An **Exponential Function with base b** is a function of the form:

$f(x) = b^x$, where $b > 0$, $b \neq 1$ is a real number.

We know the meaning of b^r if r is a rational number. What if r is irrational? What we do is we approximate the value of b^r by using rational approximate for r. For example, to approximate 5^π, we may approximate it as $5^{3.12}$, $5^{3.141}$, $5^{3.1415}$, $5^{3.14159}$.... In advance mathematics one can **define** the value of 5^π to be the *limit* of these approximations. For now, just realize that such approximation can be used to approximate the value of b^r for any positive number b and any real number r.

Since b^x can be defined for all real numbers r, the domain of an exponential function is all real numbers.

The range of b^x is all real numbers greater than 0.

Note the difference between an exponential function and a power function.

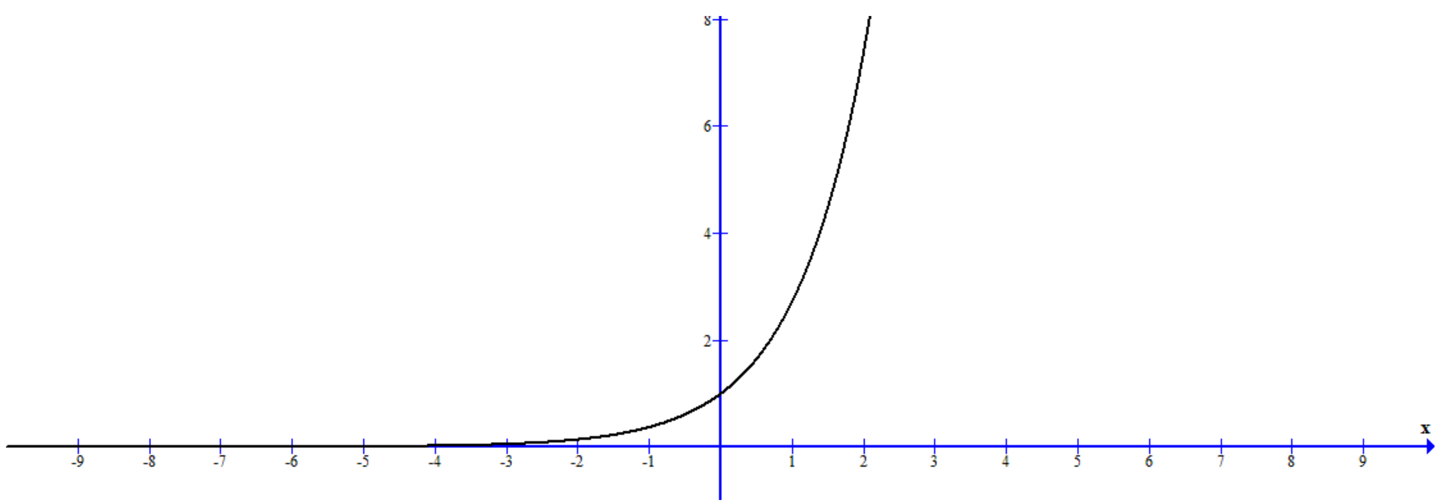
x^2 is a power function. In this function, the exponent is the constant.

2^x is an exponential function. In this function, the base is the constant but the exponent is the variable (input).

An exponential function is always positive. And if in addition $0 < b < 1$, f is a **decreasing** function. That is $f(x)$ decreases as x increases.

If $b > 1$, then f is an **increasing** function. I.e. $f(x)$ increases in value as x increases. In fact, an increasing exponential function (with any base) increases faster than *any* polynomial function.

While the property of an exponential function with an base $b > 1$ is the same, one particular useful base is the number $e \sim 2.718281828\cdots$. The exponential function definied by $f(x) = e^x$ is the **natural exponential** function.



Logarithmic Functions:

An exponential function is a one-to-one function. It has in inverse. However, the expression for the inverse of an exponential function cannot be solved by any algebraic means, therefore we do not have an algebraic expression for it. We just **define** such a function and study its property knowing that it is the inverse of b^x

Definition:

$g(x) = \log_b x$ (read "log base b of x") is the inverse function of

$f(x) = b^x$

$\log_b$ is the name of the function. x is the argument (input) to $\log_b$, and the value (output) i s$\log_b x$.

Since the range of an exponential function is all positive real numbers, the domain of a log function is all positive real numbers.

Because of the fact that $\log_b x$ is the inverse of b^x, we have this by definition:

$\log_b(b^x) = x$ for all x

$b^{\log_b x} = x$ for all $x > 0$

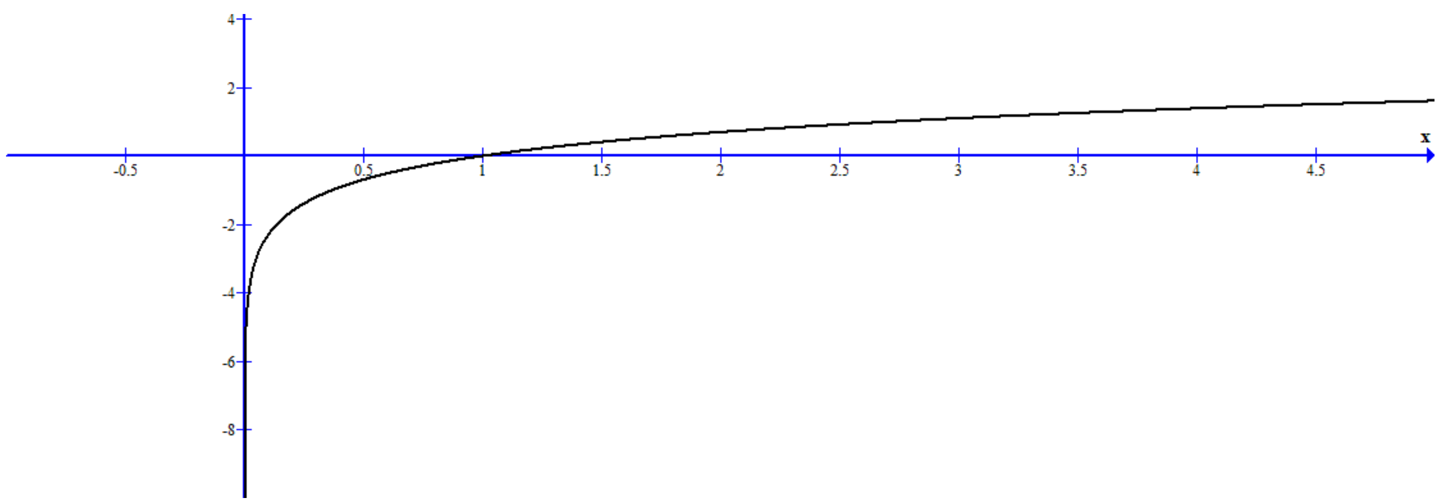
The following two equations are equivalent:

$y = b^x$

$\log_b y = x$

The logarithm with base e, which is the inverse function of e^x, is given a special name, the **natural logarithm**, and written ln. I.e.

$\log_e x = \ln x$



E.g.

Evaluate:

$\log_2 16$

Ans: Set $\log_2 16 = y$, this is the same as asking:

$2^y = 16$, what is y? From observation we see that $y = 4$, so

$\log_2 16 = 4$

E.g

$\log_5 \frac{1}{125} =?$

Ans: Since $5^{-3} = \frac{1}{125}$,

$\log_5 \frac{1}{125} = -3$

E.g.

$\log_2 x = 5$, what is x?

Ans: $2^5 = x$, so $x = 32$

$\log_x 100 = -2$, what is x?

This is the same as saying, $x^{-2} = 100$, so $x = \frac{1}{10}$

Because of the fact that logs are inverse functions of the exponential functions, they have many properties that are similar to that of the exponential functions, and can be easily proved using the definition:

Properties of Logarithm

For any real number r, any base $b > 0, a > 0$, any $x > 0, y > 0$, we have:

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

$$\log_b x^r = r \log_b x$$

Change of Base Formula:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

The change of base formula says that we can easily change from log of one base to another, so the choice of log of which base to use usually for convenience only. We like to use natural log for most of our studies because that is the most conveninent one in math and science.

Also make sure not to misuse the properties of logs, for example, the property of logarithm does **not** give us this:

$$\log(x + y) = \log x + \log y$$

To solve an equation involving exponential function, one would need to use logarithm, and to solve an equation involving logarithm, one uses exponential functions.

E.g.

Solve $3^x = 10$

Ans: We *take the log of both side*

$\log_3 3^x = \log_3 10$

Since log and exponential functions are inverse, the left hand side is just x, we have

$x = \log_3 10$

One may approach the same problem by taking the ln of both sides and use properties of logarithm:

$3^x = 10$

$\ln(3^x) = \ln 10$

$x \ln 3 = \ln 10$

$x = \dfrac{\ln 10}{\ln 3}$

E.g. Solve $\log_4(x+2) = 3$

Ans: We turn this into an equation involving exponential function:

$\log_4(x+2) = 3$

$4^3 = x + 2$

$64 = x + 2$

$x = 62$

E.g. Solve $\log_2(x+1) + \log_2(x+4) = 2$

Ans: We use property of log to combine the left hand side:

$\log_2(x+1) + \log_2(x+4) = 2$

$\log_2((x+1)(x+4)) = 2$

$\log_2(x^2 + 5x + 4) = 2$

$2^2 = x^2 + 5x + 4$

$4 = x^2 + 5x + 4$

$x^2 + 5x = 0$

$x = 0$ or $x = -5$

Notice that $x = -5$ is an **extraneous solution**, so our only solution is $x = 0$