

## Sequences:

Definition: A **sequence** is a function whose domain is the set of natural numbers or a subset of the natural numbers. We usually use the symbol  $a_n$  to represent a sequence, where  $n$  is a natural number and  $a_n$  is the value of the function on  $n$ .

Intuitively, a sequence is just an ordered list of (possibly infinitely many) numbers. Each number in a sequence is a **term** of the sequence. We usually use the letter  $i$  as the **index**, and  $a_i$  is the  **$i$ -th term** of the sequence.

A sequence may be **finite** or **infinite**.

If a sequence is finite, we sometimes write  $\{a_1, a_2, a_3, a_4, \dots, a_n\}$  to represent the sequence; If a sequence is infinite, we write  $\{a_1, a_2, a_3, \dots\}$  or  $\{a_i\}_{i=1}^{\infty}$ .

The notation  $\{a_i\}$  implies that we have a sequence whose first term is  $a_1$ , the second term is  $a_2$ , the third term is  $a_3$ ...etc. The **index**  $i$  starts from 1 (or any other positive integer) and increases by 1 each time to represent each subsequent term in the sequence.

A sequence can be represented by a formula expressed as an expression in  $i$  or in  $n$ .

If a sequence has a pattern we can also write the first few terms of the sequence and assume that the pattern continues and let the reader figure out the values of the subsequent terms.

A sequence can also be defined **recursively**, where we assign a value to the first (or first few) terms of the sequence, and the value of each term in the sequence is then defined by one or more of the preceding terms.

We can also describe a sequence verbally if there's no obvious formula or pattern that we can use to express the sequence.

Example: Consider the sequence  $\{a_i = -6\}_{i=1}^{\infty}$ . Starting with  $i = 1$ , since  $a_i = -6$ , so  $a_1 = -6$  is the first term of the sequence. If  $i = 2$ , then  $a_2 = -6$ . If  $i = 3$ , then  $a_3 = -6$ . The value of  $a_i$  is always the same value, so we have the sequence of constant terms:  $\{a_i = -6\}_{i=1}^{\infty} = \{-6, -6, -6, -6, \dots\}$

Example: Consider the sequence  $\{a_i = i\}_{i=1}^{\infty}$ . Starting with  $i = 1$ , since  $a_i = i$ , so  $a_1 = 1$  is the first term of the sequence. If  $i = 2$ , then  $a_2 = 2$ . If  $i = 3$ , then  $a_3 = 3$ . Continue in this fashion, we obtain the sequence of positive integers:  $\{a_i\}_{i=1}^{\infty} = \{1, 2, 3, 4, \dots\}$

Example: Consider the sequence  $\{a_i = i^2 - 3\}_{i=1}^{\infty}$ . Starting with  $i = 1$ , since  $a_i = i^2 - 3$ , so  $a_1 = 1^2 - 3 = -2$  is the first term of the sequence. If  $i = 2$ , then  $a_2 = 2^2 - 3 = 1$  is the second term. If  $i = 3$ , then  $a_3 = 3^2 - 3 = 6$  is the third term. If  $i = 4$ , then  $a_4 = 4^2 - 3 = 13$  is the fourth term. Continue in this fashion, we obtain the following sequence:  $\{a_i\}_{i=1}^{\infty} = \{-2, 1, 6, 13, 22, 33, 46, \dots\}$

Example: Consider the sequence  $\left\{a_i = \frac{i}{i+1}\right\}_{i=1}^{\infty}$ . Starting with  $i = 1$ , since

$a_i = \frac{i}{i+1}$ , so  $a_1 = \frac{1}{1+1} = \frac{1}{2}$  is the first term of the sequence. If  $i = 2$ , then

$a_2 = \frac{2}{2+1} = \frac{2}{3}$  is the second term. If  $i = 3$ , then  $a_3 = \frac{3}{3+1} = \frac{3}{4}$  is the third

term. If  $i = 4$ , then  $a_4 = \frac{4}{4+1} = \frac{4}{5}$  is the fourth term. Continue in this fashion,

we obtain the following sequence:  $\{a_i\}_{i=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots\right\}$

Example: Consider the sequence:  $\{2, 4, 6, 8, \dots\}$ . Assuming the pattern continues, this is the sequence of positive even integers. We can also represent this sequence using a formula:  $\{a_i = 2i\}_{i=1}^{\infty}$  or  $\{2i\}_{i=1}^{\infty}$

Example:  $\left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \dots\right\}$  Assuming the pattern continues, this is a

sequence whose terms alternate in sign. It can be expressed by  $a_i = \frac{(-1)^{i+1}}{i}$

Example: Consider the sequence defined by  $a_i = a_{i-1} + a_{i-2}$ ;  $a_1 = 1, a_2 = 1$ . This is a sequence defined recursively. The value of the first two terms is assigned to be 1 and 1, then the value of any next terms is defined to be the sum of the preceding two terms. From the definition of this sequence, we have:  $a_3 = a_2 + a_1 = 1 + 1 = 2$ ,  $a_4 = a_3 + a_2 = 2 + 1 = 3$ ,  $a_5 = a_4 + a_3 = 3 + 2 = 5$ ,  $a_6 = a_5 + a_4 = 5 + 3 = 8$ ,  $a_7 = a_6 + a_5 = 8 + 5 = 13$ . Continue in this pattern, we obtained the following sequence of numbers:

$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots\}$

This sequence is called the sequence of *Fibonacci numbers*.

Notice that when defining a sequence recursively, one can change the **initial values** of the sequence while keeping the same relationship between the terms, and we will end up with a sequence with different terms. In the previous example, if we define:  $a_i = a_{i-1} + a_{i-2}$ ;  $a_1 = 3, a_2 = 4$ , we end up with this sequence:

$\{3, 4, 7, 11, 18, 29, 47, 78, 125, \dots\}$

Example: Define a sequence recursively by:  $a_n = n(a_{n-1})$ ;  $a_0 = 1$

We obtain the terms of this sequence as:  $a_1 = 1(a_0) = 1(1) = 1$ ,  $a_2 = 2(a_1) = 2(1) = 2$ ,  $a_3 = 3(a_2) = 3(2) = 6$ ,  $a_4 = 4(a_3) = 4(6) = 24$ ,  $a_5 = 5(a_4) = 5(24) = 120$ . This sequence is used often in mathematics and is named the **factorial**, and we use the symbol  $!$ , to represent, we write:

$n! = n(n-1)!$

$0! = 1$

Notice that by definition,  $0! = 1$ ,  $1! = 1(0!) = 1(1) = 1$ ,  $2! = 2(1!) = 2(1)$ ,

$3! = 3(2!) = 3(2(1!)) = 3(2)(1)$ ,  $4! = 4(3!) = 4(3)(2)(1)$ . In general,

$$n! = n(n-1)(n-2)\cdots(3)(2)(1)$$

So we can define factorial in a different way: For any non-negative integer  $n$ ,

$$n! = 1 \quad \text{if } n = 0$$

$$n! = n(n-1)(n-2)\cdots(3)(2)(1) \quad \text{if } n \geq 1$$

Using this definition, we can evaluate:  $7! = 7(6)(5)(4)(3)(2)(1) = 5040$ .

$$11! = 11(10)(9)(8)(7)(6)(5)(4)(3)(2)(1) = 39916800$$

Example: Consider the sequence defined by:  $a_i =$  the  $i$ -th prime number.

This is the sequence  $\{2, 3, 5, 7, 11, \dots\}$ . This sequence cannot be expressed as an expression in  $i$ , but is well-defined.

Example: The sequence  $\{1, 1, 1, \dots\}$  is a sequence defined by  $a_i = 1$ .

Example: The sequence  $\{i^3\}_{i=1}^{\infty}$  is the sequence of positive perfect cubes,

$$\{a_i\} = \{1, 8, 27, 64, 125, \dots\}$$

Example: The sequence  $\frac{1}{i^2 + 1}$  is the sequence:  $\left\{\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \dots\right\}$

Example: Find an expression in  $i$  for the sequence  $\left\{\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \dots\right\}$

Ans: For each fraction, the denominator is the square of a number one greater than the numerator, so we may use:  $a_i = \frac{i}{(i+1)^2}$

Notice that we may also use:  $\left\{a_i = \frac{i-1}{i^2}\right\}_{i=2}^{\infty}$ . For any sequence,  $i$  does not have to start at 1.

Example: Find an expression in  $i$  for the sequence  $\left\{2, -\frac{3}{2}, \frac{4}{3}, -\frac{5}{4}, \frac{6}{5}, \dots\right\}$

Ans: This is a sequence where the denominator is one less than the numerator. To make the terms alternate in sign, we use a power of  $-1$ :

$$a_i = (-1)^{i+1} \left(\frac{i+1}{i}\right), \quad i \geq 1$$

## Sigma Notation

Suppose we have a sequence  $\{a_i\} = \{a_1, a_2, a_3, \dots\}$ , often times we want to add some or all of the terms of the sequence to find the sum. Instead of writing  $a_1 + a_2 + a_3 + \dots + a_i + \dots$  every time, we use the **sigma notation**,  $\Sigma$ , to represent the sum of all these terms:

Definition:  $\sum_{i=1}^n a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$

Example: Consider:  $\sum_{i=1}^8 i$ . This means that  $a_i = i$ , we start with  $i = 1$ , so  $a_1 = 1$ , then increases  $i$  by 1 each time until we get to 8, we have:

$$\sum_{i=1}^8 i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36.$$

Example: Consider:  $\sum_{i=1}^{11} i^2$ . This means that  $a_i = i^2$ , we start with  $i = 1$ , so  $a_1 = 1^2 = 1$ , then increases  $i$  by 1, we get  $a_2 = 2^2 = 4$ , then increases  $i$  by 1 again, so  $a_3 = 3^2 = 9$ . Continue in this pattern until  $i = 11$ , we have:

$$\sum_{i=1}^{11} i^2 = 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 + 121 = 506$$

Example: Consider:  $\sum_{i=5}^{12} (2i + 1)$ . This means that  $a_i = (2i + 1)$ , but this time to get the sum, we start with  $i = 5$ , so  $a_5 = 2(5) + 1 = 11$ , then increasment  $i$  by 1, we have  $a_6 = 2(6) + 1 = 13$ . Continue in this fashion until  $i = 12$ , we have:

$$\sum_{i=5}^{12} (2i + 1) = 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 = 144$$

Example: Consider:  $\sum_{i=1}^9 4$ . This means that  $a_i = 4$  for all  $i$ . In other words,

$$a_1 = 4, a_2 = 4, \dots \text{etc.}, \text{ so } \sum_{i=1}^9 4 = 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 = 36$$

Some formula involving sigma that would be useful to know:

If  $c$  is a constant,

$\sum_{i=1}^n c = nc$  (we are adding the same constant,  $c$ , for  $n$  many times, the result is  $n$  times  $c$ )

$$\sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i \text{ (this is just the distributive property)}$$

$$\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i \text{ (communtative property of addition)}$$

$$\sum_{i=1}^n (a_i - b_i) = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i$$

$$\sum_{i=1}^n i = \frac{(n)(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{(n)(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i^3 = \left[ \frac{(n)(n+1)}{2} \right]^2$$

## Arithmetic Sequence

An **arithmetic sequence** is a sequence where the **difference** between any two consecutive terms is a constant. This constant is called the **common difference**. We usually use the letter  $d$  to represent the common difference.

Example:  $\{3, 7, 11, 15, 19, 23, \cdot\}$ .

This is an arithmetic sequence, the common difference between any two consecutive terms is the number  $d = 4$ .

Example:  $\{8, 5, 2, -1, -4, -7, -10, -13, \cdot\}$

This is an arithmetic sequence, the common difference is the number  $d = -3$ .

Example: The sequence  $a_n = 2n + 1$  is an arithmetic sequence, if we look at the difference between any two consecutive terms  $a_n$  and  $a_{n+1}$ , we get:

$a_{n+1} - a_n = (2(n+1) + 1) - (2n + 1) = (2n + 2 + 1) - (2n + 1) = (2n + 3) - (2n + 1) = 2$ .  
The common difference is  $d = 2$ .

The sequence defined by the formula is:  $\{3, 5, 7, 9, 11, \dots\}$

Example: Find the  $n$ -th terms of the arithmetic sequence where the first term is  $a_1$  and common difference is  $d$ .

By definition of an arithmetic sequence, the next term of the sequence can be obtained by adding the common difference  $d$  to the preceding term, so we can obtain the term  $a_2$  by:

$$a_2 = a_1 + d$$

We can similarly obtain the third term,  $a_3$ , by:

$$a_3 = a_2 + d = (a_1 + d) + d$$

Continuing the pattern gives us:

$$a_4 = a_3 + d = (a_2 + d) + d = (a_1 + 2d) + d = a_1 + 3d$$

$$a_5 = a_4 + d = (a_1 + 3d) + d = a_1 + 4d$$

$\vdots$

$$a_n = a_1 + (n - 1)d$$

We obtained the following formula:

For an arithmetic sequence, if the first term is  $a_1$  and the common difference is  $d$ , then the  $n$ -th term  $a_n$  is given by:

$$a_n = a_1 + (n - 1)d$$

Example: Given an arithmetic sequence,  $\{a_1, a_2, a_3, a_4, \dots, a_i\}$ , find the sum of

the first  $n$  terms of the sequence, in other words, find:  $\sum_{i=1}^n a_i$

Since  $\{a_i\}$  is an arithmetic sequence, the  $i$ -th term of the sequence can be expressed as  $a_i = a_1 + (i - 1)d$ , where  $d$  is the common difference, so we write the above as:

$$\begin{aligned}\sum_{i=1}^n a_i &= \sum_{i=1}^n a_1 + (i - 1)d \\ &= \sum_{i=1}^n a_1 + \sum_{i=1}^n (i - 1)d \\ &= (n)(a_1) + d \left( \sum_{i=1}^n (i - 1) \right) \\ &= (n)(a_1) + d \left( \sum_{i=1}^{n-1} i \right) \\ &= (n)(a_1) + d \left( \frac{(n - 1)(n)}{2} \right) \\ &= (n)(a_1) + \frac{d}{2} ((n - 1)(n)) \\ &= \frac{1}{2}(n) [2a_1 + d(n - 1)]\end{aligned}$$

We just discovered the following formula:

The sum of the first  $n$  terms of an arithmetic sequence, with first term  $a_1$  and common difference  $d$ , is given by:

$$S_n = \frac{n}{2} [2a_1 + d(n - 1)]$$

Since in an arithmetic sequence, the  $n$ -term,  $a_n$ , can be expressed as  $a_n = a_1 + d(n - 1)$ , we can change the formula to be in terms of the first and  $n$ -th term:

$$\begin{aligned}S_n &= \frac{n}{2} [2a_1 + d(n - 1)] \\ &= \frac{n}{2} [a_1 + a_1 + d(n - 1)] \\ &= \frac{n}{2} [a_1 + (a_1 + d(n - 1))] \\ &= \frac{n}{2} [a_1 + a_n]\end{aligned}$$

Formula: The sum of the first  $n$ -terms of an arithmetic sequence, where the first term is  $a_1$  and the  $n$ -term is  $a_n$ , is given by:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Example: Find a formula for the given sequence, then find the value of the 25th term of the sequence, and find the sum of the first 40 terms.

$$\{-7, -2, 3, 8, 13, 18, 23, \dots\}$$

The difference between each term and the next is the constant 5, so this is an arithmetic sequence, with  $a_1 = -7$  and  $d = 5$ , using the formula, we have:

$$a_{25} = a_1 + d(25 - 1) = -7 + (5)(24) = 113$$

Since we know the common difference and the first term, we can use the formula:

$$S_n = \frac{n}{2}[2a_1 + d(n - 1)] \text{ to find the sum of the first } n \text{ terms:}$$

$$S_{40} = \frac{40}{2}[2(-7) + 5(40 - 1)] = 20[-14 + 5(39)] = 3620$$

Example: A sequence is given by  $\{a_n = 4n - 6\}$ . Find the sum of the first 31 terms of this sequence.

If we take the difference between any two terms,  $a_{n-1}, a_n$  of this sequence, we get:

$$a_n - a_{n-1} = (4n - 6) - (4(n - 1) - 6) = 4n - 6 - (4n - 4 - 6) = 4$$

The difference between any two terms of this sequence is a constant, 4. This is an arithmetic sequence with common difference  $d = 4$ . The value of the first term is  $a_1 = 4(1) - 6 = -2$ . We can now apply the formula:

$$S_{31} = \frac{31}{2}[(2)(-2) + (4)(31 - 1)] = \frac{31}{2}[-4 + 120] = 1798$$

Example: Find the sum of the first 17 terms of the following sequence:

$$\left\{2, \frac{7}{2}, 5, \frac{13}{2}, 8, \frac{19}{2}, 11, \frac{25}{2}, 14, \dots\right\}$$

The difference between any consecutive two terms is  $\frac{3}{2}$ , so this is an arithmetic sequence with common difference  $d = \frac{3}{2}$

We use the formula:

$$S_{17} = \frac{17}{2}\left[2(2) + \frac{3}{2}(17 - 1)\right] = \frac{17}{2}[4 + 24] = 238$$

Example: Find the sum of the first 35 terms of an arithmetic sequence  $\{a_n\}$  if  $a_{10} = -13$  and  $a_{16} = -49$

Ans: We need to know the the value of the first term  $a_1$  and the common difference  $d$  before we can answer the question, we use the formula:  $a_n = a_1 + (n - 1)d$ , to set up equations to solve for  $a_1$  and  $d$ :

$$a_{10} = a_1 + (10 - 1)d = -13 \Rightarrow a_1 + 9d = -13$$



$$a_{16} = a_1 + (16 - 1)d = -49 \Rightarrow a_1 + 15d = -49$$

We have a system of two equations with two unknown. Solving this system gives us:

$$a_1 + 9d = -13$$

$$a_1 + 15d = -49$$

$$-6d = 36 \Rightarrow d = -6$$

$$a_1 + 9(-6) = -13 \Rightarrow a_1 = 41$$

Now we can use the formula to find the sum of the first 35 terms:

$$S_{35} = \frac{35}{2} [2(41) + (-6)(35 - 1)] = \frac{35}{2} [82 + (-204)] = -2135$$

## Geometric Sequence

Definition: A sequence  $\{a_n\}$  is a **geometric sequence** if the **ratio** between a term and its preceding term is a (non-zero) constant. This constant is called the **common ratio** of the sequence and is usually represented by  $r$ . In other words,

$$r = \frac{a_n}{a_{n-1}}$$

Example: Consider the sequence  $\{2, 4, 8, 16, 32, 64, \dots\}$ . The ratio between any term of the sequence and its preceding term is the number 2, so this is a geometric sequence. The common ratio is the number  $r = 2$ .

Example: The sequence  $\left\{4, -\frac{4}{3}, \frac{4}{9}, -\frac{4}{27}, \frac{4}{81}, -\frac{4}{243}, \dots\right\}$ . The ratio between any term and its preceding term is the number  $-\frac{1}{3}$ , so this is a geometric sequence.

The common ratio is  $r = -\frac{1}{3}$

Example: Find the expression for the  $n$ -th term of a geometric sequence if the first term is  $a_1$  and the common ratio is  $r$ .

Ans: By definition,  $r = \frac{a_n}{a_{n-1}} \Rightarrow a_n = r a_{n-1}$ . In other words, to obtain the next term in a geometric sequence, we multiply the common ratio to the previous term, we have:

$$a_2 = a_1 r$$

$$a_3 = a_2 r = (a_1 r)(r) = a_1 r^2$$

$$a_4 = a_3 r = (a_1 r^2)r = a_1 r^3$$

$\vdots$

Looking at the pattern, we can deduce a formula for the  $n$ -th term of a geometric sequence:

Formula: If a geometric sequence has first term  $a_1$  and common ratio  $r$ , then its  $n$ -term,  $a_n$ , is given by:

$$a_n = a_1 r^{n-1}$$

Example: Find the 24th term of the sequence:

$$\left\{-\frac{4}{9}, -\frac{1}{3}, -\frac{1}{4}, -\frac{3}{16}, -\frac{9}{64}, -\frac{27}{256}, -\frac{81}{1024}, \dots\right\}$$

Ans: The ratio between each term of this sequence and the previous term is the same,  $\frac{3}{4}$ , so this is a geometric sequence with  $r = \frac{3}{4}$ ,  $a_1 = -\frac{4}{9}$ , the value of the 24th term is given by:

$$a_n = a_1 r^{n-1} \Rightarrow a_{24} = \left(-\frac{4}{9}\right) \left(\frac{3}{4}\right)^{24-1} = \left(-\frac{4}{9}\right) \left(\frac{3}{4}\right)^{23} \approx -0.0005946$$

Example: Find the sum of the first  $n$  terms of a geometric sequence  $\{a_n\}$  where the first term is  $a_1$  and the common ratio is  $r$ .

Ans: To add the first  $n$  terms of a geometric sequence with first term  $a_1$  and common ratio  $r$ , we need to find the following sum:

$$S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + a_1r^4 + \dots + a_1r^{n-1}$$

If we factor the common factor  $a_1$  in the expression we have:

$$S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + a_1r^4 + \dots + a_1r^{n-1} = a_1 (1 + r + r^2 + r^3 + r^4 + \dots + r^{n-1})$$

Consider the function  $f(x) = x^n - 1$ . Notice that  $f(1) = 1^n - 1 = 0$ . Since  $f(1) = 0$ , the factor theorem tells us that  $x - 1$  is a factor of  $f(x)$ . If we divide  $f(x)$  by  $x - 1$ , we get the following:

$$f(x) = x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + x^{n-3} + \dots + x^3 + x^2 + x + 1)$$

$$\text{In other words, } x^{n-1} + x^{n-2} + x^{n-3} + \dots + x^3 + x^2 + x + 1 = \frac{x^n - 1}{x - 1}$$

Replace  $x$  with  $r$  we can now complete the above formula:

The sum of the first  $n$  terms of a geometric sequence, if the first term is  $a_1$  and the common ratio is  $r$ , is given by:

$$S_n = a_1 (1 + r + r^2 + r^3 + r^4 + \dots + r^{n-1}) = a_1 \left( \frac{r^n - 1}{r - 1} \right)$$

Example: Find the sum of the first 8 terms of the sequence:

$$\left\{ a_n = 3 \left( \frac{2}{5} \right)^n \right\}$$

Ans: If we divide any term by its preceding term, we get:

$$\frac{a_n}{a_{n-1}} = \frac{3 \left( \frac{2}{5} \right)^n}{3 \left( \frac{2}{5} \right)^{n-1}} = \frac{2}{5}$$

This is a geometric sequence with common ratio  $r = \frac{2}{5}$  and first term  $a_1 = 3 \left( \frac{2}{5} \right)^1 = \frac{6}{5}$ . Using the formula, the sum of the first 8 terms is:

$$\begin{aligned} S_8 &= a_1 \left( \frac{r^8 - 1}{r - 1} \right) = \frac{6}{5} \left( \frac{\left( \frac{2}{5} \right)^8 - 1}{\frac{2}{5} - 1} \right) = \frac{6}{5} \left( \frac{\frac{256}{390625} - 1}{-\frac{3}{5}} \right) = \frac{6}{5} \left( \frac{-\frac{390369}{390625}}{-\frac{3}{5}} \right) \\ &= \frac{6}{5} \left( \frac{130123}{78125} \right) \approx 1.99868928 \end{aligned}$$

Example: Find an expression for the  $n$ -term of the geometric sequence  $\{a_n\}$  if  $a_3 = 4$  and  $a_6 = 108$

Ans: We need to first find the first term,  $a_1$ , and the common ratio,  $r$ . We use

the formula to set up two equations:

$$a_n = a_1 r^{n-1} \Rightarrow a_6 = a_1 r^{6-1} = 108$$

$$a_n = a_1 r^{n-1} \Rightarrow a_3 = a_1 r^{3-1} = 4$$

We get two equations:

$$a_1 r^5 = 108$$

$$a_1 r^2 = 4$$

Divide the two equations give:

$$r^3 = 27 \Rightarrow r = 3$$

Solving for  $a_1$  in the equation gives:

$$a_1 (3)^2 = 4 \Rightarrow 9a_1 = 4 \Rightarrow a_1 = \frac{4}{9}$$

So the expression for the sequence is:

$$a_n = \frac{4}{9} (3)^{n-1}$$

## Principle of Mathematical Induction:

Suppose a statement is made about some (or all) natural numbers  $n$ . Suppose we can prove the following:

1. The statement is true for  $n = 1$ .
2. If the statement is true about some natural number  $n$ , it is also true for the next natural number,  $n + 1$ .

Then the statement is true for all natural numbers.

We use the principle of mathematical induction to prove mathematical theorems and formulas about natural numbers. Notice that the first condition tells us that the statement is true for the very first number, 1. The second condition assures us that, if a statement is true for a natural number, it must be true for the next number. So since the statement is true for 1, it must be true for the next number, 2; and using the second condition again, since the statement is true for 2, it must be true for 3, and continuing in this argument, we can see that the statement must be true for all natural numbers.

In using mathematical induction to prove a statement about all natural numbers, we first prove that the statement is true for the first number,  $n = 1$ . Then we can **assume** that the statement is true for  $n$  (this is called the **induction hypothesis**), if we can then prove, from this assumption, that the statement is also true for  $n + 1$ , then we have proved the statement is true for all natural numbers.

Example: Prove the formula:

$$\sum_{i=1}^n i = 1 + 2 + 3 + 4 + \cdots + (n - 1) + n = \frac{(n)(n + 1)}{2}$$

Ans: We use mathematical induction on  $n$ . We first need to prove the the formula is true for  $n = 1$ , but this is trivial: The left hand side is:

$$\sum_{i=1}^1 i = 1$$

The right hand side is  $\frac{(1)(1 + 1)}{2} = 1$

Now we **assume** that the formula is true for  $n$ . In other words, we assume now that

$$1 + 2 + 3 + 4 + \cdots + (n - 1) + n = \frac{(n)(n + 1)}{2}$$

is true.

We must now prove that the formula is true for  $n + 1$ . In other words, we must prove this formula is true:

$$1 + 2 + 3 + 4 + \cdots + (n - 1) + (n) + (n + 1) = \frac{(n + 1)((n + 1) + 1)}{2}$$

We will use the assumption that the formula is true for  $n$ . According to the formula:

$$\begin{aligned} [1 + 2 + 3 + 4 + \cdots + (n - 1) + (n)] + (n + 1) &= \frac{(n)(n + 1)}{2} + (n + 1) \\ &= \frac{(n)(n + 1)}{2} + \frac{2(n + 1)}{2} = \frac{(n)(n + 1) + 2(n + 1)}{2} \\ &= \frac{(n + 1)[n + 2]}{2} = \frac{(n + 1)[(n + 1) + 1]}{2} \end{aligned}$$

We showed the formula is also true for  $n + 1$  if we assume that it is true for  $n$ , so we have proved the formula is true for all natural numbers by the principle of mathematical induction.

Example: Use mathematical induction to prove the sum of a geometric sequence formula:  $a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = a \left( \frac{r^n - 1}{r - 1} \right)$

Ans: We first show that the formula is true for  $n = 1$ .

If  $n = 1$ , left hand side is  $a$ . Right hand side is  $a \left( \frac{r^1 - 1}{r - 1} \right) = a$

We now assume the formula is true for  $n$ . We need to prove that it is true for  $n + 1$ . In other words, we need to prove that:

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^{(n+1)-1} = a \left( \frac{r^{n+1} - 1}{r - 1} \right)$$

By assumption, the formula is true for  $n$ , so on the left hand side we have:

$$\begin{aligned} [a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}] + ar^{(n+1)-1} &= \\ &= \left[ a \left( \frac{r^n - 1}{r - 1} \right) \right] + ar^{(n+1)-1} = \left[ a \left( \frac{r^n - 1}{r - 1} \right) \right] + ar^n = \frac{a(r^n - 1)}{r - 1} + ar^n \\ &= \frac{a(r^n - 1)}{r - 1} + ar^n \left( \frac{r - 1}{r - 1} \right) = \frac{a(r^n - 1)}{r - 1} + \frac{ar^n(r - 1)}{r - 1} \\ &= \frac{a(r^n - 1) + ar^n(r - 1)}{r - 1} = \frac{a[(r^n - 1) + r^n(r - 1)]}{r - 1} \\ &= \frac{a[r^n - 1 + r^{n+1} - r^n]}{r - 1} = \frac{a[r^{n+1} - 1]}{r - 1} = a \left( \frac{r^{n+1} - 1}{r - 1} \right) \end{aligned}$$

Suppose  $n, k$  are both non-negative integers and  $n \geq k$ , we define

$\binom{n}{k}$  (read  $n$  choose  $k$ ) to be:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Example:  $\binom{8}{3} = \frac{8!}{3!(8-3)!} = \frac{8!}{3! \cdot 5!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2} = 56$

Example:  $\binom{11}{4} = \frac{11!}{4!(11-4)!} = \frac{11!}{4! \cdot 7!} = \frac{11 \cdot 10 \cdot 9 \cdot 8}{4 \cdot 3 \cdot 2} = 11 \cdot 10 \cdot 3 = 330$

Example:  $\binom{12}{12} = \frac{12!}{12!(12-12)!} = \frac{12!}{12! \cdot 0!} = \frac{12!}{12! \cdot 1} = 1$

Example:  $\binom{7}{6} = \frac{7!}{6!(7-6)!} = \frac{7!}{6! \cdot 1!} = 7$

Example:  $\binom{14}{0} = \frac{14!}{0!(14-0)!} = \frac{14!}{1 \cdot 14!} = 1$

In general,

$$\binom{n}{n} = 1$$

$$\binom{n}{n-1} = n$$

$$\binom{n}{0} = 1$$

$$\binom{n}{1} = n$$

$$\binom{n}{k} = \binom{n}{n-k}$$

Formula:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

Proof:

$$\begin{aligned} \binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-(k-1))!} \\ &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\ &= \frac{n!}{k!(n-k)!} \cdot \frac{(n-k+1)}{(n-k+1)} + \frac{n!}{(k-1)!(n-k+1)!} \cdot \frac{k}{k} \end{aligned}$$

$$\begin{aligned}
&= \frac{n! \cdot (n - k + 1)}{k!(n - k + 1) \cdot (n - k)!} + \frac{n! \cdot k}{k \cdot (k - 1)!(n - k + 1)!} \\
&= \frac{n! \cdot (n - k + 1)}{k! \cdot (n - k + 1)!} + \frac{n! \cdot k}{k! \cdot (n - k + 1)!} = \frac{n! \cdot (n - k + 1) + n! \cdot k}{k! \cdot (n - k + 1)!} = \frac{n! [(n - k + 1) + k]}{k! \cdot (n - k + 1)!} \\
&= \frac{n! [n + 1]}{k! \cdot (n - k + 1)!} = \frac{(n + 1)!}{k! \cdot (n + 1 - k)!} = \binom{n + 1}{k}
\end{aligned}$$

For example,  $\binom{8}{6} + \binom{8}{5} = \binom{9}{6}$

For example,  $\binom{n}{3} + \binom{n}{2} = \binom{n + 1}{3}$

### Binomial Theorem:

Let  $n$  be a positive integer, then

$$\begin{aligned}
(a + b)^n = & \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \dots + \\
& \binom{n}{n-2} a^2 b^{n-2} + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n
\end{aligned}$$

Proof: We will use mathematical induction on  $n$ . If  $n = 1$ , this is trivial.

Now assume that the formula is true for  $n$ , we need to show that it is also true for  $n + 1$ .

$$(a + b)^{n+1} = (a + b)^n (a + b)$$

By the induction assumption, the formula is true for  $n$ , so  $(a + b)^n =$

$$\begin{aligned}
& \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \dots + \\
& \binom{n}{n-2} a^2 b^{n-2} + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n
\end{aligned}$$

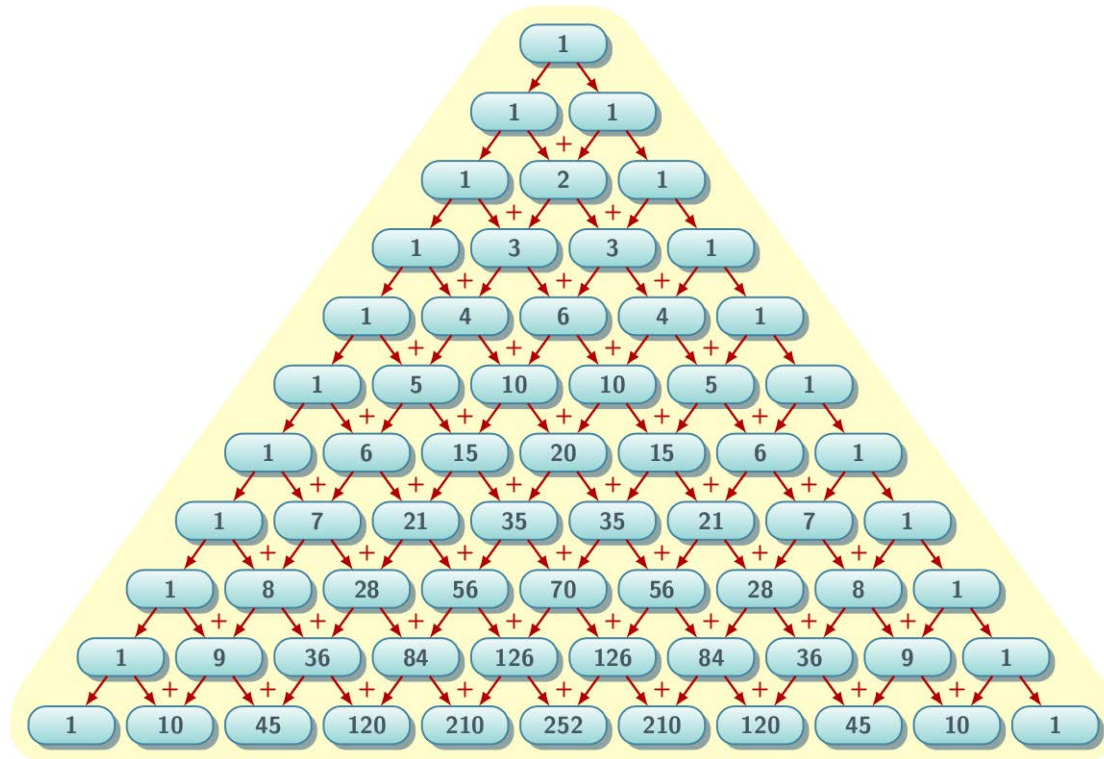
Therefore,  $(a + b)^n (a + b) =$

$$\begin{aligned}
& \left[ \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \dots + \right. \\
& \quad \left. \binom{n}{n-2} a^2 b^{n-2} + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n \right] (a + b) \\
&= \binom{n}{0} a^{n+1} + \binom{n}{1} a^n b + \binom{n}{2} a^{n-1} b^2 + \binom{n}{3} a^{n-2} b^3 + \dots + \\
& \quad \binom{n}{n-2} a^3 b^{n-2} + \binom{n}{n-1} a^2 b^{n-1} + \binom{n}{n} a b^n
\end{aligned}$$



$$\begin{aligned}
& + \binom{n}{0} a^n b + \binom{n}{1} a^{n-1} b^2 + \binom{n}{2} a^{n-2} b^3 + \binom{n}{3} a^{n-3} b^4 + \dots + \\
& \qquad \qquad \qquad \binom{n}{n-2} a^2 b^{n-1} + \binom{n}{n-1} a b^n + \binom{n}{n} b^{n+1} \\
= & \binom{n+1}{0} a^{n+1} + \binom{n+1}{1} a^{(n+1)-1} b + \binom{n+1}{2} a^{(n+1)-2} b^2 + \binom{n+1}{3} a^{(n+1)-3} b^3 + \\
& \dots + \binom{n+1}{n} a b^n + \binom{n+1}{n+1} b^{n+1}
\end{aligned}$$

Starting with  $n = 0$ , the coefficients of  $(a + b)^n$  is the entries in the  $n$ -th row of the **Pascal's Triangle**.



Example: Expand  $(x + y)^6$

Ans: Using the binomial theorem,

$$\begin{aligned}
(x + y)^6 &= \binom{6}{0} x^6 + \binom{6}{1} x^5 y + \binom{6}{2} x^4 y^2 + \binom{6}{3} x^3 y^3 + \binom{6}{4} x^2 y^4 + \binom{6}{5} x y^5 + \binom{6}{6} y^6 \\
&= x^6 + 6x^5 y + 15x^4 y^2 + 20x^3 y^3 + 15x^2 y^4 + 6x y^5 + y^6
\end{aligned}$$