \( \sin(x + h) \).

\[
 \sin(x + h) = \sin x \cos h + \cos x \sin h
\]

So the derivative becomes:
\[ f'(x) = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} \]

\[ = \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \]

Rearrange the terms and factor by collecting like-terms give us:

\[ \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \]

\[ = \lim_{h \to 0} \frac{\sin x \cos h - \sin x + \cos x \sin h}{h} \]

\[ = \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} \]

From the rules of limit, we know that if

\[ \lim_{h \to 0} \frac{\sin x (\cos h - 1)}{h} \]

and

\[ \lim_{h \to 0} \frac{\cos x \sin h}{h} \]

both exist, then

\[ \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} \]

\[ = \lim_{h \to 0} \frac{\sin x (\cos h - 1)}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h} \]

But we know that

\[ \lim_{h \to 0} \frac{\sin x (\cos h - 1)}{h} \]

\[ = \lim_{h \to 0} \sin x \cdot \frac{\cos h - 1}{h} \]

\[ = \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} = \sin x \cdot 0 = 0 \]

and

\[ \lim_{h \to 0} \frac{\cos x \sin h}{h} \]

\[ = \lim_{h \to 0} \cos x \cdot \frac{\sin h}{h} \]

\[ = \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} \]

\[ = \cos x \cdot 1 = \cos x \]
So

\[ f'(x) = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} \]

\[ = \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \]

\[ = \lim_{h \to 0} \frac{\sin x \cos h - \sin x + \cos x \sin h}{h} \]

\[ = \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \]

\[ = \lim_{h \to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h} \]

\[ = 0 + \cos x = \cos x \]

We have just derived the formula for the derivative of \textit{sine} (with respect to \textit{x}): \[(\sin x)' = \cos x\]

In other words, the derivative of \textit{sine} is \textit{cosine}.

If we use the definition and similar methods, we may derive the differentiation formula for \textit{cosine}:

\[(\cos x)' = -\sin x\]

In other words, the derivative of \textit{cosine} is the \textit{negative of sine}.

Be aware of the negative sign on \textit{sin} \textit{x} when taking the derivative of \textit{cosine}.
Once we have the derivative of \( \text{sine} \) and \( \text{cosine} \), we may easily derive the differentiation formula for the other 4 trigonometric functions, as they can all be expressed in terms of \( \text{sine} \) and \( \text{cosine} \).

Let \( f(x) = \tan x \) Find \( f'(x) \)

We use the fact that \( \tan x = \frac{\sin x}{\cos x} \) combining with the quotient rule to find the derivative of \( \text{tangent} \):

\[
(tan\ x)' = \left[ \frac{\sin\ x}{\cos\ x} \right]' = \frac{\cos\ x \cdot [\sin\ x]' - \sin\ x \cdot [\cos\ x]'}{\cos^2\ x} = \frac{\cos\ x (\cos\ x) - \sin\ x (- \sin\ x)}{\cos^2\ x} = \frac{\cos^2\ x + \sin^2\ x}{\cos^2\ x} = \frac{1}{\cos^2\ x} = \sec^2\ x
\]

In other words, the derivative of \( \text{tangent} \) is \( \text{secant squared} \).

We may use similar methods to derive the formulas for the other three functions. You should be able to do this on your own, and you \textit{should} actually do it on your own to practice your skill with differentiation. Below is a summary of the differentiation rules for the six trigonometric functions:

\[
\begin{align*}
(\sin x)' &= \cos x \\
(\cos x)' &= -\sin x \\
(tan\ x)' &= \sec^2\ x \\
(cot\ x)' &= -\csc^2\ x \\
(sec\ x)' &= \sec x \tan x \\
(csc\ x)' &= -\csc x \cot x
\end{align*}
\]
E.g. Differentiate

\[ f(x) = \sin(x^2 - 3) + \cos(2x) \]

We need to use the chain rule to differentiate the given function:

\[
\begin{align*}
    f'(x) &= \cos(x^2 - 3)(x^2 - 3)' + (-\sin(2x))(2x)' \\
    &= \cos(x^2 - 3)(2x) - \sin(2x)(2) \\
    &= 2x \cos(x^2 - 3) - 2 \sin(2x)
\end{align*}
\]

E.g. Differentiate

\[ f(x) = \sin \left( \tan \left( \frac{2x+1}{x-2} \right) \right) \]

Notice that the function is sine of tangent, not sine times tangent. This is a composition of three functions. We need to apply the chain rule, not the product rule.

\[
\begin{align*}
    f'(x) &= \cos \left( \tan \left( \frac{2x+1}{x-2} \right) \right) \cdot \left[ \tan \left( \frac{2x+1}{x-2} \right) \right]' \\
    &= \cos \left( \tan \left( \frac{2x+1}{x-2} \right) \right) \cdot \sec^2 \left( \frac{2x+1}{x-2} \right) \cdot \left[ \frac{2x+1}{x-2} \right]' \\
    &= \cos \left( \tan \left( \frac{2x+1}{x-2} \right) \right) \cdot \sec^2 \left( \frac{2x+1}{x-2} \right) \cdot \frac{(x-2)(2x+1)' - (2x+1)(x-2)'}{(x-2)^2} \\
    &= \cos \left( \tan \left( \frac{2x+1}{x-2} \right) \right) \cdot \sec^2 \left( \frac{2x+1}{x-2} \right) \cdot \frac{2(x-2) - (2x+1)(1)}{(x-2)^2} \\
    &= \cos \left( \tan \left( \frac{2x+1}{x-2} \right) \right) \cdot \sec^2 \left( \frac{2x+1}{x-2} \right) \cdot \frac{2x-4 - 2x - 1}{(x-2)^2} \\
    &= \cos \left( \tan \left( \frac{2x+1}{x-2} \right) \right) \cdot \sec^2 \left( \frac{2x+1}{x-2} \right) \cdot \frac{-5}{(x-2)^2}
\end{align*}
\]
Let \( f(x) \) be differentiable and \( x \) and let \( f^{-1}(x) \) be defined. By definition,

\[
f(f^{-1}(x)) = x.
\]

Differentiating both sides of the equation with respect to \( x \), using the chain rule on the left hand side, gives:

\[
f'(f^{-1}(x)) \cdot \frac{d}{dx}(f^{-1}(x)) = 1 \Rightarrow (f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}
\]

Using this fact, we can derive the formula for the derivative of the inverse trigonometric functions:

\[
(tan^{-1}(x))' = \frac{1}{x^2 + 1}
\]

\[
(cot^{-1}(x))' = -\frac{1}{x^2 + 1}
\]

\[
(sin^{-1}(x))' = \frac{1}{\sqrt{1-x^2}}
\]

\[
(cos^{-1}(x))' = -\frac{1}{\sqrt{1-x^2}}
\]
Implicit Differentiation

All the differentiation we have done so far we find the derivative of a function when the function is explicitly expressed in terms of another variable. For example, when we want to find the derivative of \( f(x) = \sin(x^2 + 1) \), \( f \) is explicitly expressed as a function of \( x \). When we are given that \( y = \sqrt{x} \) and we want to find \( \frac{dy}{dx} \), \( y \) is also explicitly expressed as a function of \( x \). In these cases to find the derivative is relatively simple. We just need to apply the proper differentiation rules.

In many cases in mathematics, however, one variable is related to another variable by an equation which may not necessarily explicitly express one variable in terms of the other. The equation of a circle with radius \( r \) centered at the origin, for example, is given by

\[ x^2 + y^2 = r^2 \]

In this equation, \( y \) is not explicitly expressed as a function of \( x \). Instead, \( x \) and \( y \) are related implicitly by an equation. How do we find \( \frac{dy}{dx} \) in this case and what does \( \frac{dy}{dx} \) mean?

\( \frac{dy}{dx} \) still retains its same meaning. It is the rate of change of \( y \) with respect to \( x \). Graphically, \( \frac{dy}{dx} \) is still the slope of the line tangent to the graph of the equation at any given point \((x, y)\).

How can we find \( \frac{dy}{dx} \)? One way we may try is to try to change the equation so that \( y \) is expressed in terms of \( x \) and use the differentiation formulas. i.e. We may try to solve for \( y \) in terms of \( x \). This is sometimes possible. For example in the above equation we have:

\[ x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2 \Rightarrow y = \pm \sqrt{r^2 - x^2} \]

Using the differentiation formulas we find that

\[ \frac{dy}{dx} = -\frac{y}{x} \]

regardless of the sign of \( y \).

In some cases, though, the equation relating the two quantities are so complicated that it would be extremely difficult, if not impossible to try to solve for one variable in terms of the other. For example,

Find \( \frac{dy}{dx} \) if \( \sin(x^2 + y^2) = \cos(x^2 - y^2) \)

If we want to solve for \( y \) in terms of \( x \), we will have a very difficult time with
it. Instead, we use another method, which is called the method of **implicit differentiation**. We use the idea that, since this is an equation, if we perform the same operation on both sides of the equation, then the resulting expression after whatever operation we do on both side will still be equal. In order to find \( \frac{dy}{dx} \), we **differentiate both sides of the equation with respect to** \( x \). After the differentiation, we will be able to solve for \( \frac{dy}{dx} \) in terms of \( y \) and \( x \).

Let’s do the problem just mentioned:

Find \( \frac{dy}{dx} \) if \( \sin(x^2 + y^2) = \cos(x^2 - y^2) \)

We differentiate both side of the equations **with respect to** \( x \)

\[
\frac{d}{dx} \sin(x^2 + y^2) = \frac{d}{dx} \cos(x^2 - y^2)
\]

\[
\cos(x^2 + y^2) \frac{d}{dx} (x^2 + y^2) = -\sin(x^2 - y^2) \frac{d}{dx} (x^2 - y^2)
\]

\[
\cos(x^2 + y^2) \left[ 2x \frac{d}{dx} (x) + 2y \frac{d}{dx} (y) \right] = -\sin(x^2 - y^2) \left[ 2x \frac{d}{dx} (x) - 2y \frac{d}{dx} (y) \right]
\]

Some explanation is necessary here. We actually used the chain rule to differentiate \( x^2 + y^2 \). We are assuming that both \( x^2 \) and \( y^2 \) are functions of \( x \), which they are, so we need to differentiate the **outside**, which is the **square** function, and the **inside**, which is the \( x \) or \( y \) function. Basically, we are treating each variable as a function of itself. In other words, \( x \) is a function of \( x \) in addition to being a function of \( y \), and \( y \) is a function of \( f \), in addition to being a function of \( x \).

We know that \( \frac{d}{dx} (x^2) = 2x \), so we could have skipped the \( \frac{d}{dx} (x^2) \) part, but this way of writing everything out makes it more clear, as some people tend to get confused of, when you use implicit differentiation, why do you only have \( y' \) but not \( x' \). Well, we still have \( x' \), but \( x' \) represents \( \frac{dx}{dx} = 1 \).

\[
\cos(x^2 + y^2) \left[ 2x(1) + 2y \frac{dy}{dx} \right] = -\sin(x^2 - y^2) \left[ 2x(1) - 2y \frac{dy}{dx} \right]
\]

\[
\cos(x^2 + y^2) \left[ 2x + 2y \frac{dy}{dx} \right] = -\sin(x^2 - y^2) \left[ 2x - 2y \frac{dy}{dx} \right]
\]

Once we reach this stage, we may simply use algebra to solve for \( \frac{dy}{dx} \) in terms of \( x \) and \( y \).

\[
\cos(x^2 + y^2) (2x) + \cos(x^2 + y^2) 2y \frac{dy}{dx} = -\sin(x^2 - y^2) (2x) + \sin(x^2 - y^2) 2y \frac{dy}{dx}
\]
\[
\cos(x^2 + y^2)(2x) + \sin(x^2 - y^2)(2x) = \sin(x^2 - y^2)2y \frac{dy}{dx} - \cos(x^2 + y^2)2y \frac{dy}{dx}
\]

\[
2x \cos(x^2 + y^2) + 2x \sin(x^2 - y^2) = \frac{dy}{dx}(2y \sin(x^2 - y^2) - 2y \cos(x^2 + y^2))
\]

\[
\frac{dy}{dx} = \frac{2x \cos(x^2 + y^2) + 2x \sin(x^2 - y^2)}{2y \sin(x^2 - y^2) - 2y \cos(x^2 + y^2)}
\]
Find \( \frac{dy}{dx} \) if \( xy + x^2y = x^2 - 2\cos(y) \)

We start by differentiating both side of the equation with respect to \( x \):

\[
\frac{d}{dx}[xy + x^2y] = \frac{d}{dx}[x^2 - 2\cos(y)]
\]

\( xy \) and \( x^2y \) are products, so we need to use product rule to differentiate this:

\[
y \frac{d}{dx}(x) + x \frac{d}{dx}(y) + y \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(y) = 2x \frac{d}{dx}(x) - 2(-\sin y) \frac{d}{dx}(y)
\]

\[
y(1) + x \frac{dy}{dx} + y(2x) \frac{d}{dx}(x) + x^2 \frac{dy}{dx} = 2x(1) + 2\sin y \frac{dy}{dx}
\]

\[
y + x \frac{dy}{dx} + 2xy(1) + x^2 \frac{dy}{dx} = 2x + 2\sin y \frac{dy}{dx}
\]

We may now solve for \( \frac{dy}{dx} \)

\[
x \frac{dy}{dx} + x^2 \frac{dy}{dx} - 2\sin y \frac{dy}{dx} = 2x - y - 2xy
\]

\[
\frac{dy}{dx}(x + x^2 - 2\sin y) = 2x - y - 2xy
\]

\[
\frac{dy}{dx} = \frac{2x - y - 2xy}{x^2 + x - 2\sin y}
\]

E.g.

Find \( \frac{dy}{dx} \) if \( x - y^2 = \sqrt{xy} \)

As usual, we differentiate both sides with respect to \( x \):

\[
\frac{d}{dx}[x - y^2] = \frac{d}{dx}[\sqrt{xy}]
\]

\[
\frac{d}{dx}(x) - \frac{d}{dx}(y^2) = \frac{1}{2\sqrt{xy}} \cdot \frac{d}{dx}(xy)
\]

\[
1 - 2y \frac{d}{dx}(y) = \frac{1}{2\sqrt{xy}} \left[ y \frac{d}{dx}(x) + x \frac{d}{dx}(y) \right]
\]

\[
1 - 2y \frac{dy}{dx} = \frac{1}{2\sqrt{xy}} \left[ y(1) + x \frac{dy}{dx} \right]
\]

\[
1 - 2y \frac{dy}{dx} = \frac{y}{2\sqrt{xy}} + \frac{x}{2\sqrt{xy}} \frac{dy}{dx}
\]
\[-2y \frac{dy}{dx} - \frac{x}{2\sqrt{xy}} \frac{dy}{dx} = \frac{y}{2\sqrt{xy}} - 1\]

\[\frac{dy}{dx} (-2y - \frac{x}{2\sqrt{xy}}) = \frac{y}{2\sqrt{xy}} - 1\]

\[dy = \frac{\frac{y}{2\sqrt{xy}} - 1}{-2y - \frac{x}{2\sqrt{xy}}}\]

\[dx = \frac{1 - \frac{y}{2\sqrt{xy}}}{2y + \frac{x}{2\sqrt{xy}}}\]

We may simplify the compound fraction to get:

\[\frac{dy}{dx} = \frac{2\sqrt{xy} - y}{4y\sqrt{xy} + x}\]

With the method of implicit differentiation we may find \(\frac{dx}{dy}\) instead of \(\frac{dy}{dx}\) using the exact same method. We just differentiate both side of the equation with respect to \(y\) instead of \(x\).

Find \(\frac{dx}{dy}\) if \(\frac{xy}{x + y} = y\sin x\)

This time, we differentiate both side with respect to \(y\)

\[\frac{d}{dy} \left[ \frac{xy}{x + y} \right] = \frac{d}{dy} [y\sin x]\]

\[\frac{(x + y) \frac{d}{dy} [xy] - (xy) \frac{d}{dy} [x + y]}{(x + y)^2} = \sin x \frac{d}{dy} (y) + y \frac{d}{dy} (\sin x)\]

\[\frac{(x + y)(y \frac{dx}{dy} (x) + x \frac{dx}{dy} (y)) - (xy)(\frac{dx}{dy} (x) + \frac{dx}{dy} (y))}{(x + y)^2} = \sin x (1) + y \cos x \frac{d}{dy} (x)\]

\[\frac{(x + y)(y \frac{dx}{dy} (x) + x(1)) - (xy)(\frac{dx}{dy} (x) + 1)}{(x + y)^2} = \sin x + y \cos x \frac{dx}{dy}\]

\[\frac{(x + y)(y \frac{dx}{dy} (x) + x) - (xy)(\frac{dx}{dy} (x) + 1)}{(x + y)^2} = \sin x + y \cos x \frac{dx}{dy}\]

We may now solve for \(\frac{dx}{dy}\)
\[(x + y)(y \frac{dx}{dy} + x) - (xy)(\frac{dx}{dy} + 1) = (x + y)^2 (\sin x + y \cos x \frac{dx}{dy})\]

\[(x + y)\frac{dx}{dy} + (x + y)x - xy \frac{dx}{dy} - xy = (x + y)^2 \sin x + (x + y)^2 y \cos x \frac{dx}{dy}\]

\[(x + y)\frac{dx}{dy} - xy \frac{dx}{dy} - (x + y)^2 y \cos x \frac{dx}{dy} = (x + y)^2 \sin x - (x + y)x + xy\]

\[\frac{dx}{dy}((x + y)y - xy - (x + y)^2 y \cos x) = (x + y)^2 \sin x - (x + y)x + xy\]

\[\frac{dx}{dy} = \frac{(x + y)^2 \sin x - (x + y)x + xy}{(x + y)y - xy - (x + y)^2 y \cos x}\]

Sometimes, for convenience, we may use the prime (’) notation to stand for \(\frac{dy}{dx}\) or \(\frac{dx}{dy}\). The potential danger of using the prime notation is that sometimes you don’t know if \(y’\) refers to \(\frac{dy}{dx}\) or if it refers to \(\frac{dy}{dy}\). It is ok to use the prime notation, but make sure you know which variable you are differentiating with respect to.
Higher derivatives

We have seen that, given a function \( f(x) \), the derivative of the function, \( f'(x) \), is once again a function. For example, if \( f(x) = 3x^4 - 2x^2 + 1 \), then \( f'(x) = 12x^3 - 4x \) is once again a function of \( x \). Since the derivative itself is just another function, it makes sense to take about the derivative of the derivative, or second derivative, of a function. To avoid confusion, we would sometimes refer to the derivative of the original function as the first derivative of the function. Since we use \( f'(x) \) to denote the first derivative, we use \( f''(x) \) to represent the second derivative of \( f \) with respect to \( x \). In the above example, since \( f'(x) = 12x^3 - 4x \), if we take the derivative of this function, we have

\[
 f''(x) = 36x^2 - 4 
\]

E.g. Find the first and second derivative of \( f(x) = \sin(2x) + 5x^2 \)

Differentiating the function once we get:

\[
 f'(x) = 2 \cos(2x) + 10x 
\]

Differentiating the first derivative we get:

\[
 f''(x) = -4 \sin(2x) + 10 
\]

What does the second derivative represent?

Since the first derivative represents the rate of change of the original function, and the second derivative is the derivative of the first derivative, so it must represent the rate of change of the rate of change of the original function, or, in other words, the rate of change of the first derivative. The second derivative tells us how the first derivative changes as a function of \( x \) (or whatever variable we use).

A most illustrative example comes from the motion of a particle. Remember that if \( s(t) \) represents the distance travelled by a particle along a straight line, then the rate of change of distance with respect to time, \( v(t) = s'(t) \), represents the velocity of the particle. The rate of change of the velocity of the particle with respect to time, \( s''(t) = v'(t) = a(t) \) is the acceleration of the particle. Imagine that you are driving a car. The reading on the odometer tells you your car’s velocity, which is the first derivative. If you step on the pedal and increase the velocity of your car, then you are accelerating which means your second derivative is positive. On the other hand if you step on the break then you decelerate which means your second derivative is negative.

Why do we want to find the second derivative of a function? Because the second derivative gives us information that is not available from the first derivative. We will see in later sections that the first and second derivatives help us graph the function very accurately. We must keep in mind, though, that whether we are
finding the first or second derivative of the function, the purpose of this is always to study the properties of the original function.

Since the second derivative of a function is still just another function, it makes sense to talk about the derivative of the second derivative, or the third derivative of a function. We use \( f''(x) \) to represent the third derivative of \( f \). In general, we can talk about the \( n \)-th derivative of any function \( f \), where \( n \) is any non-negative integer (the 0-th derivative of \( f \), by definition, is \( f \) itself). Once we go beyond the third derivative, we usually use the notation \( f^n(x) \) to mean the \( n \)-th derivative of \( f \), which means taking the derivative of \( f \) \( n \) times. For example, if

\[
f(x) = x^8 - x^7 + x^2
\]

then

\[
\begin{align*}
f'(x) &= 8x^7 - 7x^6 + 2x \\
f''(x) &= 56x^6 - 42x^5 + 2 \\
f'''(x) &= 336x^5 - 210x^4 \\
f^4(x) &= 1680x^4 - 840x^3 \\
f^5(x) &= 6720x^3 - 2520x^2 \\
&\vdots
\end{align*}
\]

It is possible for a function to be differentiable at a point \( a \) but its second derivative does not exist at \( a \). For example, consider

\[
f(x) = x^{5/3}
\]

then

\[
\begin{align*}
f'(x) &= \frac{5}{3}x^{2/3} \\
f''(x) &= \frac{10}{9}x^{-1/3}
\end{align*}
\]

Notice that even though \( f(0) \) and \( f'(0) \) are both defined, but \( f''(0) \) is undefined.

It is possible for a function to have a derivative up to the \( n \)-th derivative, but its \( n + 1 \)-st derivative does not exist. If a function has \( n \)-th derivative but its \( n + 1 \)-st derivative does not exist, we say that the function is \( n \)-th differentiable.

If a function has derivative of any order, we say that the function is infinitely differentiable. Most of the functions we will be dealing with in beginning calculus are infinitely differentiable in their respective domain. It turns out that for most functions, we may use the highest derivatives of a function to approximate its value at any given point.
E.g. Let \( f(x) = 124x^{27} - 45x^{13} + x^2 - 5 \). Find the 254-th derivative of \( f \).

Observe that every time you differentiate \( f \) you knock down one power of \( f \). For example,
\[
f'(x) = 3348x^{26} - 585x^{12} + 2x
\]
The derivative of the function is a polynomial of degree 26, one less than the degree of the original function. And it is easy to see that if you take the second derivative, the resulting polynomial will be of degree 25. You can see from the pattern that after you took the 27-th derivative of the function, you will end up with a polynomial of degree zero, i.e a constant. So, the 28-th derivative of \( f \) will be zero, and since the derivative of 0 is 0, the \( n \)-th derivative of \( f \) for any \( n \geq 28 \) will be zero. Therefore,

\[
f^{254}(x) = 0
\]

E.g. Let \( f(x) = \sin x \). Find the 83rd derivative of \( f \).

Just like in the previous example, you do not want to use brute force. This time, let us observe the pattern:
\[
\begin{align*}
f(x) &= \sin x \\
f'(x) &= \cos x \\
f''(x) &= -\sin x \\
f'''(x) &= -\cos x \\
f^4(x) &= \sin x
\end{align*}
\]
So after we take the derivative of \( \sin x \) four times it goes back to itself again. This means the derivatives of \( \sin x \) just repeats every four times. When 83 is divided by 4, the remainder is 3, so
\[
f^{83}(x) = f'''(x) = -\cos x
\]
Related Rates

Derivative is a rate of change. Sometimes, when one quantity changes, another quantity changes according to how the first quantity changes. For example, imagine you pump air into a balloon. As the balloon increases in volume, its radius also increases. The rate of increase of the radius is related to the rate of increase of the balloon. This is what we meant by related rates.

When we talk about related rates, we are always referring to the rate of change of a quantity with respect to time. In other words, we are always looking for \( \frac{dq}{dt} \), where \( q \) is the quantity whose rate of change (with respect to time) we are interested.

In solving any related rate problem, try the following:

1. Ask yourself, for which quantity am I trying to find its rate of change. In other words, you are being asked for \( \frac{dx}{dt} \), what does \( x \) represent?

2. What are the other quantities that are also present in the problem, are they constant or does any of them change? If the quantity changes in value, represent them with a variable. Try to find the rate of change (\( \frac{d}{dt} \)) of these quantities, if they are given.

3. Is there relationship, either given in the problem, or known from math formula, geometry, physics, pythagorean theorem...etc, that can help me write an equation that relates those quantities in (1) and (2).

4. Differentiate both sides of the equation with respect to time, \( t \).

5. Plug in the values of the quantity at the given instance.

6. Solve the equation for \( \frac{dx}{dt} \).
E.g.
The radius of a circle is increasing at a rate of 3 meters per second. How fast is the area of the circle increasing when the radius is 10 meters?

Ans: You may try to use the guideline given above:

(1) What is the quantity being asked for? The question asks how fast the area of the circle is changing. Let \( A \) represent the area of the circle, then I am being asked for \( \frac{dA}{dt} \).

(2) The radius of the circle is also a quantity present in the problem. We are told that the radius is increasing, so it is not a constant, it must be represented by a variable, let’s use \( r \). In addition, we know that the radius is increasing at 3 meters per second, so \( \frac{dr}{dt} = 3 \text{ m/s} \).

(3) The relationship between the radius and area of a circle is given by:
\[
A = \pi r^2
\]
This is a formula that I know from geometry.

(4) Differentiate both sides of the equation with respect to time gives:
\[
\frac{dA}{dt} = 2\pi r \frac{dr}{dt}
\]

(5) At the moment when the radius is 10 meters, we have: \( r = 10 \). so \( \frac{dA}{dt} = 2\pi(10)3 = 60\pi \text{ m}^2/\text{s} \).

(6) The equation is already in terms of the unknown, \( \frac{dA}{dt} \), we have
\[
\frac{dA}{dt} = 60\pi \text{ m}^2/\text{s}.
\]
E.g. Two people start travel at the same place at the same time. One heads north with a speed of 4 miles per hour. One heads east with a speed of 3 miles per hour. At what rate is the distance between them increasing two hours after they began?

Ans:

(1) The two people are moving, so the distance between them is always changing. Let $h$ represent the distance between these two people, We are being asked for the rate this distance changes, so we are looking for $\frac{dh}{dt}$.

(2) Since both of the two people are moving, the distance they have travelled (from the starting point) are also variables. We will denote the distance travelled by the first person with the variable $y$, and denote the distance travelled by the second person by $x$. Since the first person has a speed of 4 miles per hour, $\frac{dy}{dt} = 4$. The second person has a speed of 3 miles per hour, so $\frac{dx}{dt} = 3$.

(3) We know from the pythagorean theorem that the relationship between $h$, $x$, and $y$ is given by:

$$h^2 = x^2 + y^2$$

(4) Differentiating both side with respect to time $t$ gives:

$$\frac{d}{dt}(h^2) = \frac{d}{dt}(x^2 + y^2)$$

$$2h \frac{dh}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

(5) After two hours, $x = 6$, $y = 8$, so $h = 10$, and we have:

$$2(10) \frac{dh}{dt} = 2(6)(3) + 2(8)(4)$$

(6) Solving for $\frac{dh}{dt}$ gives:

$$\frac{dh}{dt} = \frac{2(6)(3) + 2(8)(4)}{2(10)} = \frac{36 + 64}{20} = 5$$

After two hours, the distance between the two person is increasing at a rate of 5 miles per hour.
Water is leaking out of the bottom of an inverted cone at a rate of $4\text{m}^3/\text{sec}$. The base radius of the cone is 5m and its height is 8m. At what rate is the height of the water dropping when the height is at 3m?

Ans:

(1) The question asks for the rate the height of the water is changing. If we use $h$ to represent the height of the water, then we are looking for $\frac{dh}{dt}$.

(2) Since water is leaking, the volume and radius of the water (inside the cone) are both changing. They must be represented by variables. Let’s use $V$ for the volume and $r$ for the radius of the water. We also know from the given that water is leaking out of the cone, so $\frac{dV}{dt} = -4\text{m}^3/\text{s}$. We use the negative sign to indicate that the volume of the water is decreasing. It is also important to note that the cone itself (the container) is not changing. The radius of the container cone is always 5m and the height of the container cone is always 8m.

(3) What information do I know that I can set up an equation that relates these variables ($h, V, r$)?

The relationship between the volume of the cone and its base area and height is given by a geometry formula:

$$V = \frac{1}{3} \pi r^2 h$$

Problem with this equation is that it also involves the variable $r$. While we know $\frac{dV}{dt}$, we do not know $\frac{dr}{dt}$, so we would like to remove the variable $r$ in the equation. We can do this by using our knowledge of similar triangles. We use the fact that the larger triangle, formed by the container cone with base 5 and height 8, is similar to any smaller triangle, formed by the water with base $r$ and height $h$. Using ratios from similar triangles, we have

$$\frac{r}{h} = \frac{5}{8} \Rightarrow r = \frac{5h}{8}$$
Substituting this value for $r$ in the above equation gives:

$$V = \frac{1}{3} \pi \left( \frac{5h}{8} \right)^2 h = \frac{25\pi h^3}{192}$$

(4) Differentiating both sides with respect to time we get:

$$\frac{dV}{dt} = \frac{25\pi}{192} \left( 3h^2 \cdot \frac{dh}{dt} \right) = \frac{25\pi h^2}{64} \cdot \frac{dh}{dt}$$

(5) When the height of the water is at 3m, we have

$$-4 = \frac{25\pi (3)^2}{64} \cdot \frac{dh}{dt}$$

(6) Solving for $\frac{dh}{dt}$ gives:

$$\frac{dh}{dt} = \frac{-4(64)}{25(9)\pi} = -\frac{256}{225\pi} \text{ m/sec.}$$

The height of the water is decreasing at a rate of $\frac{256}{225\pi}$ m/sec.

In a typical related-rate problem, you will be given quantities whose values are changing, and you will be asked for the rate of change of a quantity at a particular time or value. When trying to solve such a problem, set up an equation with all the quantities as variables, then differentiate with respect to time, then plug in the value of the quantity at the given value. Do not start the equation with the given value for the quantity, because then you are treating the given quantity as a constant, which does not work.
Linear Approximations and Differentials

As we have discussed the derivative of a function gives the slope of the tangent line to a function. We also mentioned that the tangent line to a function at a point \( a \) is the best approximation of \( f \) by a line at point \( a \). That means if we were to try to approximate the value of \( f \) at or near the point \( a \) using a straight line, then the tangent line to \( f \) at \( a \) is the straight line that best approximates this.

What is the equation of the line tangent to \( f \) at a point \( a \)? The point we have is \((a, f(a))\), and the slope of the tangent line is \( f'(a) \). Using the point slope form of the equation of a line we have:

\[
y - f(a) = f'(a)(x - a) \Rightarrow y = f(a) + f'(a)(x - a)
\]

This is the equation of the line tangent to \( f \) at a point \( a \). It is the line that best approximates \( f \) at the point \( a \), and is called the linear approximation of \( f \) at \( a \). We use the notation:

\[
L(x) = f(a) + f'(a)(x - a)
\]

\( L(x) \) is the linear approximation of \( f \) at \( a \). Sometimes we say that the approximation is centered at \( a \) or that \( a \) is the center of the approximation.
E.g.

Let \( f(x) = \sqrt{x} \). Find the linear approximation of \( f \) at 4.

Since \( a = 4 \), using the linear approximation formula, we have: \( L(x) = f(4) + f'(4)(x - 4) \). In order to find \( f'(4) \), we must take the derivative of \( f \).

\[
f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}
\]

Also, \( f(4) = \sqrt{4} = 2 \), so the linear approximation of \( \sqrt{x} \) at 4 is:

\[
L(x) = 2 + \frac{1}{4}(x - 4)
\]

This means that, at the point \((4, 2)\), the line that best approximates \( \sqrt{x} \) has the equation \( y = 2 + \frac{1}{4}(x - 4) \). We use the notation

\[
\sqrt{x} \approx 2 + \frac{1}{4}(x - 4) \quad \text{when } x \text{ is near 4}
\]

to mean that the line \( 2 + \frac{1}{4}(x - 4) \) is an approximation of \( \sqrt{x} \) for \( x \) near the number 4.

We can use this linear approximation to estimate (approximate) the values of \( f(x) = \sqrt{x} \) near the point 4. For example, \( L(4.1) = 2 + \frac{1}{4}(4.1 - 4) = 2.025 \). The actual value of \( \sqrt{4.1} \) is 2.0248. So we are accurate up to 2 places after the decimal.

Naturally, the closer the point we choose to the center (\( a \)), the better the approximation. Since 4.1 is relatively close to 4, we get a rather good approximation. If we try 5, we have \( L(5) = 2 + \frac{1}{4}(5 - 4) = 2.25 \). The actual value of \( \sqrt{5} \) is 2.236. We are only accurate upto 1 place after the decimal.

For the \( \sqrt{x} \) function, we can easily find values like \( \sqrt{1}, \sqrt{4}, \sqrt{9}, \sqrt{16}, \ldots \) etc. Any time we want to approximate the value of \( \sqrt{x} \) at a particular value, we want to choose \( a \) to be one of the value we can easily evaluate \( \sqrt{a} \) and that \( a \) is nearest to
the value in question. For example, if we want to approximate the value of $\sqrt{15.8}$, then we want to choose $a = 16$ since 16 is closest to 15.8 amount all the perfect squares. On the other hand, if we want to approximate $\sqrt{26}$, we would want to use $a = 25$.

E.g.
Find the linear approximation of $f(x) = \sin(x)$ at 0, and use that to approximate $\sin 0.1$.

We note that $f'(x) = \cos(x)$, $f'(0) = \cos(0) = 1$. $f(0) = \sin(0) = 0$.

So the formula for the linear approximation of $\sin x$ at 0 gives:

$$L(x) = f(0) + f'(0)(x - 0) = 0 + 1(x - 0) = x$$

That means, at values of $x$ near 0,

$$\sin x \approx x$$

Using $L$ to approximate $\sin(0.1)$ we have: $L(0.1) = 0.1$. The actual value of $\sin(0.1)$ is 0.0998.
Differentials

A concept that is closely related to linear approximation is the concept of **differentials**. Remember that by the definition of derivatives,

\[
\frac{dy}{dx} = f'(x)
\]

As we mentioned, \(dy\) and \(dx\) are *not* numbers. They are just symbols to represent the derivative of a function. However, we can *define* them to be numbers by assigning a (usually very small) value to \(dx\) then define:

\[
dy = f'(x)dx
\]

\(dy\) and \(dx\) are called the **differentials**.

If we use the \(f(x)\) notation, with \(y = f(x)\), we can also say that \(df = f'(x)\ dx\)

In the above formula, \(dx\) is the *independent variable*, while \(dy\) or \(df\) is the *dependent variable* that depends on the values of \(dx\) and \(x\).

E.g. Let \(f(x) = x^2\). Find the differential of \(f\).

**Ans:** Since \(f'(x) = 2x\), we have \(df = 2x\ dx\).

E.g. Let \(y = \sin(x)\), Find the differential of \(y\).

**Ans:** Since \(\frac{dy}{dx} = \cos x\), \(dy = \cos x\ dx\)
What is the differential?

Let $f$ be a function. Let $(x_0, f(x_0))$ be a point on the graph of $f$. If $\Delta x$ represents the change in $x$ and $\Delta y$ the change in $y$ of $f$, then we know that

$$\Delta y = f(x_0 + \Delta x) - f(x_0)$$

That is, $\Delta y$ represents the changes in $f$ when $x_0$ changes by $\Delta x$ amount.

On the other hand, if we write

$$df = f'(x)dx$$

and use $dx = \Delta x$, then $df$ represents the change in the tangent line when $x_0$ changes by $\Delta x$ amount. The difference between $\Delta y$ and $df$ is the difference between the function $f$ and the tangent line when $x$ changes by $\Delta x$. In general, the smaller the $\Delta x$, the smaller the difference between $\Delta y$ and $df$, which means the better the approximation if we were to use $df$ to approximate the value of $f$. 
E.g.
Find $df$ and $\Delta y$ for $f(x) = x^2 - 5x + 5$ at $x = 2$ and $\Delta x = dx = 0.1$.
Ans: Since $f'(x) = 2x - 5$. We have

$$df = (2x - 5)dx$$

At $x = 2$, $\Delta x = dx = 0.1$, we have

$$df = (2(2) - 5)(0.1) = -1(0.1) = -0.1$$

$$\Delta y = f(x + \Delta x) - f(x) = f(2 + 0.1) - f(2)$$
$$= (2.1)^2 - 5(2.1) + 5 - (2^2 - 5(2) + 5) = -1.09 - (-1) = -0.09$$

So the difference between $df = (-0.1)$ and $\Delta y(-0.09)$ is 0.01. If we use the differential to approximate $f(2.1)$, we will get

$$f(2.1) = f(2 + 0.1) \approx f(2) + dy = -1 + -0.1 = -1.1$$