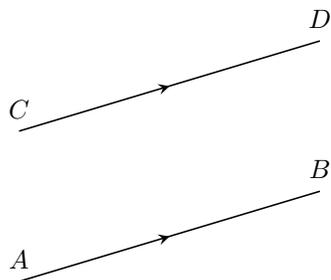


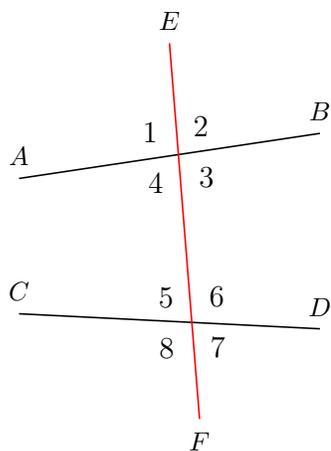
Parallel Lines:

Definition: We say that two lines (on the same plane) are **parallel** to each other if they never intersect each other, regardless of how far they are extended on either side. Pictorially, parallel lines run along each other like the tracks of a train.



Lines \overline{AB} and \overline{CD} are parallel to each other. We use the symbol \parallel to represent two lines being parallel. We write $\overline{AB} \parallel \overline{CD}$ to denote \overline{AB} is parallel to \overline{CD} . We use little arrows on the two lines to indicate that they are parallel to each other.

A **transversal** of two (or more) lines is another line that intersects the two lines.



In the picture above, line \overline{EF} is a transversal of lines \overline{AB} and \overline{CD} . It intersects the two lines and forms 8 angles with the two lines. We name the relationship of the angle pairs based on their position with respect to each other and to the lines \overline{AB} and \overline{CD} .

The angles $\angle 1$, $\angle 2$, $\angle 7$, $\angle 8$ are **exterior** angles because they are on the *outside* of lines \overline{AB} and \overline{CD} .

The angles $\angle 3$, $\angle 4$, $\angle 5$, $\angle 6$ are **interior** angles because they are on the *inside* of lines \overline{AB} and \overline{CD} .

Corresponding Angles are angles that are on the same side of the transversal and on the same side of each intersected line. In the picture above,

$\angle 2$ and $\angle 6$ are corresponding angles.

$\angle 3$ and $\angle 7$ are corresponding angles.

$\angle 1$ and $\angle 5$ are corresponding angles.

$\angle 4$ and $\angle 8$ are corresponding angles.

Alternate Interior Angles are interior angles on opposite sides of the transversal.

$\angle 3$ and $\angle 5$ are alternate interior angles.

$\angle 4$ and $\angle 6$ are alternate interior angles.

Alternate Exterior Angles are exterior angles on opposite sides of the transversal.

$\angle 2$ and $\angle 8$ are alternate exterior angles.

$\angle 1$ and $\angle 7$ are alternate exterior angles.

Same-Side Interior Angles are interior angles on the same side of the transversal.

$\angle 4$ and $\angle 5$ are same-side interior angles.

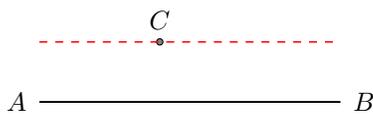
$\angle 3$ and $\angle 6$ are same-side interior angles.

Same-Side Exterior Angles are exterior angles on the same side of the transversal.

$\angle 2$ and $\angle 7$ are same-side exterior angles.

$\angle 1$ and $\angle 8$ are same-side exterior angles.

Parallel Postulate: Given a line and given a point not on the line, there is one and only one line that can be drawn that contains the given point and is parallel to the given line.

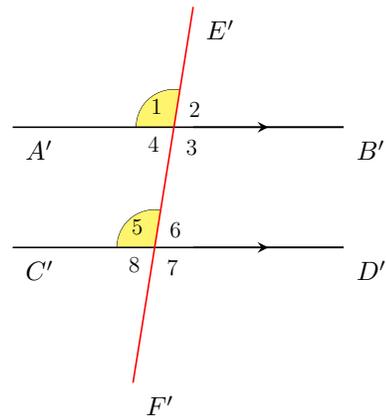
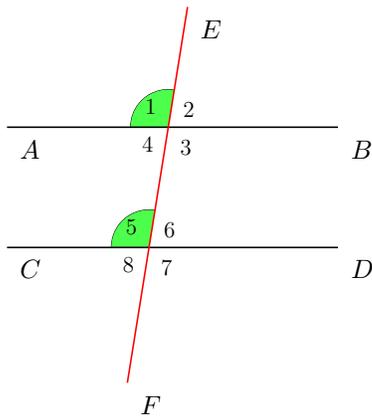


This is the fifth (and most controversial) postulate of Euclid's. It says that if \overline{AB} is a line, and point C is a point not on the line, then there is one and only one line containing C and is parallel to \overline{AB} . Using this postulate, we can prove the following theorems:

Theorem: Corresponding Angles: If two lines are cut by a transversal that makes a pair of corresponding angles congruent, then the two lines are parallel.

The converse of this theorem is also true:

If two parallel lines are cut by a transversal, then their corresponding angles are congruent to each other



The corresponding angle theorem says that, if $\angle 1 \cong \angle 5$, then $\overline{AB} \parallel \overline{CD}$ (picture on the left)

The converse of the theorem says that, if $\overline{AB} \parallel \overline{CD}$, then $\angle 1 \cong \angle 5$, (picture on the right)

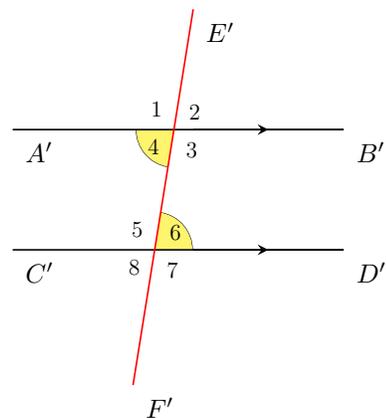
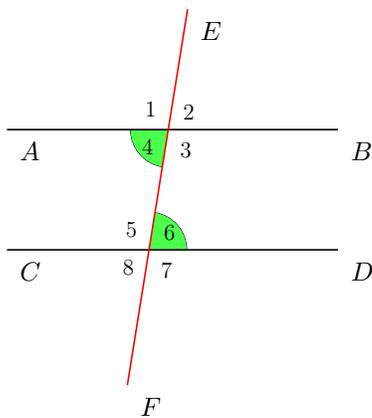
Notice that there are multiple corresponding angle pairs. The theorem says that as long as we have *one* pair of corresponding angles congruent, the lines will be parallel, so we could prove the two lines \overline{AB} and \overline{CD} are parallel as long as we can prove $\angle 1 \cong \angle 5$, or $\angle 2 \cong \angle 6$, or $\angle 4 \cong \angle 8$, or $\angle 3 \cong \angle 7$.

The converse of the theorem says that, when two lines are parallel, then **all** corresponding angles are congruent, so given that $\overline{AB} \parallel \overline{CD}$, then $\angle 1 \cong \angle 5$, and $\angle 2 \cong \angle 6$, and $\angle 4 \cong \angle 8$, and $\angle 3 \cong \angle 7$.

Alternate Interior Angles: If two lines are cut by a transversal that make a pair of alternate interior angles congruent to each other, then the two lines are parallel.

The converse of this theorem is also true:

If two parallel lines are cut by a transversal, then the alternate interior angles are congruent



The alternate interior angle theorem says that, if $\angle 4 \cong \angle 6$, then $\overline{AB} \parallel \overline{CD}$ (picture

on the left)

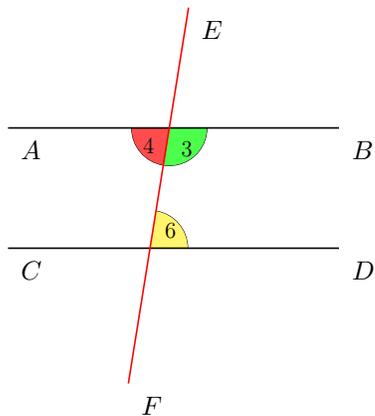
The converse of the theorem says that, if $\overline{AB} \parallel \overline{CD}$, then $\angle 4 \cong \angle 6$, (picture on the right)

Notice that there are two pairs of alternate interior angles. The theorem says that as long as we have *one* pair of alternate interior angle congruent, the lines will be parallel, so the two line will be parallel if we could prove $\angle 4 \cong \angle 6$, or $\angle 3 \cong \angle 5$.

The converse of the theorem says that, when two lines are parallel, then **all** corresponding angles are congruent, so given that $\overline{AB} \parallel \overline{CD}$, then $\angle 4 \cong \angle 6$, and $\angle 3 \cong \angle 5$.

Theorem: If a transversal intersecting two lines making a pair of same side interior angle supplementary, then the two lines are parallel

Proof: In the picture below, transversal \overline{EF} intersects lines \overline{AB} and \overline{CD} , making $\angle 3$ and $\angle 6$ supplementary, we want to prove that $\overline{AB} \parallel \overline{CD}$.



Statements	Reasons
1. \overline{EF} is a transversal to \overline{AB} and \overline{CD} $\angle 3$ and $\angle 6$ are supplementary	1. Given
2. $m\angle 3 + m\angle 4 = 180^\circ$	2. \angle addition postulate
3. $\angle 3$ and $\angle 4$ are supplementary	3. def. of supp. \angle
4. $\angle 4 \cong \angle 6$	4. \angle 's supp. to the same angle are \cong
5. $\overline{AB} \parallel \overline{CD}$	5. Alternate Interior Angles

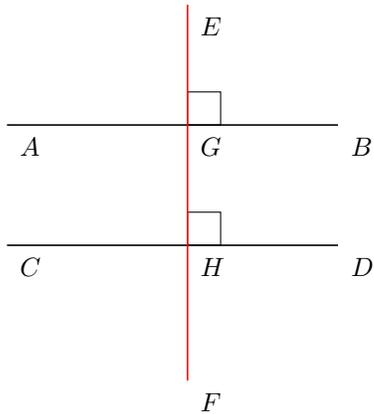
The converse of this statement is also true. In other words,

If two parallel lines are intersected by a transversal, the same side interior angles are supplementary

Theorem: If two lines are perpendicular to the same line, then they

are parallel to each other

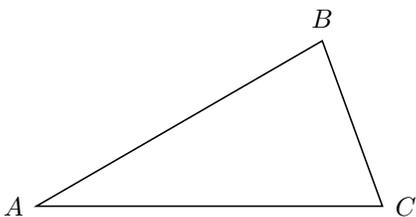
Proof: In the picture below, lines \overline{AB} and \overline{CD} are perpendicular to the same line, \overline{EF} . We need to prove that $\overline{AB} \parallel \overline{CD}$



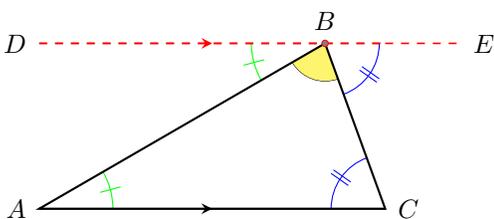
Statements	Reasons
1. $\overline{AB} \perp \overline{EF}, \overline{CD} \perp \overline{EF}$	1. Given
2. $\angle EGB$ and $\angle GHD$ are right angles	2. Def. of \perp lines
3. $\angle EGB \cong \angle GHD$	3. All right angles are \cong
4. $\overline{AB} \parallel \overline{CD}$	4. Corresponding Angles

Theorem: Sum of Interior Angles of a Triangle: The sum of the three interior angles of any triangle is always equal to 180° (two right angles)

Proof: Given $\triangle ABC$, we must prove that $m\angle A + m\angle B + m\angle C = 180^\circ$



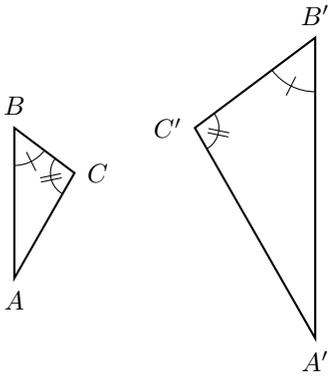
To prove this theorem, we use the idea we used before. We will introduce something new into the problem. We will construct the line \overline{DE} that contains point B and is parallel to \overline{AC} . The existence of this line is guaranteed by the parallel postulate.



Statements	Reasons
1. $\triangle ABC$ is a triangle	1. Given
2. Construct \overline{DE} so that \overline{DE} contains B and $\overline{DE} \parallel \overline{AC}$	2. Parallel Postulate
3. $m\angle DBA + m\angle ABC + m\angle EBC = 180^\circ$	3. Angle Addition Postulate
4. $\angle DBA \cong \angle CAB$, $\angle EBC \cong \angle ACB$	4. Alternate interior angles
5. $m\angle CAB + m\angle ABC + m\angle ACB = 180^\circ$	5. Substitution

An immediate consequence of the triangle interior sum theorem is the following:

If two angles of one triangle is congruent to two angles of another triangle, then the third angle must also be congruent



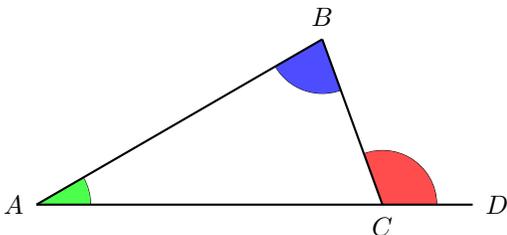
Example: In triangles $\triangle ABC$ and $\triangle A'B'C'$, since $\angle B \cong \angle B'$ and $\angle C \cong \angle C'$, it follows that $\angle A \cong \angle A'$.

This theorem says that if any two triangles have two angles congruent to each other, then all three angles must be congruent. Notice that two triangles need *not* be congruent even if all their angles are congruent.

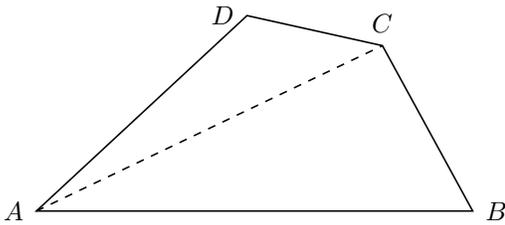
We also have this fact:

Theorem: The sum of any two interior angles of a triangle is equal to the opposite exterior angle.

In the picture below, $m\angle A + m\angle B = m\angle BCD$



Theorem: Sum of Interior Angles of a Quadrilateral: The sum of the interior angles of a quadrilateral is 360°



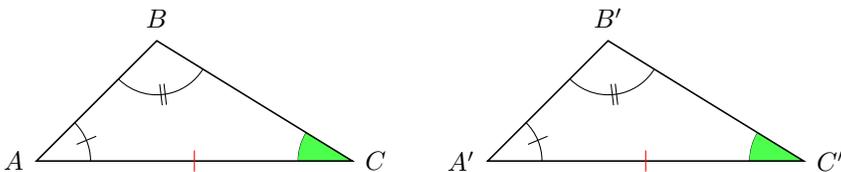
In picture above, notice that \overline{AC} divides quadrilateral $ABCD$ into two triangles, $\triangle ACD$ and $\triangle ACB$. The sum of the interior angles of each triangle is 180° , so the sum of the interior angles of a quadrilateral is 360°

In general, for a polygon with n sides, the sum of its interior angles is equal to $(n - 2)180^\circ$

We can use the triangle interior angle sum theorem to prove another triangle congruence theorem:

Angle-Angle-Side (AAS): If two angles of a triangle is congruent to two angles of another triangle, and a corresponding side (not necessarily between the two angles) is also congruent, then the two triangles are congruent.

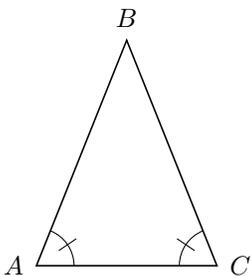
The AAS congruence is a more general version of the ASA. In the picture below, if we have $\angle B \cong \angle B'$, $\angle A \cong \angle A'$, and $\overline{AC} \cong \overline{A'C'}$, then since two angles of $\triangle ABC$ is congruent to two angles of $\triangle A'B'C'$, the third angle must also be congruent, i.e. $\angle C \cong \angle C'$, so $\triangle ABC \cong \triangle A'B'C'$ because of ASA.



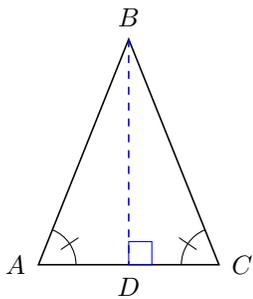
We can now prove the converse of the isoceses triangle theorem:

In a triangle if two angles are congruent, then the two sides opposite the two angles are also congruent

Proof: Given $\triangle ABC$, if $\angle A \cong \angle C$, we need to prove that $\overline{BA} \cong \overline{BC}$



We will again introduce something new. We will construct the altitude \overline{BD} that contains B and is perpendicular to \overline{AC}

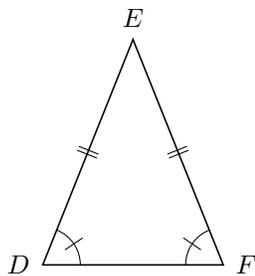
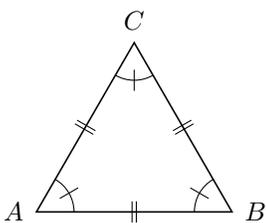


Statements	Reasons
1. $\angle A \cong \angle C, \overline{BD} \perp \overline{AC}$	1. Given; Def. of Altitude
2. $\angle BDA$ and $\angle BDC$ are right angles.	2. Def. of perpendicular lines
3. $\angle BDA \cong \angle BDC$	3. All Right Angles are \cong
4. $\overline{BD} \cong \overline{BD}$	4. Reflexive
5. $\triangle BAD \cong \triangle BCD$	5. AAS
6. $\overline{BA} \cong \overline{BC}$	6. CPCTC

The above theorem leads us to the next two theorems:

In an isosceles triangle, the altitude (to the non-congruent side) bisects the base. In the above picture, \overline{BD} is the altitude to isosceles triangle $\triangle ABC$, it bisects \overline{AC} .

Theorem: If a triangle has all three interior angles congruent, then the triangle is equilateral



Together with the result from isosceles triangles, we have the following:

If a triangle is isosceles (two side congruent), then its base angles are congruent.

Conversely, if a triangle has two angles congruent, then the triangle is isosceles.

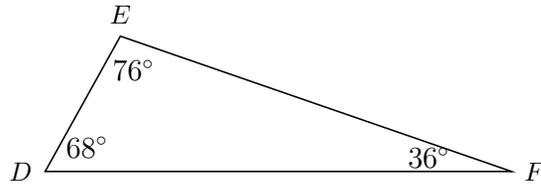
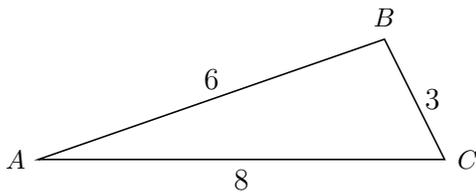
If a triangle is equilateral, then all three of its interior angles are congruent.

If all three interior angles of a triangle is congruent, then the triangle is equilateral.

Theorem: In any triangle, the greater side subtends the greater angle; and the greater angle is subtended by the greater side.

In other words, in a triangle, the longest side is the side opposite the largest

angle. Conversely, the largest angle is the one opposite the longest side. The shortest side is the side opposite the smallest angle, and the smallest angle is the one opposite the shortest side.



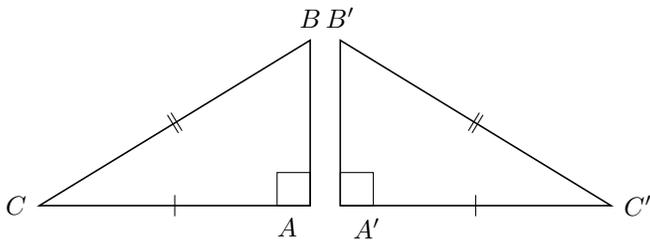
In $\triangle ABC$, side \overline{AC} is the longest at 8 units long, this implies that $\angle B$, the angle opposite \overline{AC} , is the largest of the three angles. Side \overline{AB} is the second longest side at 6 units long, so $\angle C$, the angle opposite \overline{AB} , is the second largest angle. Side \overline{BC} , at 3 units long, is the shortest, therefore $\angle A$ opposite \overline{BC} is the smallest of the three angles.

In $\triangle DEF$, $\angle E$, at 76° , is the largest angle, this means \overline{DF} , the side opposite $\angle E$, is the longest side. $\angle F$ is the smallest of the three angles, meaning that \overline{ED} , the side opposite $\angle F$, is the shortest side.

Triangle Inequality: The sum of the length of any two sides of a triangle is greater than the third side.

Hypotenuse-Leg (HL), right-triangle only:

If the hypotenuse of two right-triangles is congruent and one of their legs is congruent, then the two right-triangles are congruent.



In the right triangles above, hypotenuses $\overline{BC} \cong \overline{B'C'}$, and leg $\overline{AC} \cong \overline{A'C'}$ therefore, $\triangle ABC \cong \triangle A'B'C'$ because of HL.

While we will not prove the theorem, imagine that we put the two triangles side by side so that \overline{AB} and $\overline{A'B'}$ are on top of each other (they become the same line). This gives us the isosceles triangle $\triangle CBC'$. This allows us to see that $\angle C \cong \angle C'$, which will be enough to establish congruence of the two triangles (using SAS).

Note that the HL congruence theorem can only be used if we are dealing with right triangles.

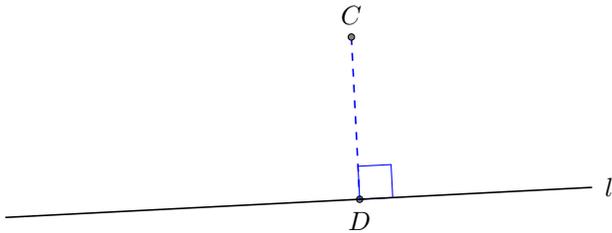
Distance

Given any two points, we say that the **distance** between the two points is the length of the line segment with the two points as end-points.



The *distance* between points A and B is the length of \overline{AB}

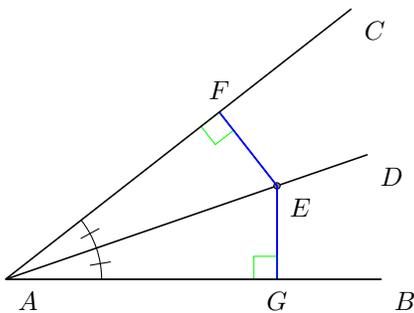
Given a line and a point not on the line, the **distance between the point and the line** is the perpendicular distance between the point and the line. In other words, it is the length of the line segment that contains the given point and is perpendicular to the given line.



In the picture above, line l is a line, point C is a point not on l . The *distance* between point C and line l is the length of \overline{CD} , where D is the point on line l where $\overline{CD} \perp l$

Theorem: Any point on the bisector of an angle is equidistant from the two sides.

Proof: In the picture, \overline{AD} is the bisector of $\angle CAB$. E is a point on \overline{AD} . We need to prove that the (perpendicular) distance between E and \overline{AC} is equal to the (perpendicular) distance between E and \overline{AB} . In other words, we need to prove that $\overline{EF} \cong \overline{EG}$, where $\overline{AC} \perp \overline{EF}$ and $\overline{AB} \perp \overline{EG}$.

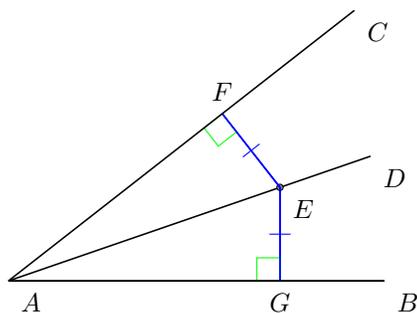


Statements	Reasons
1. \overline{AD} bisects $\angle CAB$, E is on \overline{AD}	1. Given
2. $\angle FAE \cong \angle GAE$	2. Def. of angle bisector
3. $\angle EFA \cong \angle EGA$	3. All Right Angles are \cong
4. $\overline{AE} \cong \overline{AE}$	4. Reflexive
5. $\triangle FAE \cong \triangle GAE$	5. AAS
6. $\overline{EF} \cong \overline{EG}$	6. CPCTC

The converse of this theorem is also true. In other words,

Theorem: If a point is equidistant to both sides of an angle, then the point lies on the bisector of that angle.

Proof: Given $\angle CAB$, and given point E equidistant from \overline{AC} and \overline{AB} (meaning that $\overline{EF} \cong \overline{EG}$), we need to prove that $\angle GAE \cong \angle FAE$.



Statements	Reasons
1. $\overline{EF} \cong \overline{EG}$, E is on \overline{AD}	1. Given
2. $\angle EFA$ and $\angle EGA$ are right angles.	2. Def. of dist. of a point to a line
3. $\overline{AE} \cong \overline{AE}$	3. Reflexive
4. $\triangle FAE \cong \triangle GAE$	4. HL
5. $\angle FAE \cong \angle GAE$	5. CPCTC
6. \overline{AE} bisects $\angle FAG$	6. Def of \angle bisector

What we just proved allows us to prove the following:

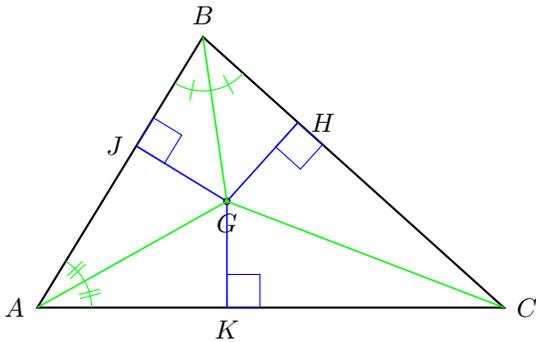
Theorem: The three angle bisectors of a triangle intersect in a point inside the triangle (**the incenter**). This point is equidistant from the three sides of the triangle.

Proof: In the picture below, let point G be the intersection of the angle bisectors of $\angle ABC$ and $\angle BAC$. In other words, \overline{BG} is the bisector of $\angle ABC$ and \overline{AG} is the bisector of $\angle BAC$.

Let \overline{GJ} , \overline{GK} , and \overline{GH} be the distance from point G to \overline{AB} , \overline{AC} , and \overline{BC} ,

respectively.

Let the line segment \overline{CG} be drawn. We must prove that \overline{CG} is the bisector of $\angle BCA$, and that $\overline{GJ} \cong \overline{GK} \cong \overline{GH}$



Statements	Reasons
1. \overline{BG} bisects $\angle ABC$ \overline{AG} bisects $\angle BAC$	1. Given
2. $\overline{GJ} \cong \overline{GH}$, $\overline{GJ} \cong \overline{GK}$	2. Point on angle bisector is equidistant to sides
3. $\overline{GK} \cong \overline{GH}$	3. Substitution
4. \overline{GC} is bisector of $\angle ACB$	4. Point equidistant to sides is on bisector of angle