Complex Numbers:

Definition: A **complex number** is a number of the form:

$z = a + bi$

where a, b are real numbers and i is a symbol with the property:

$i^2 = -1$. You may treat $i = \sqrt{-1}$

We can treat i as a variable in an algebraic expression and all algebraic rules are still to be followed in operations involving a complex number.

The **complex number system** is an **extension** of the real number system. All real numbers are complex numbers (by having $b = 0$), but there are complex numbers that are not real, e.g. $2 - 3i$.

We add or subtract two complex numbers by adding and substracting the corresponding real and complex part of the number.

E.g.

$z = 4 + 3i$, $w = 1 - 2i$

$z + w = (4 + 1) + (3 + -2)i = 5 + i$

$z - w = (4 - 1) + (3 - (-2))i = 3 + 5i$

To multiply two complex numbers, we multiply them like we multiply binomials, with the understanding that $i^2 = -1$

Example:

$z = -2 + 5i$, $w = 4 + 2i$

$zw = (-2 + 5i)(4 + 2i) = -8 - 4i + 20i + 10i^2 = -8 + 16i + 10(-1)$

$= -8 + 16i - 10 = -18 + 16i$

To divide a complex number by a real number is to multiply the reciprocal of the real number to the complex number.

Example:

$z = 3 - 4i$, $r = 6$,

$$\frac{z}{r} = \frac{3 - 4i}{6} = \frac{1}{6}(3 - 4i) = \frac{3}{6} - \frac{4}{6}i = \frac{1}{2} - \frac{2}{3}i$$

How about dividing a complex a number by another complex number? In order to do this, we first need another concept.

Definition: If $z = a + bi$ is a complex number, then $\overline{z} = a - bi$ is the **complex conjugate** of z.

E.g. If $z = 4 - 5i$, then $\overline{z} = 4 + 5i$

E.g. If $z = 3 + 2i$, then $\overline{z} = 3 - 2i$

E.g. If $z = 12i$, then $\overline{z} = -12i$

E.g. If $z = 8$, then $\overline{z} = 8$

Notice that a real number is its own conjugate.

If $z = a + bi$, then $\overline{z} = a - bi$, and

$z \cdot \overline{z} = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 - b^2 i^2 = a^2 - b^2(-1) = a^2 + b^2.$

In other words, the product of a complex number z with its complex conjugate $\overline{z}$ is always a real number.

To divide two complex numbers, say $\frac{z}{w}$, multiply the numerator and denominator by the complex conjugate of the denominator. The result will make the denominator into a real number, and we can divide accordingly.

E.g. Let $z = 3 - 2i$, $w = 4 + 3i$, find $\frac{z}{w}$

Ans: We multiply $\frac{z}{w}$ by the fraction $\frac{\overline{w}}{\overline{w}}$

$$\frac{z}{w} = \frac{z}{w} \cdot \frac{\overline{w}}{\overline{w}} = \frac{3 - 2i}{4 + 3i} \cdot \frac{4 - 3i}{4 - 3i} = \frac{(3 - 2i)(4 - 3i)}{(4 + 3i)(4 - 3i)} = \frac{12 - 9i - 8i + 6i^2}{16 - 9i^2}$$

$$= \frac{12 - 6 - 17i}{16 + 9} = \frac{6 - 17i}{25} = \frac{6}{25} - \frac{17}{25}i$$

Using complex numbers we can provide solutions to equations like:

$x^2 + 4 = 0$

Notice that this equation has no solution in the real number system, but $x = -2i$ and $x = 2i$ solve the equation.

In general, for any quadratic equation of the form $ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$, we have the two solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the **discriminant**, $b^2 - 4ac$, is less than 0, then the equation has two **complex solution**

If $b^2 - 4ac > 0$, then the equation has two **real solution**

If $b^2 - 4ac = 0$, then the equation has one **real solution**

Example: Solve the equation: $3x^2 - x + 2 = 0$

Ans: The solutions are:

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4(3)(2)}}{2(3)} = \frac{1 \pm \sqrt{-23}}{6} = \frac{1}{6} \pm \frac{\sqrt{23}}{6} i$$

Definition: A **polynomial of degree n (with real coefficients)** is a function of the form:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_n, a_{n-1}, a_{n-2} \cdots a_2, a_1, a_0$ are real numbers, and $a_n \neq 0$

The number a_n is the **leading coefficient** of the polynomial.

Using long division of polynomials, we can divide a polynomial of higher degree by a polynomial of lower degree to find the **quotient** and **remainder**.

Example: Divide $(x^4 - 2x^3 + 5x^2 - 1)$ by $(x^2 + x + 1)$

Ans:

$$\begin{array}{r|rrrrr}
 & & & x^2 & -3x & +7 \\ \hline
x^2 + x + 1 & x^4 & -2x^3 & +5x^2 & & -1 \\
 & -x^4 & -x^3 & -x^2 & & \\ \cline{2-4}
 & & -3x^3 & +4x^2 & & \\
 & & 3x^3 & +3x^2 & +3x & \\ \cline{3-5}
 & & & 7x^2 & +3x & -1 \\
 & & & -7x^2 & -7x & -7 \\ \cline{4-6}
 & & & & -4x & -8
\end{array}$$

The quotient is $(x^2 - 3x + 7)$ and the remainder is $(-4x - 8)$.

Example: Divide $(2x^3 - x^2 - x + 2)$ by $(x - 4)$

Ans:

$$\begin{array}{r|rrrr}
 & & 2x^2 & +7x & +27 \\ \hline
x - 4 & 2x^3 & -x^2 & -x & +2 \\
 & -2x^3 & +8x^2 & & \\ \cline{2-3}
 & & 7x^2 & -x & \\
 & & -7x^2 & +28x & \\ \cline{3-4}
 & & & 27x & +2 \\
 & & & -27x & +108 \\ \cline{4-5}
 & & & & 110
\end{array}$$

The quotient is $(2x^2 + 7x + 27)$ and the remainder is (110)

Notice that when the divisor is a linear polynomial (of degree 1), the remainder is a constant polynomial of degree 0 (a number).

Using this method we can arrive at the:

Division Algorithm:

Suppose $P(x)$, $D(x)$ are polynomials and degree of $D(x)$ is less than or equal to degree of $P(x)$, then there exist **unique** polynomials $Q(x)$ and $R(x)$ such that:

$P(x) = D(x) \cdot Q(x) + R(x)$

and degree of $R(x)$ is strictly less than degree of $D(x)$.

If $R(x) = 0$ is the **zero polynomial**, we say that $D(x)$ is a **factor** of $P(x)$

Example: Given $P(x) = x^5 - 2x^4 + 2x^2 - x - 1$ and $D(x) = x^2 + x + 2$, we have:

$$
\begin{array}{r|rrrrrr}
 & x^3 & -x^2 & -3x & +1 & & \\
\hline
x^2 - x + 2 & x^5 - 2x^4 & & + 2x^2 & - x - 1 & & \\
 & -x^5 + x^4 - 2x^3 & & & & & \\
\cline{2-2}
 & -x^4 - 2x^3 + 2x^2 & & & & & \\
 & x^4 - x^3 + 2x^2 & & & & & \\
\cline{2-2}
 & -3x^3 + 4x^2 - x & & & & & \\
 & 3x^3 - 3x^2 + 6x & & & & & \\
\cline{2-2}
 & x^2 + 5x - 1 & & & & & \\
 & -x^2 + x - 2 & & & & & \\
\cline{2-2}
 & 6x - 3 & & & & &
\end{array}
$$

In other words, $Q(x) = x^3 - x^2 - 3x + 1$ and $R(x) = 6x - 3$.

$Q(x)$ is the quotient, and $R(x)$ is the remainder, and we can write:

$(x^5 - 2x^4 + 2x^2 - x - 1) = (x^2 - x + 2)(x^3 - x^2 - 3x + 1) + (6x - 3)$

If we divide a polynomial $P(x)$ by a linear factor $(x - r)$, the division algorithm tells us that there are polynomials $Q(x)$ and $R(x)$ such that

$P(x) = (x-r)Q(x)+R(x)$, where degree of $R(x)$ is less than degree of $(x-r)$. Since degree of $(x - r)$ is 1, degree of $R(x)$ must be 0. In other words, $R(x)$ is a constant polynomial. Notice that $P(r) = (r - r)Q(r) + R(r) = 0(Q(r)) + R(r) = R(r)$.

But $R(x)$ is a constant, so $P(r)$ must be the remainder, $R(x)$. We have the:

Remainder Theorem:

When a polynomial $P(x)$ is divided by $x - r$, the remainder is $P(r)$.

Example: If $p(x) = x^4 + x^3 - x^2 - 2x + 3$. Since

$p(2) = (2)^4 + (2)^3 - (2)^2 - 2(2) + 3 = 16 + 8 - 4 - 4 + 3 = 19$,

the remainder theorem tells us that when p is divided by $x - 2$, the remainder is

19. This is indeed the case:

$$
\begin{array}{r|rrrrr}
 & x^3 & +3x^2 & +5x & +8 & \\
\hline
x-2 & x^4 & +x^3 & -x^2 & -2x & +3 \\
 & -x^4 & +2x^3 & & & \\
\cline{2-3}
 & & 3x^3 & -x^2 & & \\
 & & -3x^3 & +6x^2 & & \\
\cline{3-4}
 & & & 5x^2 & -2x & \\
 & & & -5x^2 & +10x & \\
\cline{4-5}
 & & & & 8x & +3 \\
 & & & & -8x & +16 \\
\cline{5-6}
 & & & & & 19
\end{array}
$$

Example: If $p(x) = 2x^5 - 2x^4 + x^3 - x - 1$. Since

$p(-1) = 2(-1)^5 - 2(-1)^4 + (-1)^3 - (-1) - 1 = -2 - 2 - 1 + 1 - 1 = -5$,

the remainder theorem says that when $p(x)$ is divided by $(x - (-1)) = (x + 1)$, the remainder is -5.

$$
\begin{array}{r|rrrrrr}
 & 2x^4 & -4x^3 & +5x^2 & -5x & +4 & \\
\hline
x+1 & 2x^5 & -2x^4 & +x^3 & & -x & -1 \\
 & -2x^5 & -2x^4 & & & & \\
\cline{2-3}
 & & -4x^4 & +x^3 & & & \\
 & & 4x^4 & +4x^3 & & & \\
\cline{3-4}
 & & & 5x^3 & & & \\
 & & & -5x^3 & -5x^2 & & \\
\cline{4-5}
 & & & & -5x^2 & -x & \\
 & & & & 5x^2 & +5x & \\
\cline{5-6}
 & & & & & 4x & -1 \\
 & & & & & -4x & -4 \\
\cline{6-7}
 & & & & & & -5
\end{array}
$$

Factor Theorem:

A real number r is a **root (or zero)** of a polynomial $p(x)$ if and only if $x - r$ is a factor of $p(x)$

Example: Let $p(x) = x^4 - 3x^2 + x - 6$. Since $p(2) = 0$, the factor theorem tells us that $(x - 2)$ is a factor of $p(x)$. In other words, when $p(x)$ is divided by $(x - 2)$, the remainder is 0.

$$
\begin{array}{r}
x^3 + 2x^2 \quad + x + 3 \\
x - 2 \overline{\big) \; x^4 \qquad\quad - 3x^2 \quad + x - 6} \\
\underline{- x^4 + 2x^3} \qquad\qquad\qquad\quad \\
2x^3 - 3x^2 \qquad\qquad \\
\underline{- 2x^3 + 4x^2} \qquad\qquad \\
x^2 \quad + x \qquad \\
\underline{- x^2 + 2x} \qquad \\
3x - 6 \\
\underline{- 3x + 6} \\
0
\end{array}
$$

We have: $(x^4 - 3x^2 + x - 6) = (x - 2)(x^3 + 2x^2 + x + 3)$

Let $p(x)$ be a polynomial, if $(x-r)^k$ is a factor of p but $(x-r)^{k+1}$ is not a factor of p, then we say that r is a root of p **with multiplicity** k.

Example: Let $p(x) = (x-2)^3(x+1)^2(x+4)$.

p has three roots, namely 2, -1, and -4. The root 2 has a multiplicity of 3, the root -1 has a multiplicity of 2, and the root -4 has a multiplicity of 1.

One of the major problem of algebra is finding solution(s) of polynomial equations. In polynomial equations where the coefficients involve one kind of real numbers, the solutions may be a (possibly more complicated) different kind of real numbers.

For example:

$2x - 1 = 0$

The is an equation where the coefficients are integers, but the solution, $\frac{1}{2}$, is not an integer. We need to involve rational numbers to solve polynomial equations where the coefficients are only integers.

$x^2 - 2 = 0$

This is an equation where the coefficients are rational numbers, but the solutions, $\pm\sqrt{2}$, are not rational numbers. They are irrational numbers. We need to involve irrational numbers to solve polynomial equations where the coefficients are only rational numbers.

$x^2 + 1 = 0$

This is an equation where the coefficients are all real numbers, but the solutions, $\pm i$, are not real numbers, they are complex numbers. We need to involve the complex numbers to solve polynomial equations involving only real numbers.

For polynomial equations involving complex numbers, we may ask the same question. Do we need to involve something beyond the complex number system to solve polynomial equations involving complex numbers? To answer this question, we have the following:

Fundamental Theorem of Algebra:

Any polynomial (whose coefficients are real or complex numbers) of degree n has n complex roots, counting multiplicity.

The significance of the fundamental theorem of algebra is that it tells us that the complex number system is **algebraically closed**. In other words, all polynomial equations involving complex numbers as coefficients, the solutions will also be complex. We do not have to worry about inventing any new kind of numbers to accomodate the solutions.

Rational Roots Theorem:

Consider the equation:

$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots a_2 x^2 + a_1 x + a_0$ be a polynomial of integer coefficients (i.e. $a_n, a_{n-1}, \cdots a_2, a_1, a_0$ are all integers), then:

If $r = \dfrac{p}{q}$ is a **rational root** of f, then p is a factor of a_0 and q is a factor of a_n.

The rational roots theorem gives us a way of looking for the **candidates** of the rational roots of a polynomial of integer (and rational) coefficients. For example,

$f(x) = 4x^3 - 5x^2 + 2x - 21$

Since the only factors of 4 are $\pm4, \pm2, \pm1$, and the only factors of -21 are $\pm21, \pm7, \pm3, \pm1$, the only **candidates** of rational roots of f are:

$\pm21, \pm\frac{21}{4}, \pm\frac{21}{2}, \pm7, \pm\frac{7}{4}, \pm\frac{7}{2}, \pm3, \pm\frac{3}{4}, \pm\frac{3}{2}, \pm1, \pm\frac{1}{4}, \pm\frac{1}{2}$

These are the only **candidate** for rational roots of the polynomial. We only need to test these numbers for the existance of possible rational roots of f. None of them has to be a root, however.

Also note that the rational roots theorem gives no information about irrational or complex roots. A polynomial may still have irrational or complex roots regardless of what the rational roots theorem says about the existance of rational roots.

We say that a polynomial p (of real coefficient) has a **variation in sign** if two consecutive (non-zero) coefficients of p have opposite sign.

Example:

$p(x) = 7x^4 + 6x^2 + x + 9$ has 0 variation in sign

$p(x) = -3x^3 - 2x^2 - 3x - 1$ has 0 variation in sign

$p(x) = 2x^6 - 4x^5 - 3x^3 - 12$ has 1 variation in sign

$p(x) = x^4 - 2x^3 + 5x^2 - 2x + 9$ has 4 variations in sign

Decarte's Rule of Sign Let $p(x)$ be a polynomial of real coefficients, then:

The number of **positive real roots** of p is equal to the number of variations in sign of $p(x)$ or less than that by an even number.

The number of **negative real roots** of p is equal to the number of variations in sign of $p(-x)$ or less than that by an even number.

Example:

Let $p(x) = 2x^5 - 3x^4 - x^3 + x - 1$

Since $p(x)$ has 3 variations in sign, Decarte's rule of sign tells us that p has 3, or 1, positive real root.

Notice that $p(-x) = 2(-x)^5 - 3(-x)^4 - (-x)^3 + (-x) - 1 = -2x^5 - 3x^4 + x^3 - x - 1$

$p(-x)$ has 2 variations in sign, Decarte's rule of sign tells us that p has 2, or 0, negative real root.