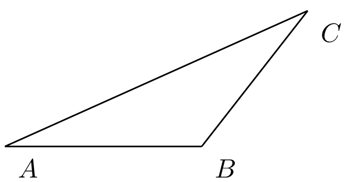
**Triangles:**

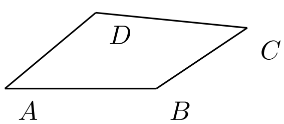
A **polygon** is a closed figure on a plane bounded by (straight) line segments as its sides. Where the two sides of a polygon intersect is called a **vertex** of the polygon.

A polygon with three sides is a **triangle**.

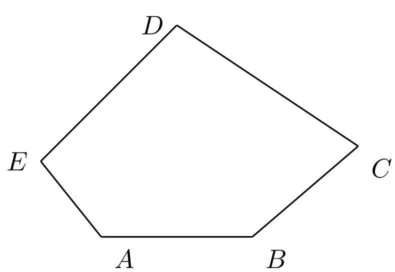



We name a triangle by its three vertices. The above is $\triangle ABC$. Notice the use of the little triangle next to the vertices to indicate we are referring to the triangle instead of the angle.

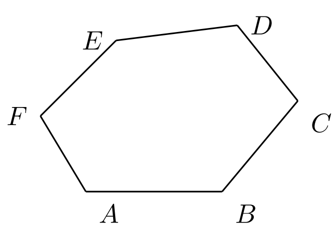
A polygon with four sides is a **quadrilateral**.



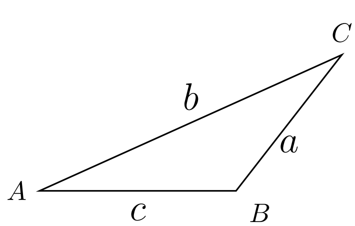

A polygon with five sides is a **pentagon**.




A polygon with six sides is a **hexagon**.

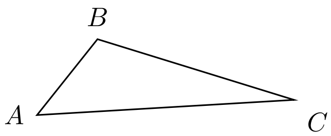



In a given triangle, we use capital letters for the vertex of the triangles, and use the lower-case letter for the side opposite the vertex. So in $\triangle ABC$, the side opposite vertex $A$ is side $a$, the side opposite vertex $B$ is side $b$.



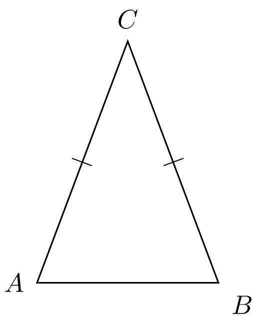
We name triangles by the nature of its sides and also the nature of its angles:

A triangle where all three sides are unequal is a **scalene triangle**:



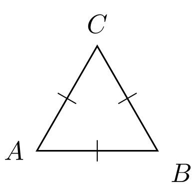

The above $\triangle ABC$ is scalene.

A triangle where at least two of its sides is equal is an **isoceles triangle**:



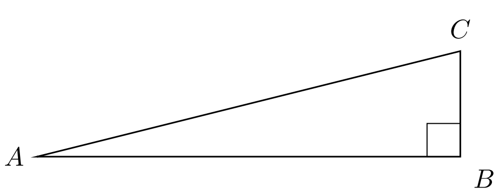

The above $\triangle ABC$ is isoceles. $\overline{AC} \cong \overline{BC}$

A triangle where all three sides are the same is an **equilateral triangle**.



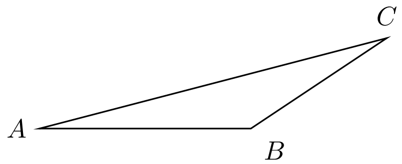

The above $\triangle ABC$ is equilateral. $\overline{AB} \cong \overline{BC} \cong \overline{AC}$

A triangle where one of its angle is right is a **right triangle**. In a right-triangle, the side that is opposite the right-angle is called the **hypotenuse** of the right-triangle. The other two sides are the **legs** of the right-triangle.




The above $\triangle ABC$ is right. $\overline{AC}$ is the hypotenuse, $\overline{BC}$ and $\overline{AB}$ are the two legs.

A triangle where one of its angle is obtuse is an **obtuse triangle**:

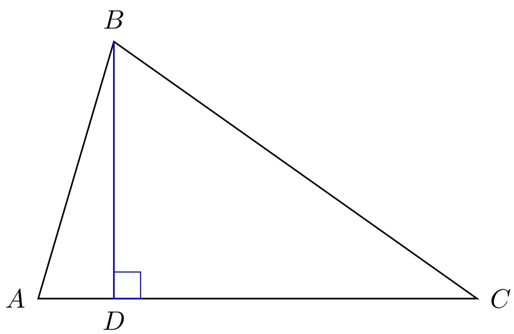



The above $\triangle ABC$ is obtuse, since $\angle B$ is obtuse.

A triangle that does not have any obtuse angle (all three angles are acute) is called an **acute triangle**.

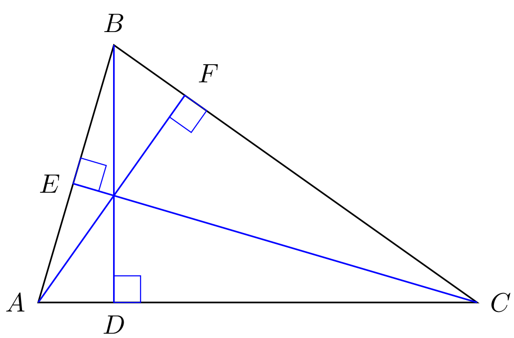
**Altitude of a Triangle**

In a triangle, if through any vertex of the triangle we draw a line that is perpendicular to the side opposite the vertex, this line is an **altitude** of the triangle. The line opposite the vertex where the altitude is perpendicular to is the **base**.




In $\triangle ABC$ above, $\overline{BD}$ is an altitude. It contains vertex $B$ and is perpendicular to $\overline{AC}$, which is the base.

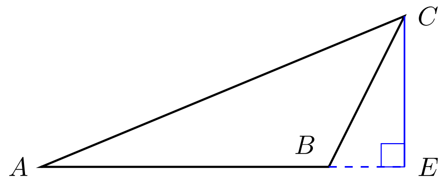
Notice that any triangle always have three altitudes, one through each of the vertex and is perpendicular to the opposite side:




In the above $\triangle ABC$, $\overline{BD}$, $\overline{CE}$, and $\overline{AF}$ are all altitudes of the triangle. Notice that all three of the altitudes intersect at the same point. This is always the case and the point of intersection is called the **orthocenter** of the triangle.

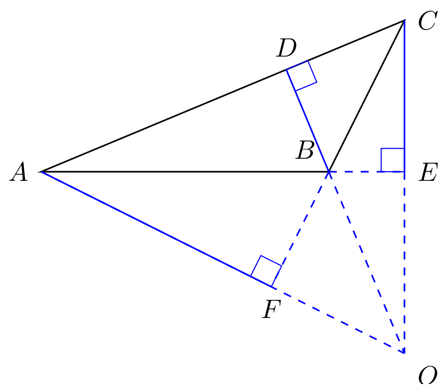
The altitude and orthocenter of a triangle have important geometrical properties which will be discussed.

The altitude of a triangle does *not* have to lie inside the triangle. If we have an obtuse triangle, its altitudes will lie outside of the triangle.



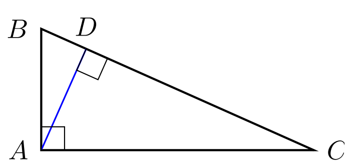

In the above obtuse $\triangle ABC$, $\overline{CE}$ is an altitude which lies outside of the triangle, with $\overline{AE}$ being the base.

If a triangle is obtuse, its orthocenter also lies outside of the triangle:




Notice that in the obtuse $\triangle ABC$ above, the orthocenter, $O$, is outside of the triangle.

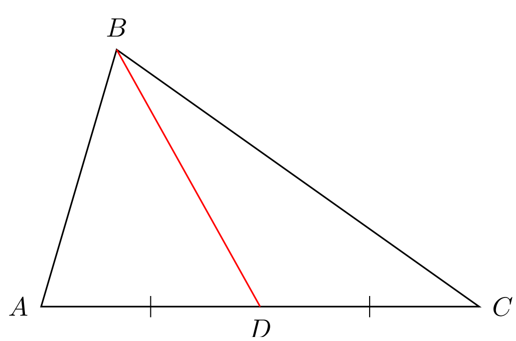
Also note that if a triangle is right, then two of its sides will also be its altitude, and the orthocenter is the vertex of the right angle:




In the above $\triangle ABC$, $\overline{AB}$ is the altitude with base $\overline{AC}$, and $\overline{AC}$ is the altitude with base $\overline{AB}$. $\overline{AD}$ is the altitude with base $\overline{BC}$. Point $A$ is the orthocenter.

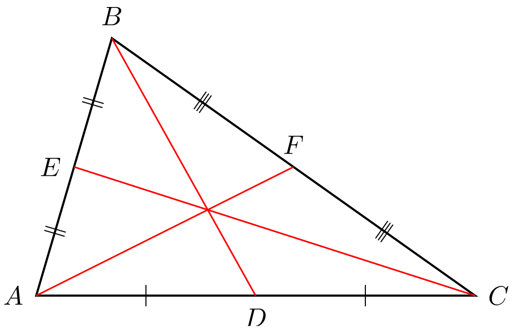
**Median of a Triangle:**

In any triangle, if through one of its vertex we draw a line that **bisects** the opposite side, this line is called a **median** of the triangle.



In $\triangle ABC$ above, $\overline{BD}$ bisects $\overline{AC}$ in $D$ ($\overline{AD} \cong \overline{DC}$), so by definition, $\overline{BD}$ is a median of $\triangle ABC$

Just like altitudes, each triangle has three medians, each through a vertex and bisects a side.




In the above $\triangle ABC$, $\overline{BD}$ is a median that bisects $\overline{AC}$.

$\overline{CE}$ is a median that bisects $\overline{AB}$.

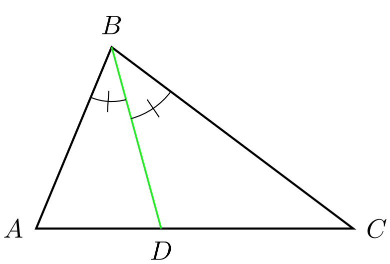
$\overline{AF}$ is a median that bisects $\overline{BC}$.

The three medians of a triangle intersects at a single point. This point is called the **centroid** of the triangle. The medians and the centroid of a triangle have important geometric properties which will be discussed.

The medians and centroid of a triangle always lie inside the triangle, even if the triangle is obtuse.

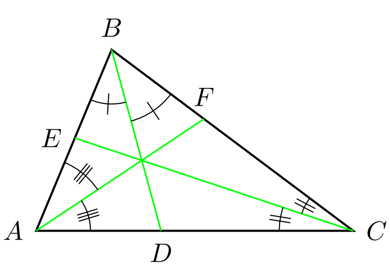
**Angle Bisectors**

An **angle bisector** of a triangle is a line that bisects an angle of the triangle and intersects the opposite side.




In $\triangle ABC$ above, $\overline{BD}$ is an angle bisector of $\angle ABC$

Like altitudes and medians, each triangle has three angle bisectors, one for each of the angles.



In $\triangle ABC$ above, $\overline{BD}$ is an angle bisector of $\angle ABC$, $\overline{CE}$ is an angle bisector of $\angle ACB$, and $\overline{AF}$ is an angle bisector of $\angle BAC$

The three angle bisectors of a triangle also intersect at the same point, called the **incenter** of the triangle. This point has geometric properties to be discussed later.

**Congruent Triangles:**

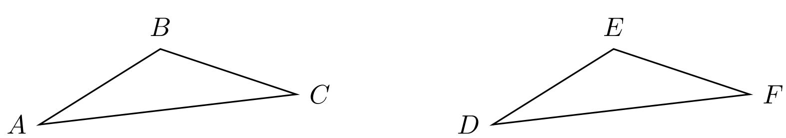
Two triangles are **congruent** to each other if all three of their sides and all three of their angles are equal to each other. Geometrically, congruent triangles have the exact same shape and size, and if we put congruent triangles on top of one another, they will fit perfectly. Algebraically, if two triangles are congruent, this means that their sides have the same length and their angles have the same measurement.

If two triangles are congruent to each other, then their corresponding parts (including sides and angles) are congruent. In other words, if $\triangle ABC \cong \triangle DEF$, then this means that

$\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, and $\overline{BC} \cong \overline{EF}$.

In addition, $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$.

This fact will be used many times in proofs involving congruent triangles. We use the acronym **CPCTC** to stand for *Corresponding Parts of Congruent Triangles are Congruent*.




Basically, if two triangles are congruent, then for all intends and purposes, they are the same triangle.

Ways to prove triangles are congruent:

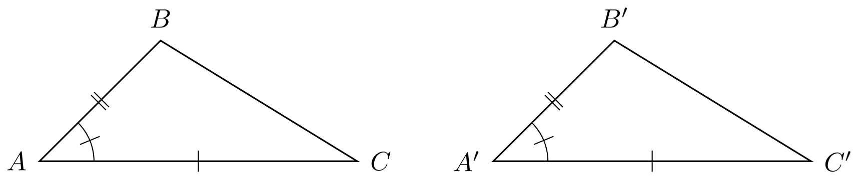
**Side-Side-Side (SSS)**

If all three sides of a triangle is congruent to all three sides of another triangle, the two triangles are congruent.


In the above two triangles, $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, and $\overline{AC} \cong \overline{A'C'}$ therefore, $\triangle ABC \cong \triangle A'B'C'$ because of SSS.

**Side-Angle-Side (SAS):**

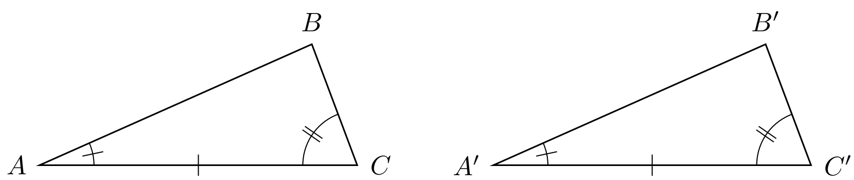
If two sides of a triange is congruent to two sides of another triangle, and the angle formed by the two sides is also congruent, then the two triangles are congruent.




In the above two triangles, $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$, and $\angle A \cong \angle A'$, therefore, $\triangle ABC \cong \triangle A'B'C'$ because of SAS.

**Angle-Side-Angle (ASA):**

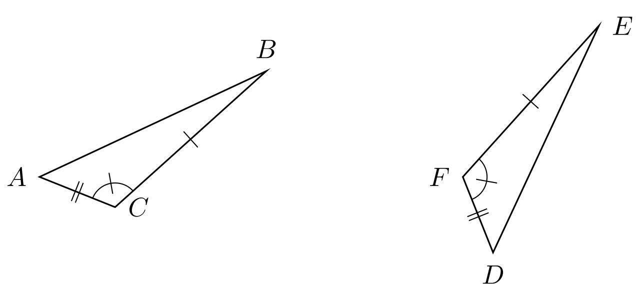
If two angles of a triange is congruent to two angles of another triangle, and the side between the two angles is also congruent, then the two triangles are congruent.




In the above two triangles, $\overline{AC} \cong \overline{A'C'}$, $\angle A \cong \angle A'$, and $\angle C \cong \angle C'$,

therefore, $\triangle ABC \cong \triangle A'B'C'$ because of ASA.

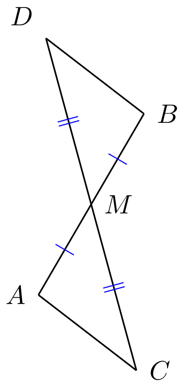
It is important to note that, when denoting two triangles being congruent, the vertices and sides where the congruence is marked must be correctly corresponded.




In the above, $\overline{AC} \cong \overline{DF}$, $\overline{CB} \cong \overline{FE}$, $\angle ACB \cong \angle DFE$, therefore by **SAS**, $\triangle ACB \cong \triangle DFE$.

It will be *incorrect* to say that triangle $ACB$ is congruent to triangle $FDE$, since this is not the corresponance of the congruent sides and angles.

Example: In the picture below, $\overline{AB}$ and $\overline{CD}$ bisect each other at $M$. Prove that $\triangle AMC \cong \triangle BMD$

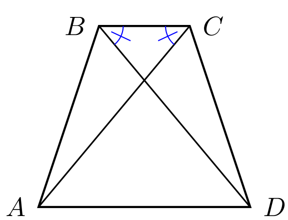



Proof:

| | Statements | | Reasons |
|---|---|---|---|
| 1. | $\overline{AB}$ and $\overline{CD}$ bisects each other at $M$. | 1. | given |
| 2. | $\overline{AM} \cong \overline{MB}$, $\overline{CM} \cong \overline{MD}$ | 2. | definition of segment bisector |
| 3. | $\angle AMC \cong \angle BMD$ | 3. | vertical angles are $\cong$ |
| 4. | $\triangle AMC \cong \triangle BMD$ | 4. | SAS |

Example:

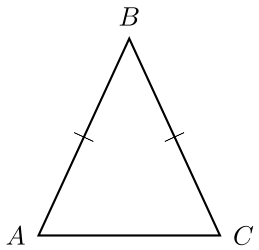
In quadrilateral $ABCD$, $\overline{AC} \cong \overline{BD}$ and $\angle ACB \cong \angle DBC$. Prove $\overline{AB} \cong \overline{DC}$




Proof:

| | Statements | | Reasons |
|---|---|---|---|
| 1. | $\overline{AC} \cong \overline{DB}$, $\angle ACB \cong \angle DBC$ | 1. | Given |
| 2. | $\overline{BC} \cong \overline{BC}$ | 2. | reflexive |
| 3. | $\triangle ABC \cong \triangle DCB$ | 3. | SAS |
| 4. | $\overline{AB} \cong \overline{DC}$ | 4. | CPCTC |

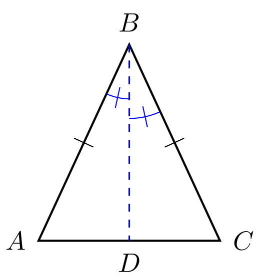
Example: $\triangle ABC$ is an isoceles triangle with $\overline{AB} \cong \overline{CB}$. Prove that $\angle A \cong \angle C$.




To do this problem, we need to **introduce something extra**.

The idea of introducing something new in order to solve a math problem is a technique that is used often in mathematics, and you should be aware of it and try to use it on your own. Sometimes a mathematics problem will be difficult to solve on its own, but if we introduce a new variable, or a new number into the situation, it may help make the problem more clear and easier to approach. In the case with geometry problems, we often have to construct a new segment bisector or a perpendicular, or extend a line, or some other additional information into the problem. Sometimes this will make the problem easier to see or allows us to use something that was not available before.

For this particular problem, we draw (construct) the angle bisector to $\triangle ABC$ that bisects $\angle B$ and intersects $\overline{AC}$ at $D$. The reason is because we wanted to introduce two (congruent) triangles so we can use CPCTC.




Proof:

| | Statements | | Reasons |
|---|---|---|---|
| 1. | $\triangle ABC$ is isoceles with $\overline{AB} \cong \overline{CB}$<br>$\overline{BD}$ bisects $\angle ABC$ | 1. | Given; construction |
| 2. | $\angle ABD \cong \angle CBD$ | 2. | definition of angle bisector |
| 3. | $\overline{BD} \cong \overline{BD}$ | 3. | reflexive |
| 4. | $\triangle ABD \cong \triangle CBD$ | 4. | SAS |
| 5. | $\angle BAD \cong \angle BCD$ | 5. | CPCTC |

Note that in this example, we *assumed* that the angle bisector $\overline{BD}$ exists and we can contruct it as the way we used in the problem. The existence of the angle bisector is guaranteed by the earlier postulates on lines and angles. The construction of the angle bisector can be done using rulers and compass.

It should be noted that, in mathematics in general, as long as we know that a mathematical object exists, we can freely use it as if we already *have* it. In (Euclidean) geometry, however, the convention is that we *do not use* a geometrical object unless we can *contruct it* by using ruler and compass. Without going into details about construction, we will approach a geometry problem by assuming that if a geometrical object exists, we can use it.

The above problem we just did proved the following:

**Theorem:** In an isoceles triangle, the bases angles (the angles on the opposite sides of the congruent sides) are congruent.

This easily leads to another fact about equilateral triangles:

**Theorem:** In an equilater triangle, all three angles are congruent.

The converse of the above two statements are also true, but we will wait till later to prove them.