Maximum and Minimum

Definition:

Definition:

Let f be defined in an interval I. For all x, y in the interval I. If
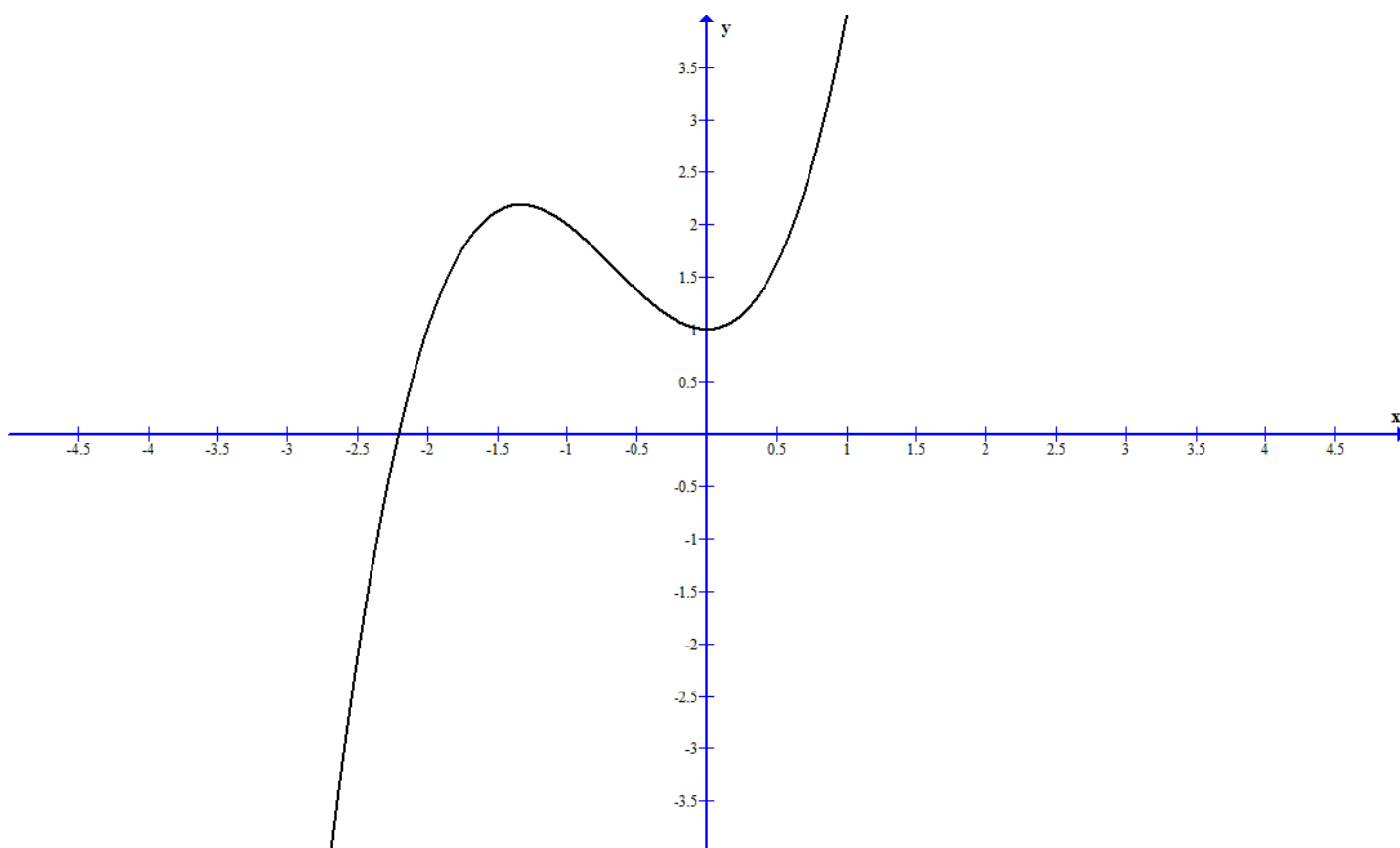
$$f(x) < f(y) \quad \text{whenever } x < y$$

then we say f is **increasing** in the interval I.

If

$$f(x) > f(y) \quad \text{whenever } x < y$$

then we say f is **decreasing** in the interval I.

Graphically, a function is *increasing* if its graph goes *uphill* when x moves from left to right; and if the function is *decresing* then its graph goes *downhill* when x moves from left to right. Notice that a function may be increasing in part of its domain while decreasing in some other parts of its domain.

For example, consider $f(x) = x^2$.
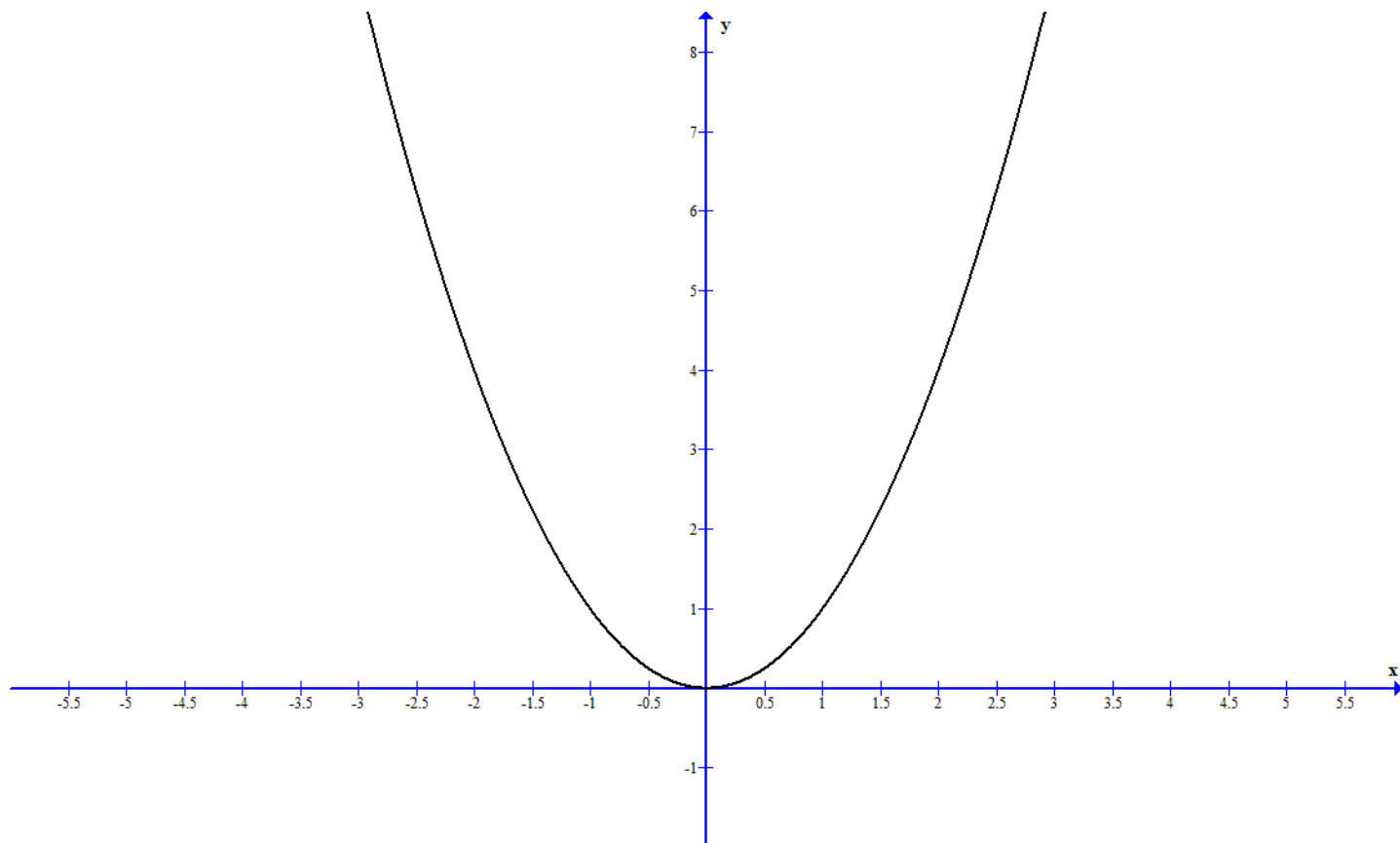
Notice that the graph of f goes downhill before $x = 0$ and it goes uphill after $x = 0$. So $f(x) = x^2$ is decreasing on the interval $(-\infty, 0)$ and increasing on the interval $(0, \infty)$.

Consider $f(x) = \sin x$.
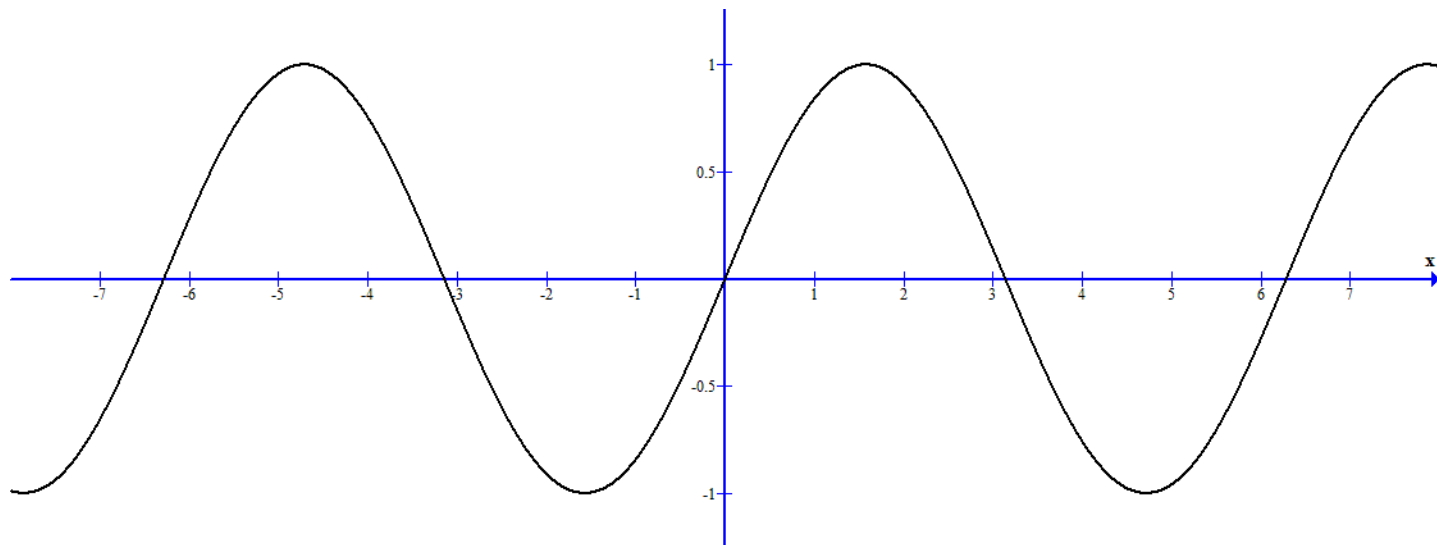
f is increasing on the intervals $(-\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{3\pi}{2}, \frac{5\pi}{2})$, $(\frac{7\pi}{2}, \frac{9\pi}{2})$...etc, while it is decreasing on the intervals $(\frac{\pi}{2}, \frac{3\pi}{2})$, $(\frac{5\pi}{2}, \frac{7\pi}{2})$, $(\frac{9\pi}{2}, \frac{11\pi}{2})$...etc. In general, $f = \sin x$ is increasing on any interval of the form $(\frac{(2n+1)\pi}{2}, \frac{(2n+3)\pi}{2})$, where n is an *odd* integer. $f(x) = \sin x$ is decreasing on any interval of the form $(\frac{(2m+1)\pi}{2}, \frac{(2m+3)\pi}{2})$, where m

is an *even* integer.

What about a constant function? Is a constant function an increasing function or decreasing function? Well, it is like asking when you walking on a flat road, as you going uphill or downhill? From our definition, a constant function is *neither increasing nor decreasing.*

For a line $y = mx + b$, notice that it is increasing if its slope is positive, and it is decreasing if its slope is negative. Since the derivative of a line is its slope, we see that, at least for a line, if its derivative is positive it is increasing, and if its derivative is negative it is decreasing. We may wonder if this observation applies to every function. If we observe the two functions x^2 and $\sin x$ we just considered, we see that it is true for the two functions also. Our observation leads us to this fact:

If $f'(x)$ is positive on an interval I, then f is increasing on I.

If $f'(x)$ is negative on an interval I, then f is decreasing on I.

E.g. Consider $f(x) = x^3 + 2x^2 + 1$. Since $f'(x) = 3x^2 + 4x$, we see that $f'(x)$ is positive on $(-\infty, -\frac{4}{3}) \cup (0, \infty)$ and $f'(x)$ is negative on the interval $(-\frac{4}{3}, 0)$. Hence, f is increasing on $(-\infty, -\frac{4}{3}) \cup (0, \infty)$ and f is decreasing on $(-\frac{4}{3}, 0)$.
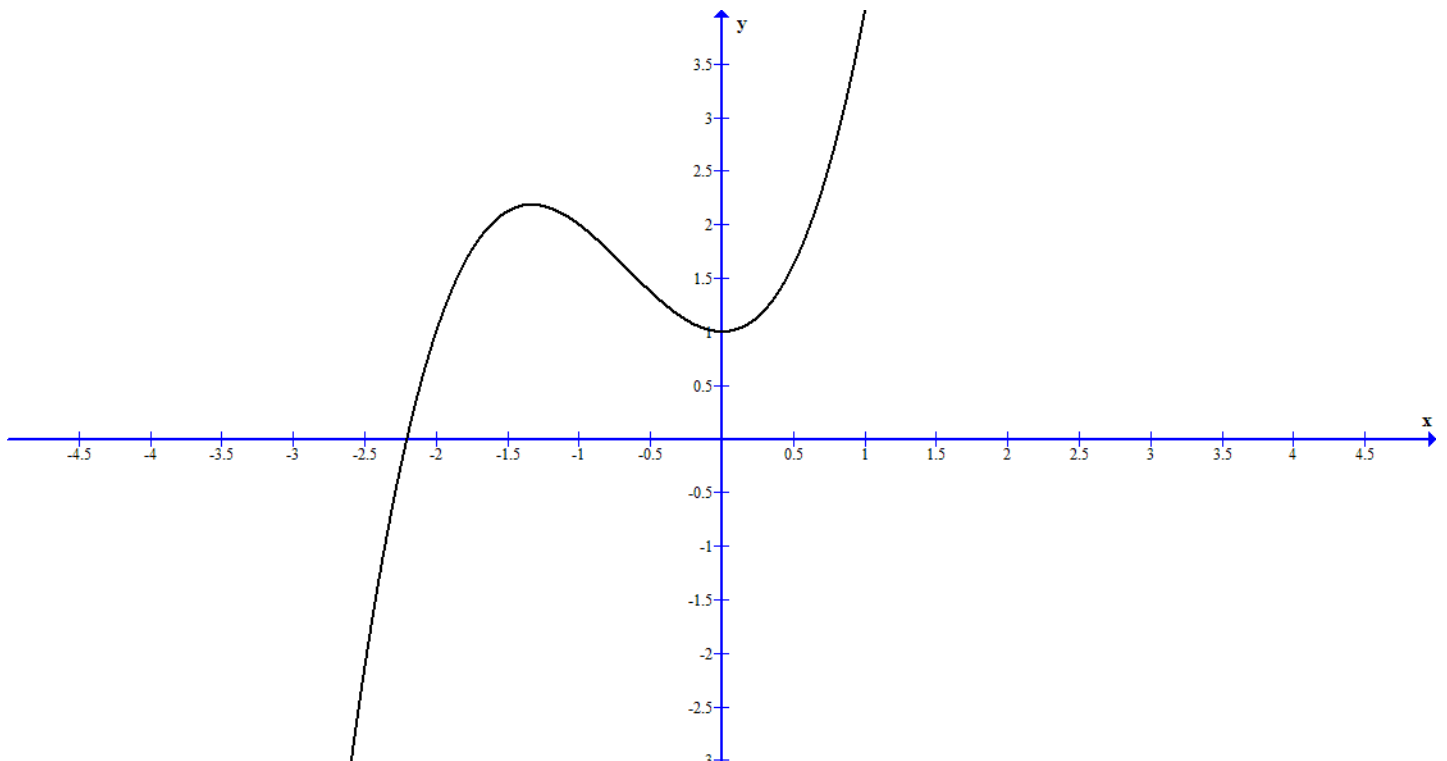
We are now ready to define the maximum and minimum of a function and how to find them:

Definition: Let f be a function. We say that f has an **absolute maximum** (or **global maximum**) at c if $f(c) \geq f(x)$ for all x in the domain of f. If f has an absolute maximum at c, then $f(c)$ is called the **maximum value** of f.

Let f be a function. We say that f has an **absolute minimum** (or **global minimum**) at c if $f(c) \leq f(x)$ for all x in the domain of f. If f has an absolute minimum at c, then $f(c)$ is called the **minimum value** of f.

Definition: Let f be a function. We say that f has a **relative maximum** (or **local maximum**) at c if there is an open interval containing c such that $f(c) \geq f(x)$ for all x in the open interval.

Let f be a function. We say that f has a **relative minimum** (or **local minimum**) at c if there is an open interval containing c such that $f(c) \leq f(x)$ for all x in the open interval.

Intuitively, the *global maximum* is the largest value of f over its whole domain. For example, think of the highest mountain in the world (Mount Everest). Since Mount Everest is the highest mountain in the *world*, there's no other mountain that's taller, so Mount Everest is a *global maximum*. Now consider the Rocky mountain. The Rocky Mountain is *not* the tallest mountain *globally*. However, if we just consider the state of Colorado, then the Rocky Mountain is the highest mountain within the state of Colorado. That is, *locally* within the state of Colorado, the Rocky Mountain is the maximum, so the Rocky Mountain is a *local maximum*.

local max

local and gobal max
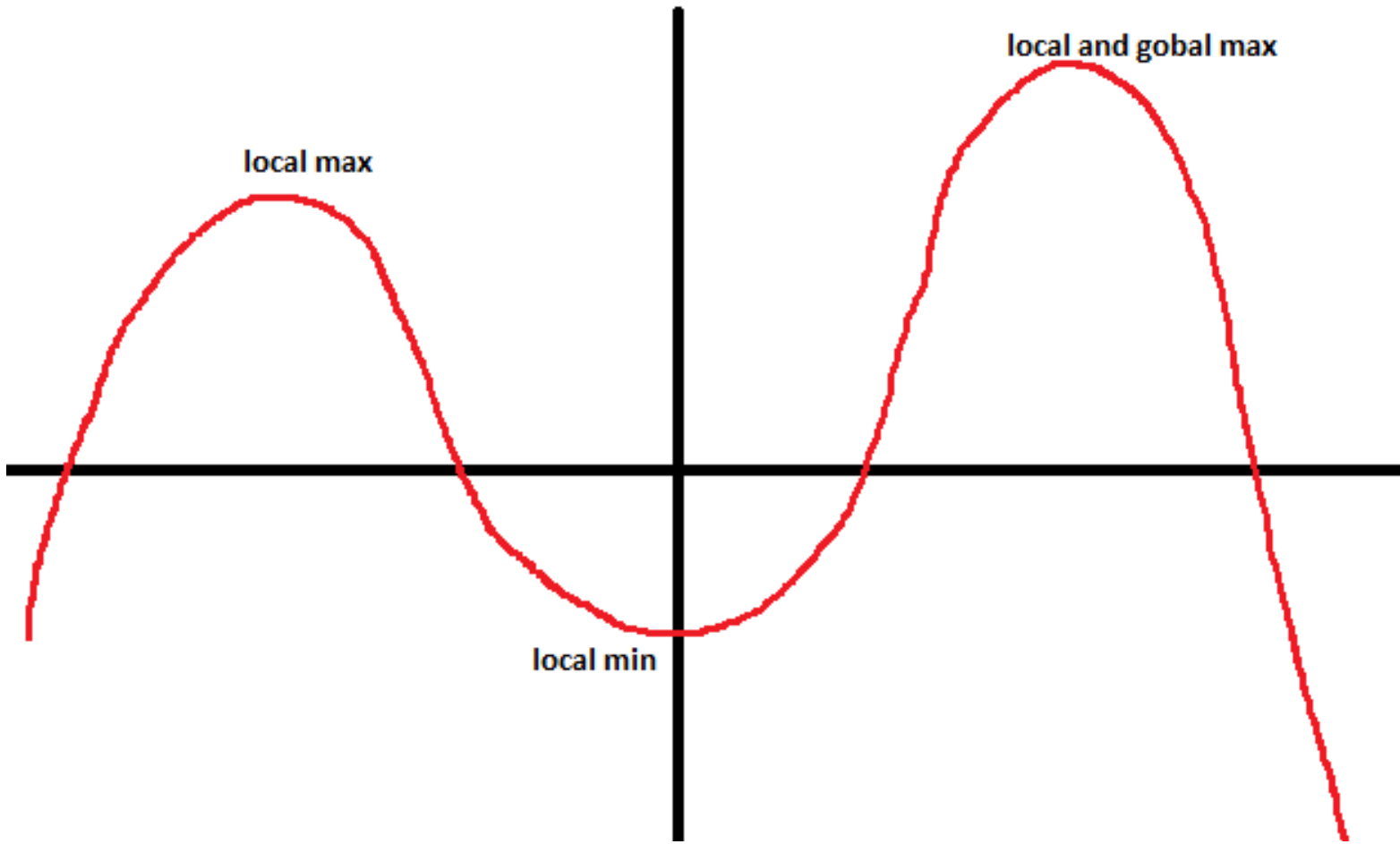
local min

The graph of the above function has a local minimum, a local maximum and also the global maximum, but the function does not have a global minimum.
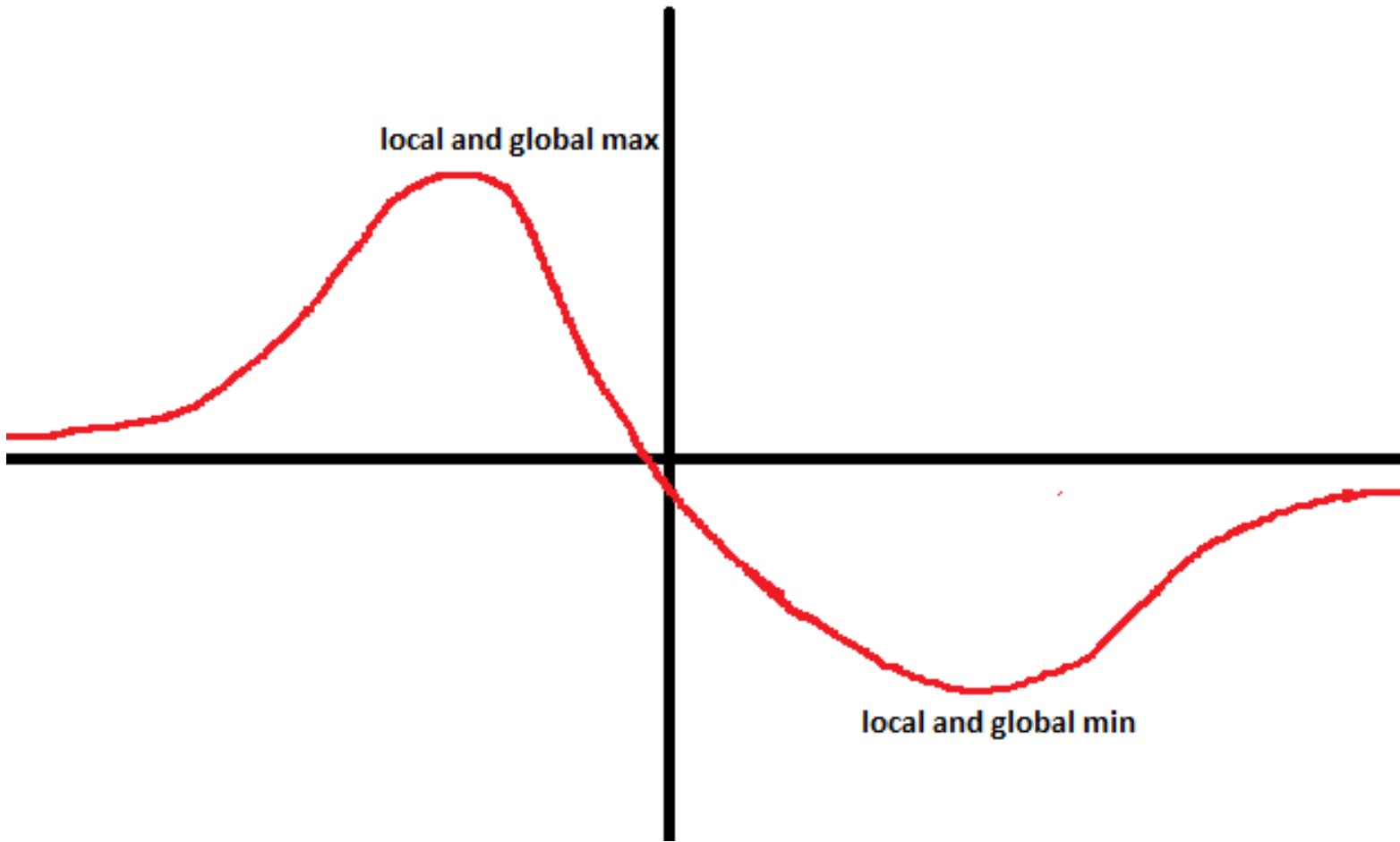
The graph of the above function has a local and global minimum, and a local and global maximum
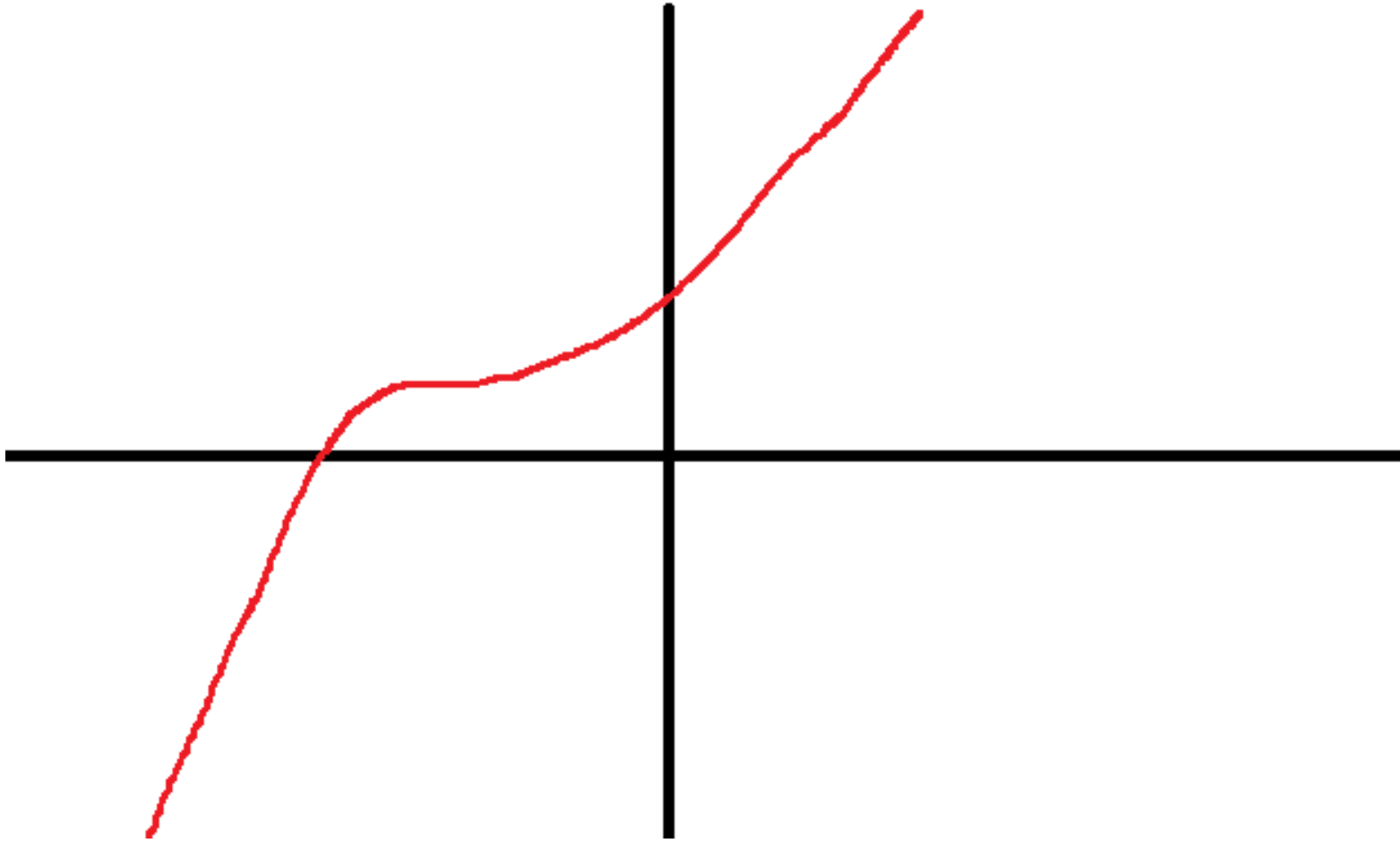
The graph of the above function has no maximum or minimum of any kind

The local maximum appears on the graph of a function as a *peak* (the top of the mountain), while the locall minimum appears on the graph of a function as a *valley*.

Notice that any global max is also a local max. (If you are the tallest person in the world, you must be the tallest person within your neighborhood.) On the otherhand, if you are the tallest person within your neighborhood, you may not be the tallest person in the world. Therefore, a local maximum is not necessarily a global maximum.

An important point to note is that a local maximum within an open interval might not necessarily be larger than another point that is not a local maximum.

Notice also that a function does not have to have any global or local maximum, or global or local minimum.

Example: $f(x) = 3x + 4$

f has no local or global max or min.
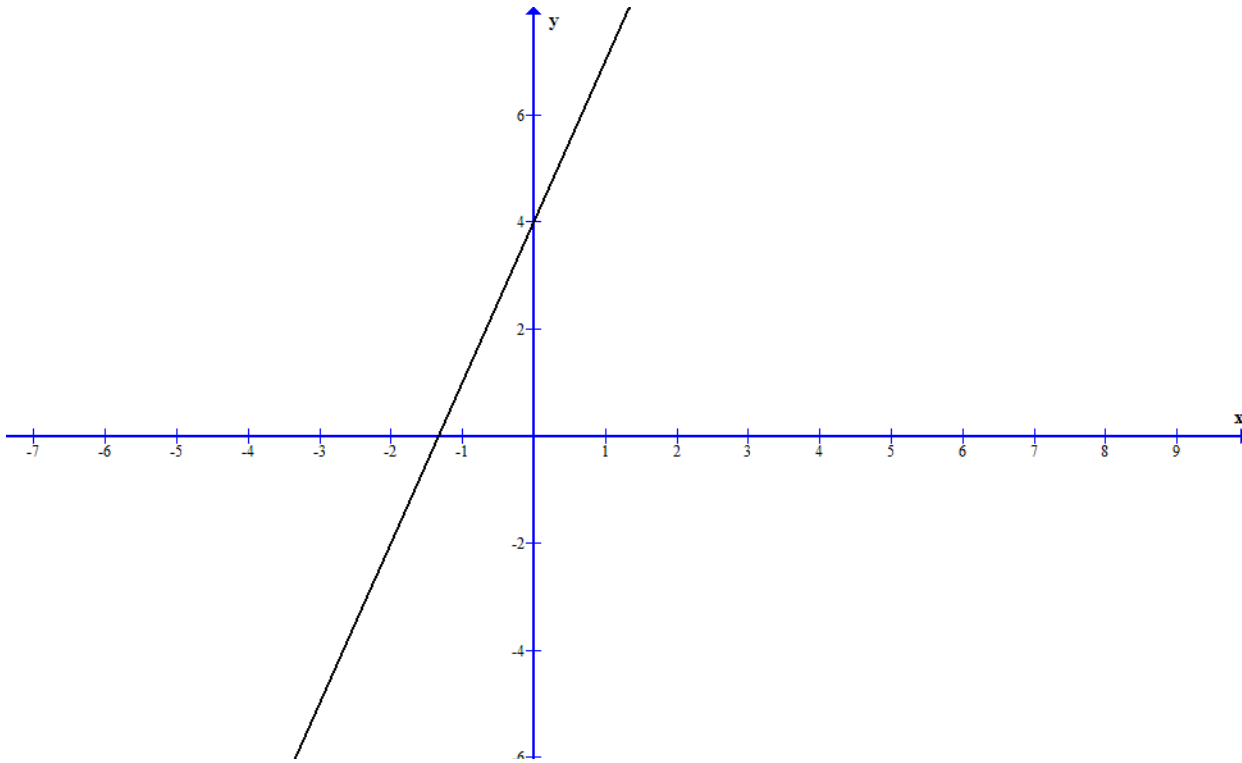
Example

$$f(x) = x^2 - 4x + 4$$

f has a global (therefore also local) minimum at $x = 2$. Minimum value of f is $f(2) = 0$.
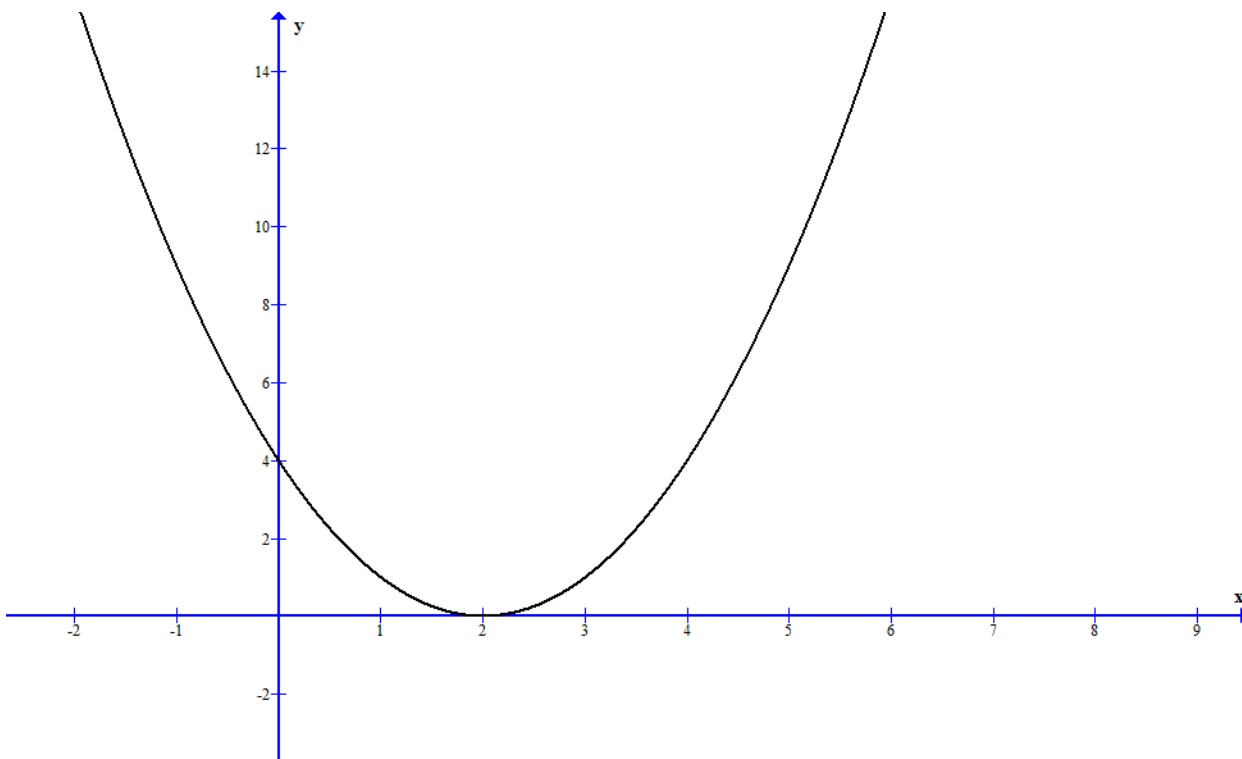
Example: $f(x) = x^3 + 2x^2 + x$

f has a local maximum at $x = -1$ and a local minimum at $x = -\frac{1}{3}$ Local maximum value of f is $f(-1) = 0$. Local minimum value of f is $f(-\frac{1}{3}) = -\frac{4}{27}$.
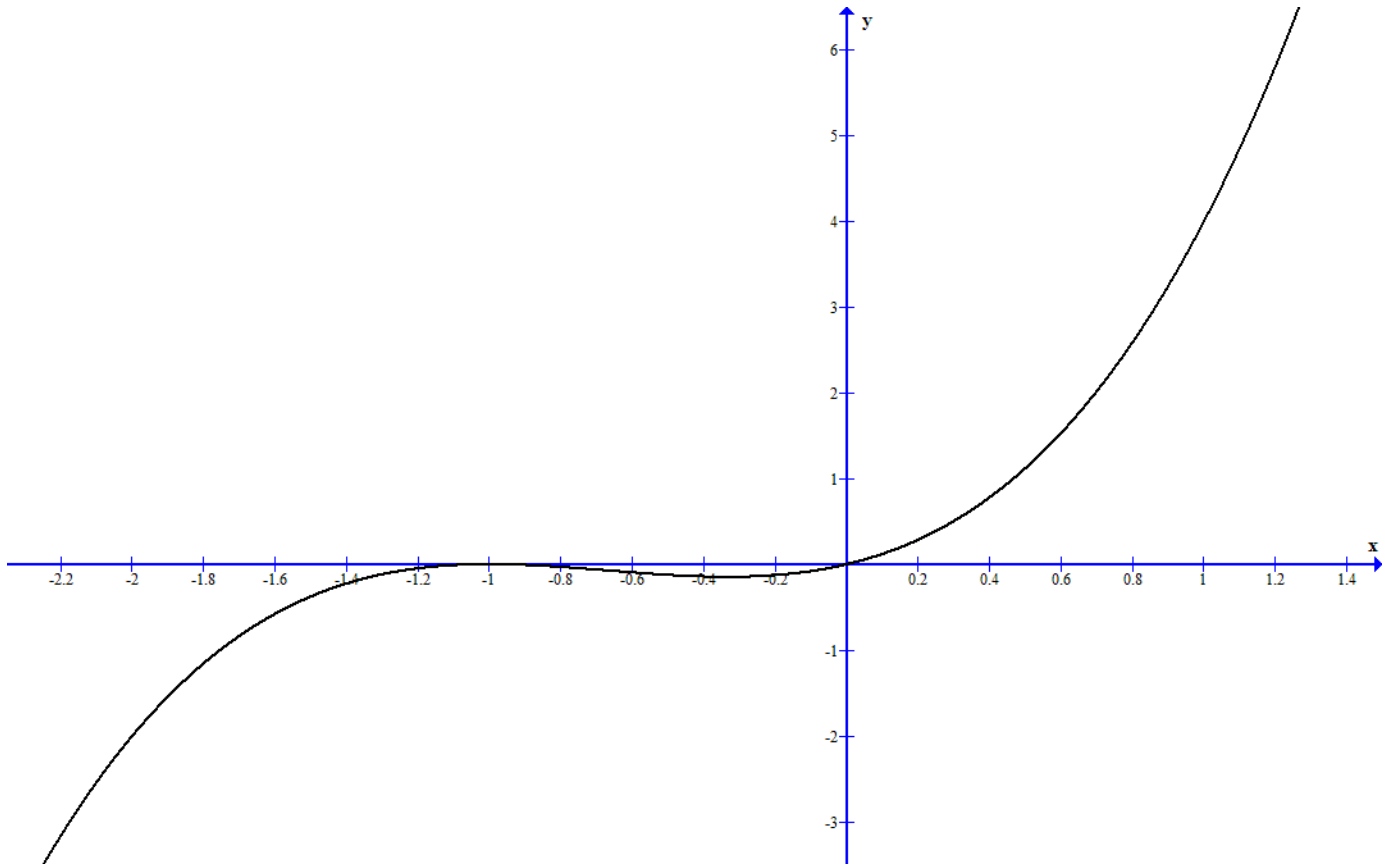
Example: $f(x) = \sin x$

f has global (local) maxima at $\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$ Maximum value of f is 1 f has global (local) minima at $\frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \ldots$ Minimum value of f is -1
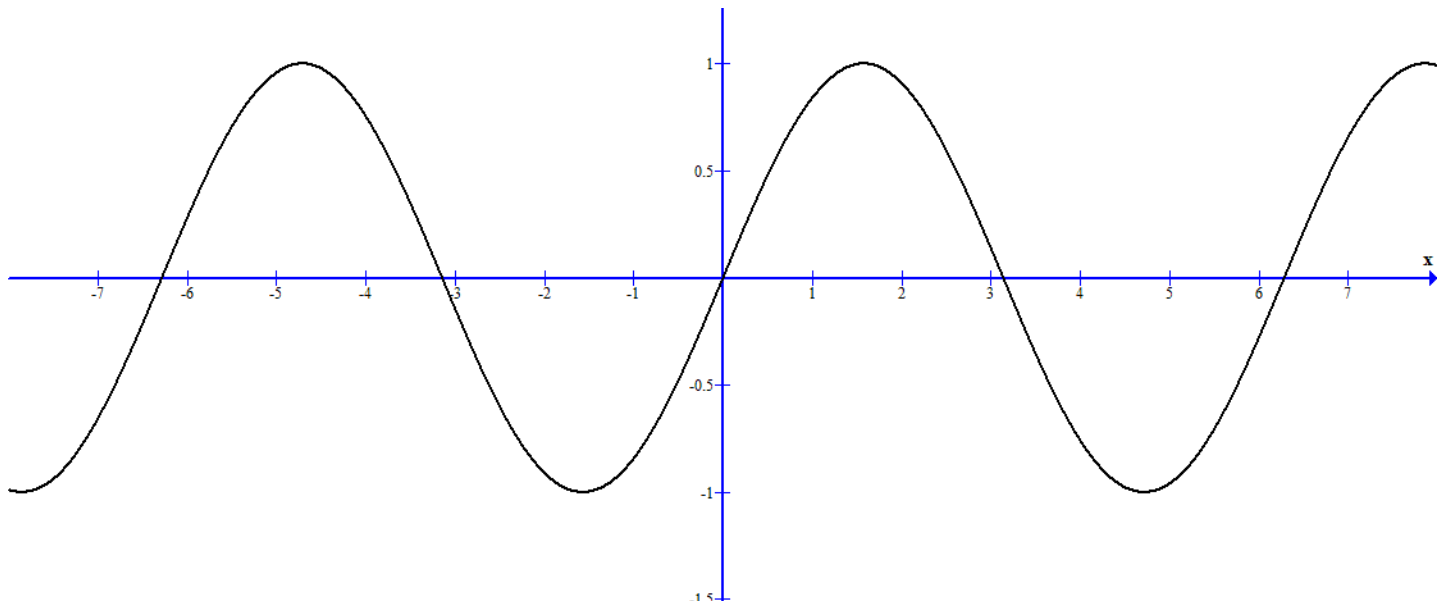
From the example of $\sin x$ we see that, while there is only one *value* for a global maximum or minimum, such value can be achieved at *many* different x values. For $\sin x$, the maximum *value* is 1. That means the *value* of $\sin x$ never exceeds 1. However, there are (infinitely) many x's for which $\sin x = 1$. Think of it this way: Let say the tallest human being is 3 meters tall, which means no human being can be taller than 3 meters. However, there can be more than one person at 3 meters tall.
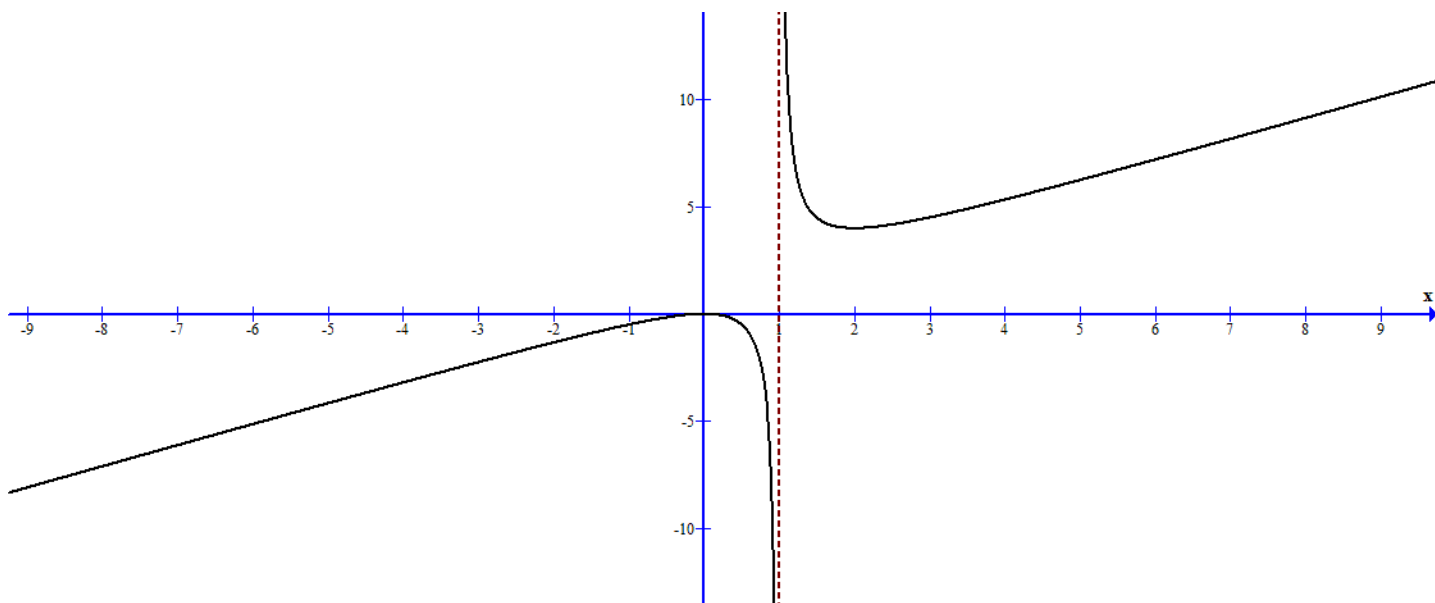
Example: $f(x) = \frac{x^2}{x-1}$

f has local maximum at $x = 0$. Local maximum value of f is $f(0) = 0$.

f has local minimum at $x = 2$. Local minimum value of f is $f(2) = 4$.

This example shows that *a local minimum may have a value that is greater than a local maximum.* In the example, the local maximum value of f is 0, while the local minimum value of f is 4. This is perfectly fine. The important point here is that these are *local* minimum or local maximum. They are only largest (or smalles) *within a confined domain.* You may be the tallest person in your family, but you may be shorter than the shortest NBA player.

Considering that the derivative is a *rate of change*, we saw that if a function is going uphill (increasing), then its derivative is positive, and if the function is going downhill, then its derivative is negative. What if the function is at the peak (local maximum) or valley (local minimum)? What should be the derivative of a function at a local maximum or local minimum? Well, if you are going uphill you are *increasing* (positive derivative), if you are going downhill you are *decreasing* (negative derivative). It makes sense that if you reached the peak (maximum) or valley (minimum) that you are neither moving up or down, then you are stationary. i.e, your rate of change is 0. This is indeed the case:

Fermat's Theorem: If f has *local maximum* or *local minimum* at c, and f is differentiable at c, then $f'(c) = 0$.

This theorem says that if a function has a local maximum or local minimum at a point c, then its tangent at that point (if it has a tangent) *must* be a horizontal line. (Slope is 0). We saw that this is true with all the examples above. Intuitively, since you are at the peak or valley, you are neither moving up or down, so your velociy (rate of change) must be 0.

It is important to note that what the theorem says is that if f has a local maximum or local minimum at c and $f'(c)$ exists then it must be 0. The theorem does **not** say that if $f'(c) = 0$, then f has a local maximum or minimum at c. It is possible that $f'(c) = 0$ but that c is neither a local maximum or local minimum value for f. Consider this example:

Let $f(x) = x^3$. Notice that $f'(x) = 3x^2$, so $f'(0) = 0$. However, f has neither a maximum nor minimum at 0. f just happen to have a horizontal tangent at that particular point.
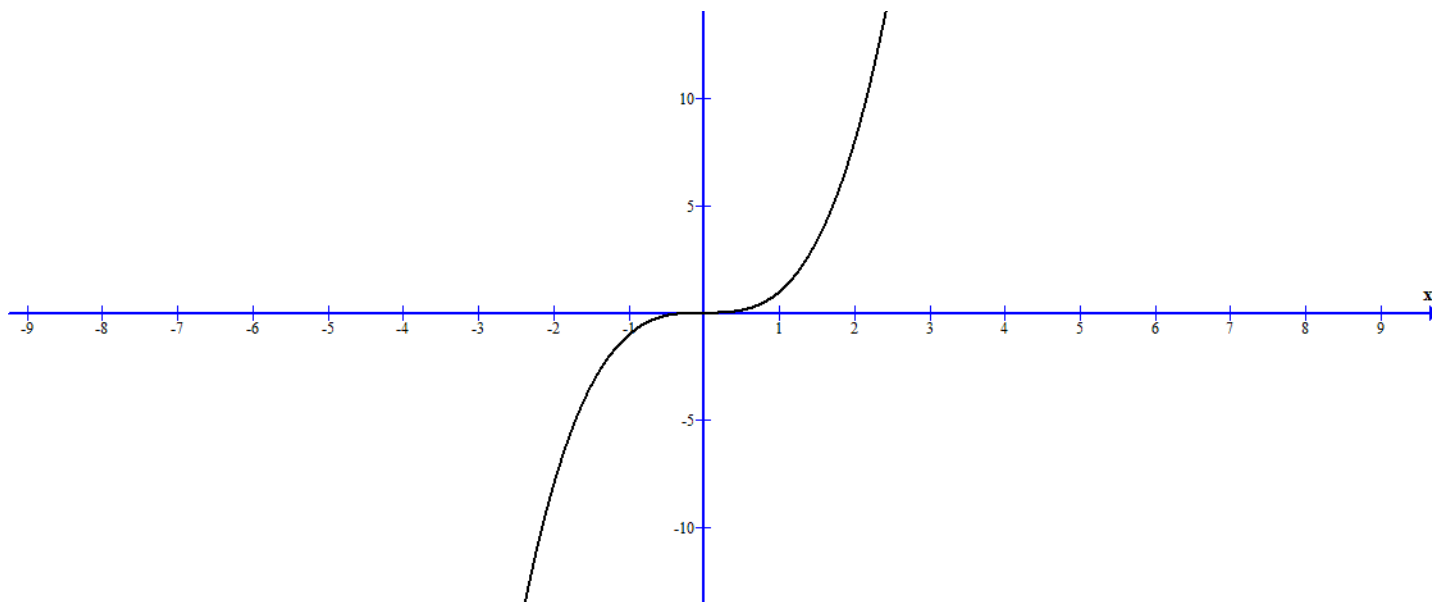
Notice also that the theorem does not say anything about c if f is *not* differentiable at c. It is *possible* that f may have a local maximum or local minimum at c but that $f'(c)$ is undefined, as the next example illustrates:

$f(x) = |x|$. $f'(0)$ does not exist (f has a sharp corner there), however $x = 0$ is a local minimum of f.
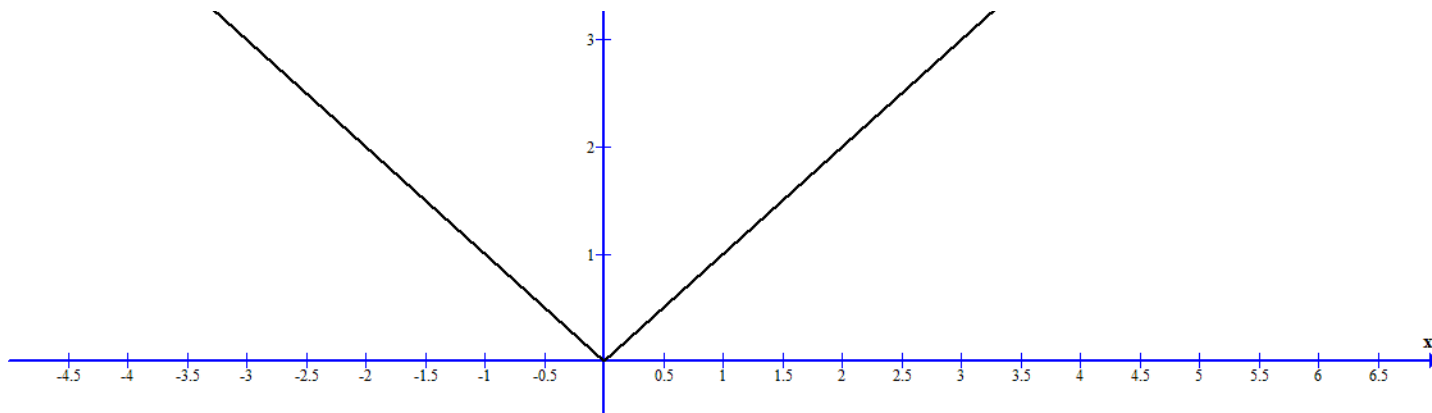
This example shows that even if $f'(c)$ is undefined, f may still have a local minimum or local maximum at c. In general, the local maxima or minima of a function occurs only in locations where the derivative of a function is equal to 0 $(f'(0))$ or if the derivative is undefined.

Definition:

Let a number c be in the domain of f. (That means $f(c)$ exists). c is called a **critical number** of f if $f'(c) = 0$ or if $f'(c)$ is undefined.

What the definition says is that the critical numbers of a function occurs where f has a horizontal tangent ($f'(c) = 0$) or when f is not differentiable at c, like when f has a sharp corner.

Example: $f(x) = (x - 2)^2$

f has a critical number at 2 since $f'(2) = 0$.

Example: $f(x) = x^{2/3}$

f has a critical number at 0 because $f'(0)$ is undefined

Fermat's Theorem tells us that if f has a local maximum or local minimum at c, and if f is differentiable at c, then its derivative must be 0. We also saw from example that f could have a local maximum or local minimum at c if $f'(c)$ is undefined. From the definition of critical numbers, we saw that:

If f has a local maximum or local minimum at c, then c is a cirtical number of f.

This gives us a way to find the local minima and local maxima of a function. (Actually, we can only find the **candidates** for the local minima and local maxima of a function at this moment. We will see later how we can determine if the candidate are local maxima, local minima, or neither.)

To find the local maxima and local minima of a function f, take the derivative of the the function, find the critical numbers of f. (That is, find the points c where $f'(c)$ is equal to 0 or $f'(c)$ is undefined.) The critical numbers are the *candidates* for the local maxima or minima of f. Since the critical numbers c occurs only when the derivative of f is equal to 0 or is undefined at c, this means that, algebraically, to find the maxima and minima of a function f, we first find f', then set f' to 0 to solve for the points c where $f'(c) = 0$. We also look for the points where f' is undefined. These are the *candidate* for local maxima and local minima.

Note that I have been emphasizing that the critical numbers are only *candidate* for local maxima and local minima. This means that even if f has a critical number at c, it does not necessarily mean that the critical number is a local maximum or minimum. But if we were to find any local maxima or local minima of f, we must find the critical numbers first.

Example: $f(x) = \frac{x^2}{x-1}$ Find the critical numbers of f
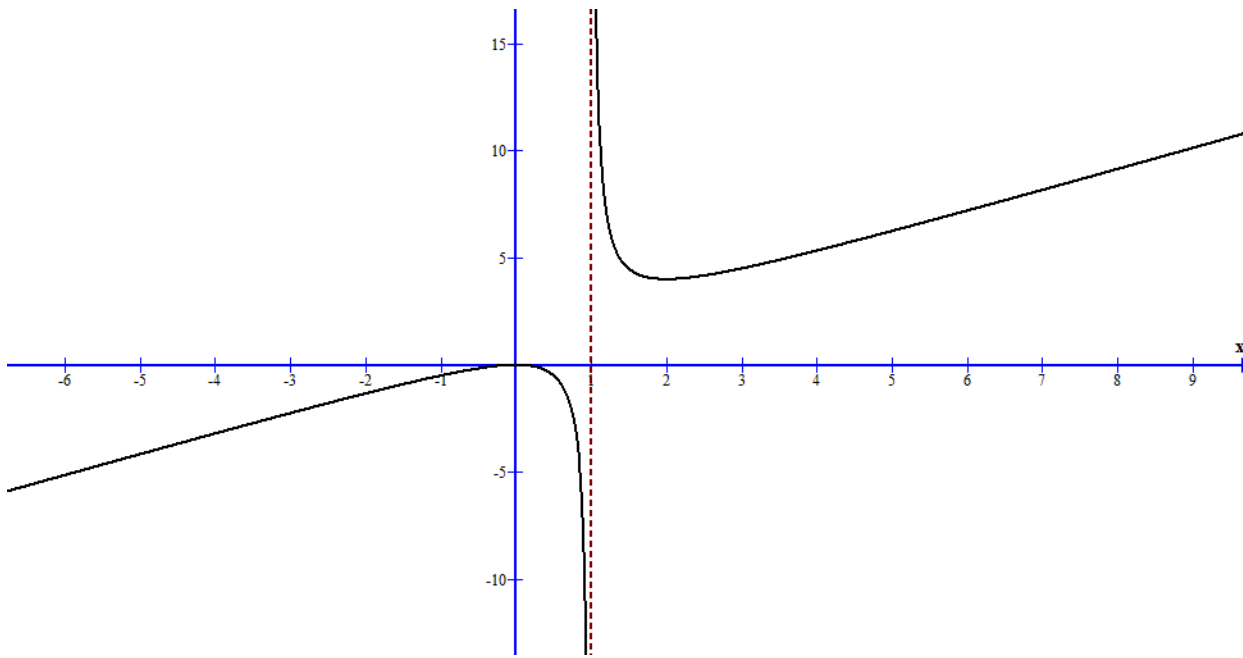
To find the critical numbers of f, we must take its derivative and set it equal to 0.

$$f'(x) = \frac{x^2 - 2x}{(x-1)^2}$$

Setting $f'(x) = 0$ and solving for the equation we have:

$$\frac{x^2 - 2x}{(x-1)^2} = 0$$

Notice that the left hand side is a fraction. A fraction is equal to 0 if and only if its numerator is equal to 0, so

$$\frac{x^2 - 2x}{(x-1)^2} = 0 \Leftrightarrow x^2 - 2x = 0 \Leftrightarrow x = 0 \text{ or } x = 2$$

So f has two cirtical numbers, $x = 0$ and $x = 2$. f has a horizontal tangent at these two points.

How about the point $x = 1$, where $f'(1)$ is undefined? Even though f' is not defined at $x = 1$, but since $f(1)$ is also not defined. $x = 1$ is *not* in the domain of f, so we do not consider $x = 1$ as a critical number of f. However, $x = 1$ is a **vertical asymptote** of f and later when we consider the graph of f, knowing the vertical asymptote of a function will help us better graph a function.

What are the critical numbers 0 and 2? Are they local maximum, local minimum, or neither? From the graph of f we know that f has a local maximum at $x = 0$ and a local minimum at $x = 2$. We will see later how we can determine if a critical point is a local maximum, local minimum, or neither without the graph.

Example: $f(x) = x^{2/3}$ Find the critical numbers of f
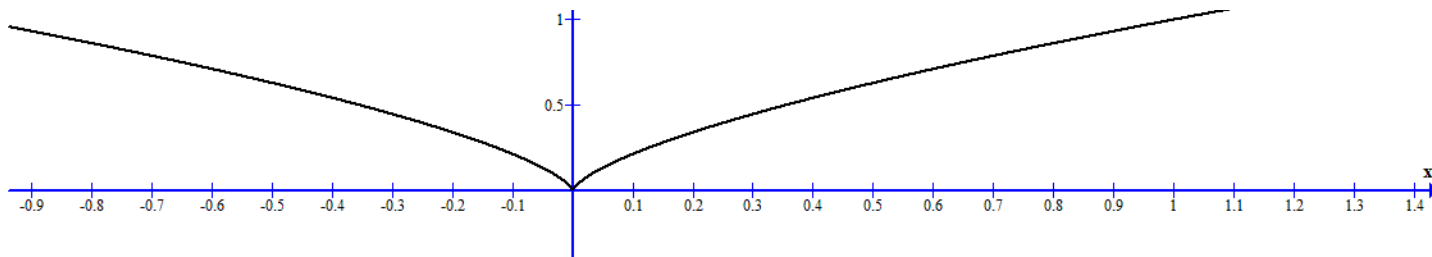
Differentiate f we get

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

Notice that $f'(0)$ is undefined, even though $f(0) = 0$ is defined, so $x = 0$ is a critical point of f. It turns out f has a local (and also global) minimum at 0.

Thus far we have considered the critical numbers of a function f, and we said that the critical numbers are the candidates for local maximum and local minimum. How about global maximum and global minimum? How do we find the global maximum and minimum of a function, if they exist?

In general, if the domain of a function is not a closed interval, the function may not even have a global maximum or global minimum. (For example, a line). Even in the case a function does have a global maximum or a global minimum, we might not be able to determine that maximum or minimum easily. We may need to graph the function.

On the other hand, if we confine the domain of a function to be within a closed interval, then a continuous function will always have a minimum and maximum:

Extreme Value Theorem:

Let f be a continuous function on a closed interval $[a, b]$, then f has a *global maximum* and a *global minimum* inside $[a, b]$. That is, there exists a number C such that $f(C) \geq f(x)$ for all x inside $[a, b]$, ($f(C)$ is the global maximum), and there exists a number c such that $f(c) \leq f(x)$ for all x inside $[a, b]$. ($f(c)$ is the global minimum).

While a function might not necessarily have global maximum or minimum, the *Extreme Value Theorem* tells us that a continuous function inside a closed interval *must* have a global maximum and a global minimum. This theorem is not easy to prove but we can see that from some graphs:
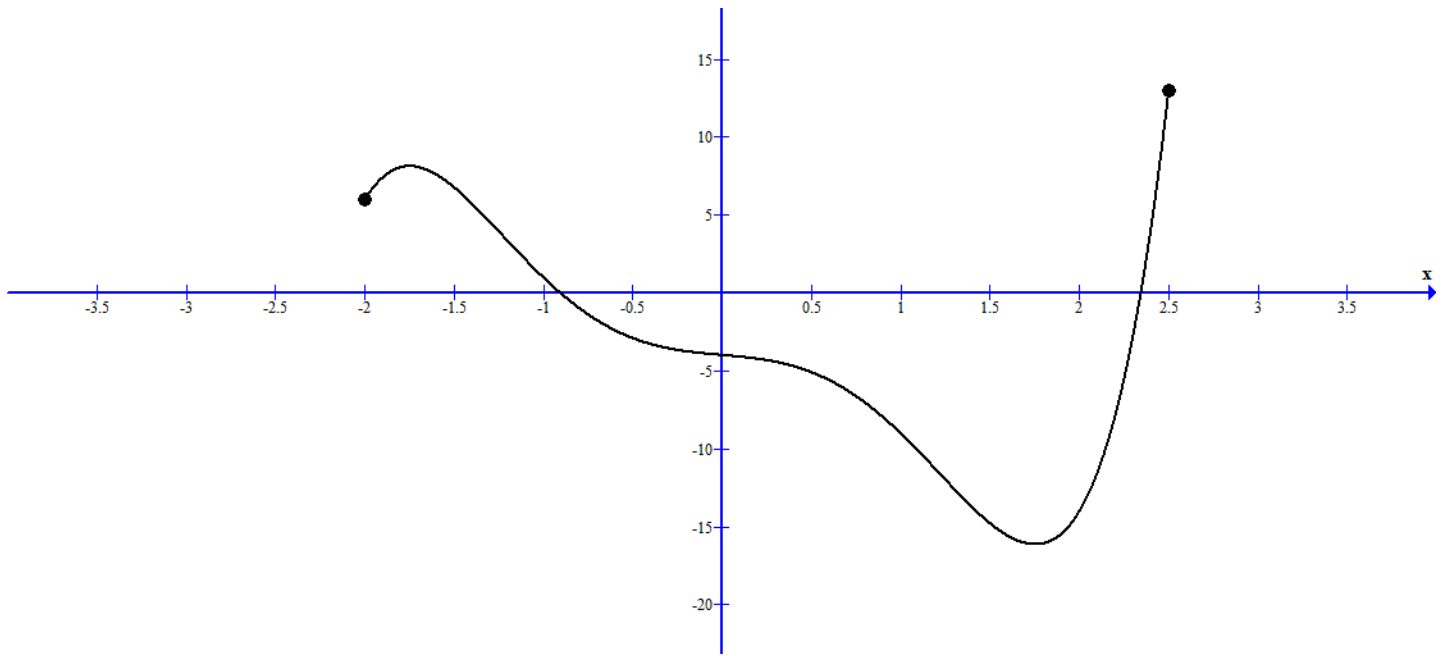
The above function is continuous inside a closed interval. It satisfies the extreme value theorem.

Some functions fail to have a maximum or minimum inside a closed interval because they are not continuous, and a continuous function may not have a maximum or minimum if its domain is not confined within a closed interval.
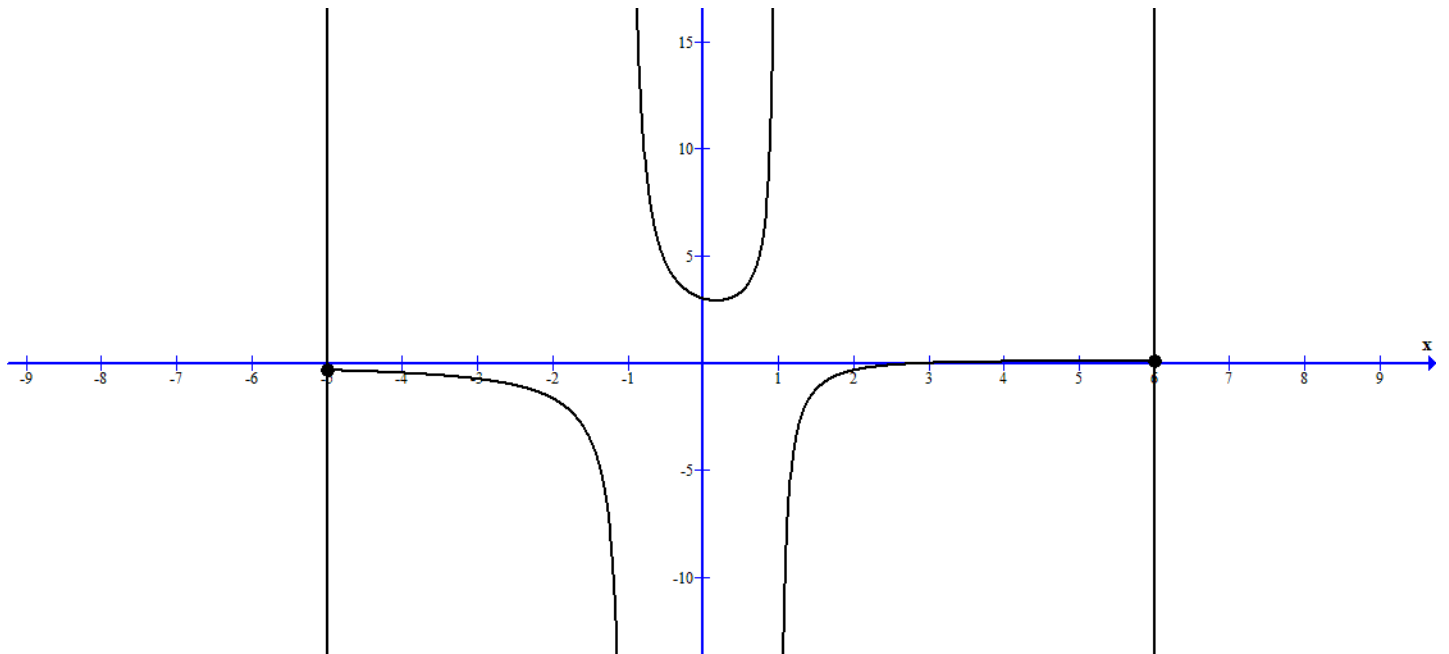
The above function is discontinuous inside the interval from -5 to 6, it does not have any global max or min inside that interval.
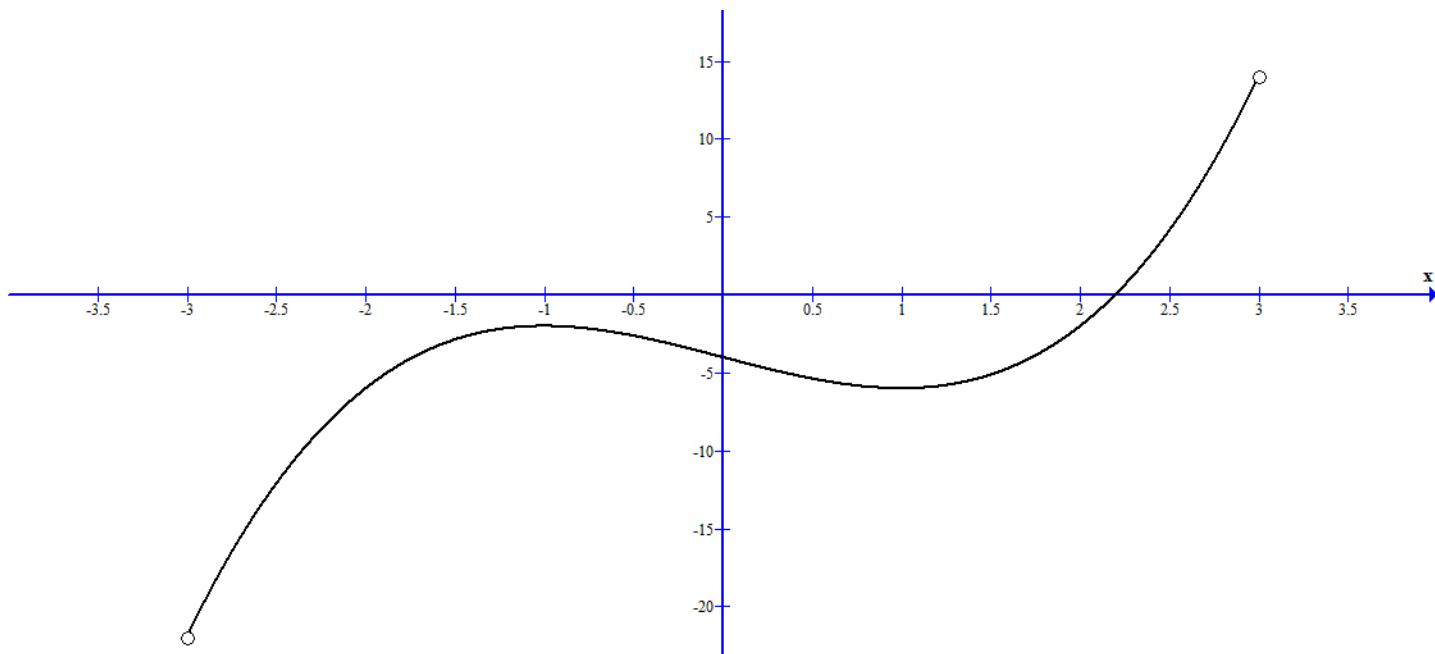
The above function is continuous inside the open interval $(-3, 3)$, but it does not have any global min or max inside this open interval.

To find the global maximum and global minimum of a continuous function f inside a closed interval $[a, b]$, we need to find all the local maximum and local minimum of the function, and also find the values of the function at the boundary points (that is, find $f(a)$ and $f(b)$. The largest of the values is the global maximum and the smallest of the values is the global minimum.

Example: Find the global maximum and global minimum of $f(x) = x^3 - 3x + 1$ on the interval $[0, 3]$.

We first find all the critical numbers of f, as these are the candidates for local maximum and local minimum:

$$f'(x) = 3x^2 - 3$$

Setting derivative to 0 we get:

$$3x^2 - 3 = 0 \Rightarrow x = \pm 1$$

The critical points of f are $x = 1$ and $x = -1$. However, since -1 is *not* inside the interval $[0, 3]$, we do not consider it. The only three points we need to consider are $(1, f(1))$, $(0, f(0))$, and $(3, f(3))$. $x = 1$ is a critical number of f, and $x = 0$ and $x = 3$ are the boundary points. Since $f(0) = 1$, $f(1) = -1$, and $f(3) = 19$, we see that $f(3) = 19$ is the largest value among the three, so $f(3) = 19$ is the global maximum value, and $f(1) = -1$ is the global minimum value.

Notice that if we changed the closed interval on which the function is defined, we may have different global maximum and global minimum. For example, for the above functions, if we changed the intervals to $[-3, 2]$ instead, then both critical

points, $x = -1$ and $x = 1$, should be considered. And the four points we need to consider are $(-3, f(-3))$, $(-1, f(-1))$, $(1, f(1))$, and $(2, f(2))$. $x = -1$ and $x = 1$ are critical points, while $x = -3$ and $x = 2$ are boundary points. In this case, since $f(-3) = -17$ is the smallest value among the four, -17 is the global minimum value. Notice that both $f(2)$ and $f(-1)$ are equal to 3 and is the largest value among the four, so 3 is the global maximum value, and this value is achieved by f on two different x values.
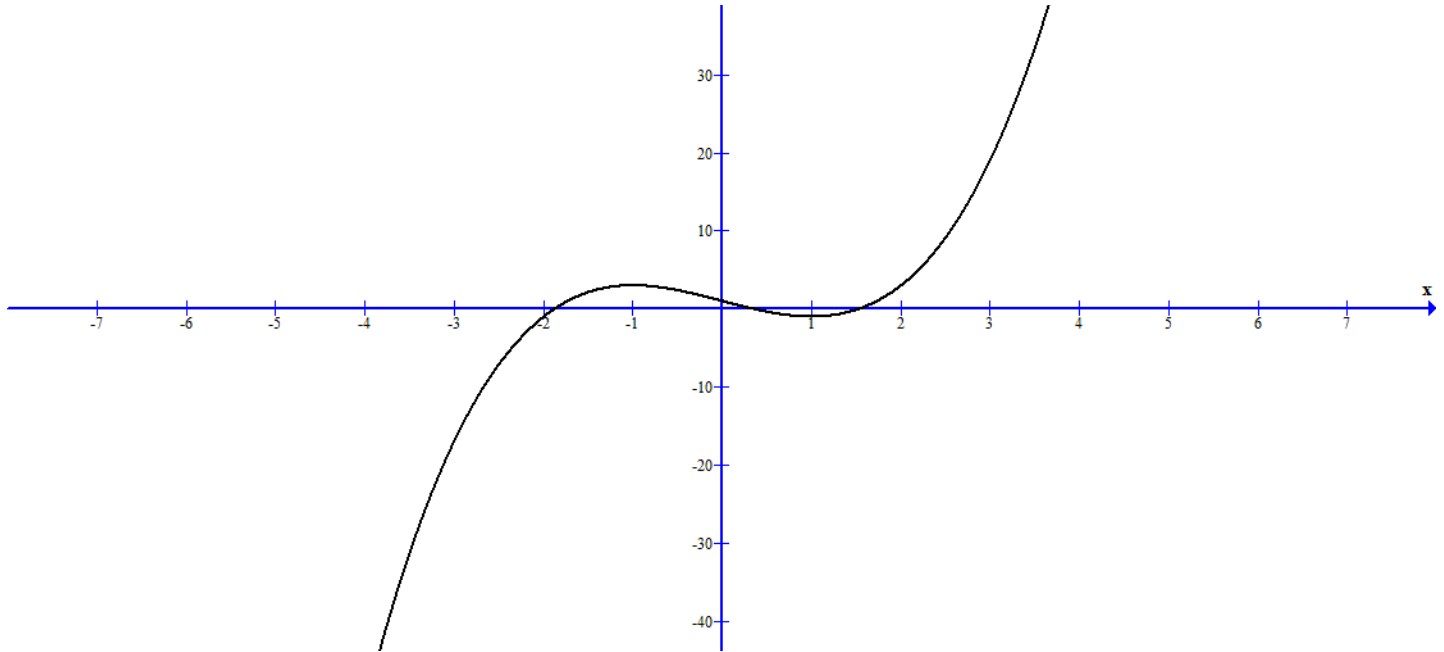
What does it help to be able to find the global maximum and minimum values of a function only within a confined domain (a closed interval)?

In mathematics, where most functions take on the whole real line as its domain, or at least an unbounded subset of the real line, to be able to find the maximum or minimum of a function over only a bounded interval is not too practical. However, in most applications, where the functions are used to model real world phenomenom, then the domain of the functions are more than often bounded by some physical constrain. For example, a function might be used to model the profit of a factory in terms of its production level. In which case the domain of the function cannot be negative numbers (it doesn't make sense to produce negative many products), and most likely due to limitations on market demand or labor, or raw material, the production level will not exceed too high a volume either. In another example, notice that if we have $V = \frac{4\pi r^3}{3}$, then as a function of r, the domain of V is all real numbers. However, if the formula is used to model the volume of a sphere that we want to build in a space ship, then the physical constrain limits that r must be positive and cannot be too large.

When a function is used to model a physical phenomenom, the physical constrain usually puts some artificial boundary on the function. In which case it does make sense to find the *global maximum* and *global minimum* of a function within a closed interval. This idea will be useful to us when we consider the applications

of finding maximum and minimum. For example, one might want to know the production level that would maximize profit, or the dimension of a box that would maximize volume...etc.

The Mean Value Theorem

Consider this scenerio: you are going from San Francisco to Sacremento. Lets say you started your trip at 12:00 Midnight, and you reached your destination at 2:00am, and the distance of your trip is 100 miles.
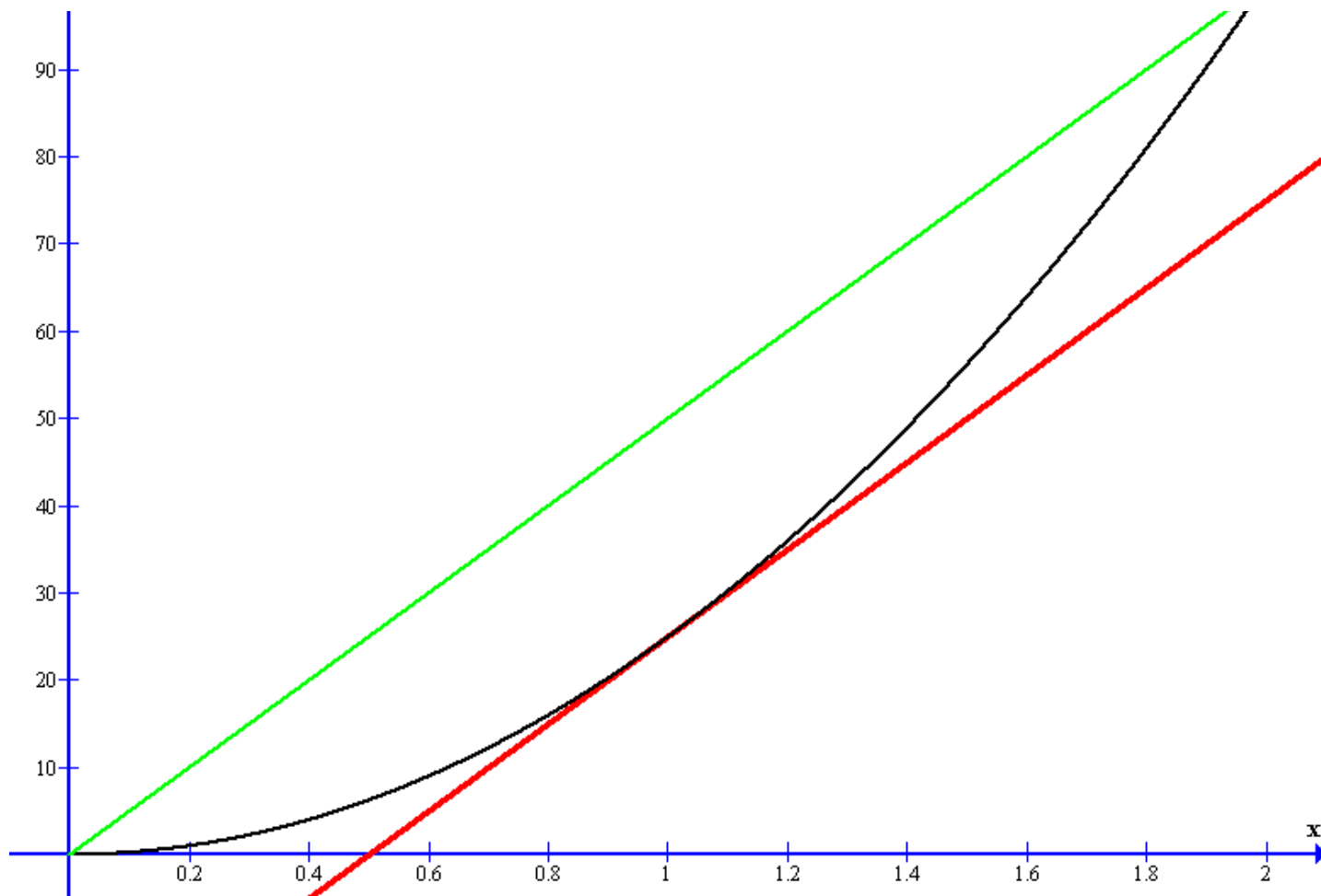
What is your *average velocity* over the trip? It took you 2 hours to travel 100 miles, so your average velocity is $\dfrac{100}{2} = 50$ miles per hour. Does this mean that you were always driving at 50 mph in all 2 hours of your trip? Probably not. You may be driving faster at some time, and may be driving slower at some other time. However, one point is clear: since your *average velocity* is 50 miles per hour over the trip, there must be at least **one instance** in the trip at which you are driving at **exactly** 50 miles per hour. After all, if you are always driving at below 50 mph, there's no way you could average a 50 mph velocity (since you will be too slow), so you must speed up to a faster velocity. If you were to speed up to higher than 50 mph, during your speeding up, there must be a point in which you will be driving at **exactly** 50 mph before you can speed up to, say 60 mph or some faster speed.

This analogy gives us the idea behind the mean value theorem, which says that: if the *average rate of change* of a differentiable funtion over a given interval is equal to a particular number, then there must be some point within the interval

in which the *instantaneous rate of change* of that function is exactly equal to that number.

The Mean Value Theorem Let f be a function such that f is continuous on the closed interval $[a,b]$, and that f is differentiable on the open interval (a,b), then there exists a number c inside the open interval (a,b) (i.e. $a < c < b$) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
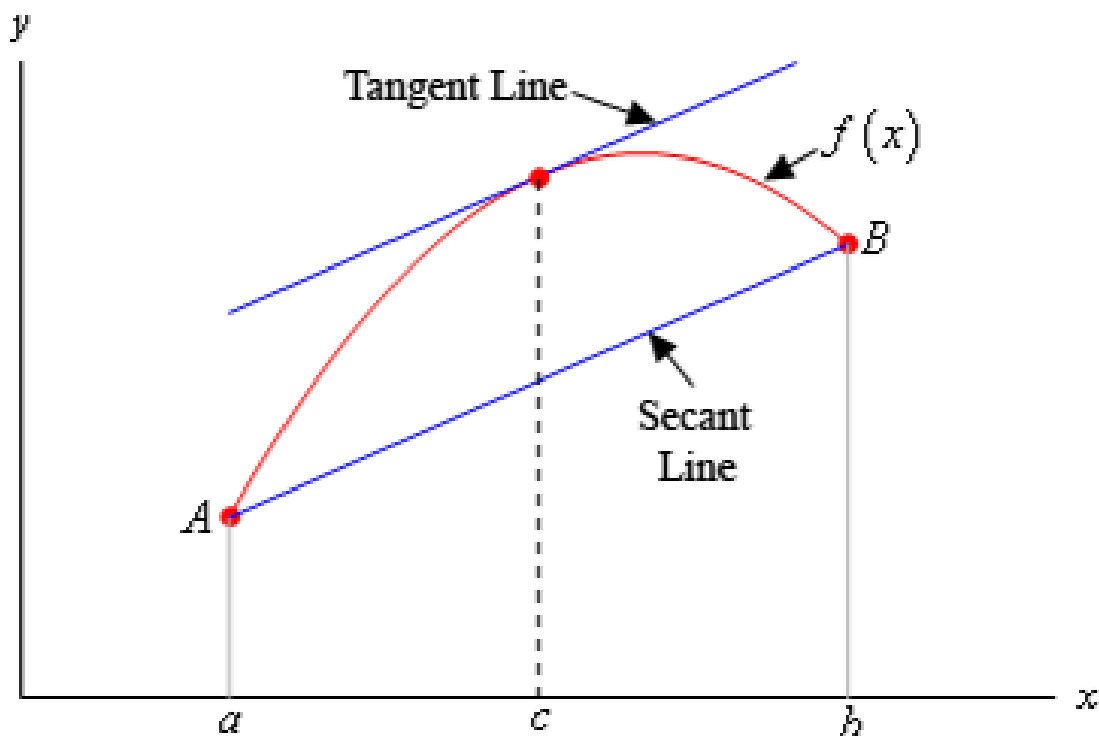
In the equation of the Mean Value Theorem, the fraction $\dfrac{f(b) - f(a)}{b - a}$ on the right hand side is the slope of the line containing the end points of the function over the interval. That is, it is the slope of the line containing the points $(a, f(a))$ and $(b, f(b))$. It represents the *average rate of change* of the function over the interval (a,b). The expression $f'(c)$ is the slope of a tangent line to f at some point c. It represents the *instantaneous rate of change* of f at point c. The Mean Value Theorem says that any function f that is differentiable over an interval must have a point c inside the interval such that the derivative of f at c, $f'(c)$, is equal to the slope of the secant line containing the points of f at the two end points of the interval.

The mean value theorem is what we call an *existance* theorem. That is, it tells you that something *exists* without telling you how to find them. In fact, while we know that if f satisfies such a condition then such a c must exists, we do not know how to find the value of c. Sometimes it may be possible or even easy to find the c that works, but sometimes it may be extremely difficult or even

impossible to find such a c. Imagine the situation we mentioned before, that you travelled 100 miles in 2 hours, so you know that your average velocity over the 2 hours is 50 miles per hour. You also know that you must be driving at exactly 50 mph at *some* point in your trip. However, if I ask you *exactly when* were you driving at 50 mph in your tirp? You probably cannot easily answer this question.

Example:

Let $f(x) = x^2 - 3x + 4$. Consider f in the interval $[2, 5]$. Notice that f is differentiable in $[2, 5]$, so the Mean Value Theorem says that there is a point c between 2 and 5 such that

$$f'(c) = \frac{f(5) - f(2)}{5 - 2} = 4$$

To solve for c, we take the derivative of f and set it equal to 4:

$$f'(x) = 2x - 3$$

Setting the derivative equal to 4 we get:

$$2x - 3 = 4 \Rightarrow x = \frac{7}{2} = 3.5$$

Notice that 3.5 is indeed between 2 and 5.
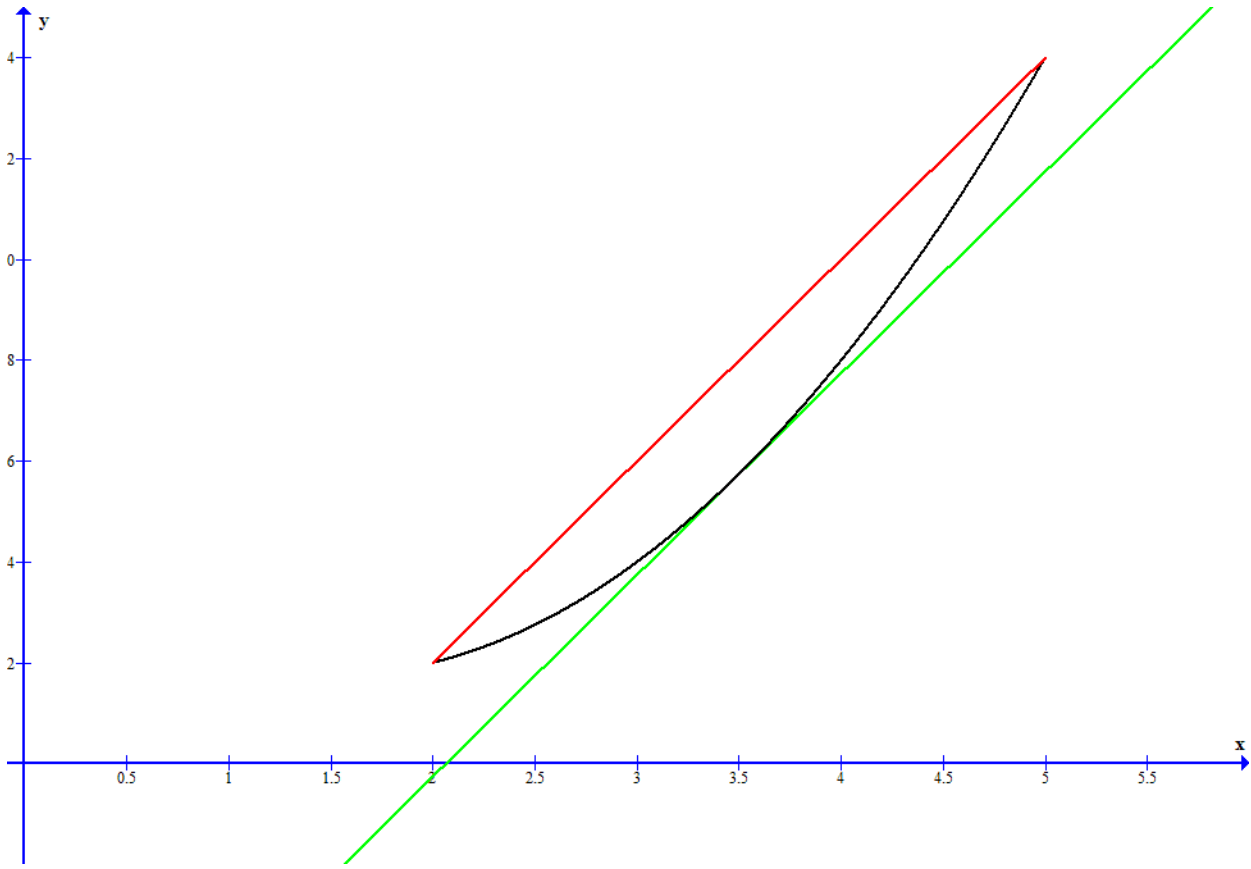
We know that the derivative of a constant function is equal to zero. Is the converse of this statement true? That is, if a function f has a derivative of zero, does this necessarily mean that f is a constant function?

Theorem:

If f is a differentiable function such that $f'(x) = 0$ for all x inside an interval (a, b), then f is a constant function in (a, b)

Proof: Let $r < t$ be any two numbers inside the open interval (a, b). The mean value theorem tells us that there exists a number, say s, where $r < s < t$, such that:

$$f'(s) = \frac{f(t) - f(r)}{t - r}$$

But $f'(s) = 0$ since s is inside (a, b), so

$$\frac{f(t) - f(r)}{t - r} = 0 \Rightarrow f(t) - f(r) = 0 \Rightarrow f(t) = f(r)$$

This shows that if $f'(x) = 0$ for all x inside (a, b), then $f(t) = f(r)$ for any r, t inside the interval (a, b). In other words, $f(x)$ is a constant function inside (a, b)

Graphing

In order to draw the graph of a function f accurately, we need to know much information about the function. The derivatives of a function gives us these information. We already saw that the local maxima and local minima of a function must occur at critical numbers. We also saw that on the intervals where f' is positive, f is increasing; and f is decreasing where f' is negative. What else can we find out about a function from its derivatives.

We saw that we could find the critical numbers of a function f, but we have yet to learn how to classify the critical numbers. How do we determine if a critical number is a local maximum, a local minimum, or neither?

Let f be a function with a critical number at c. Suppose that f' is positive before reaches c, and that f' is negative after passing c. This means that f is increasing before reaching c, and then f is decreasing after passing c.
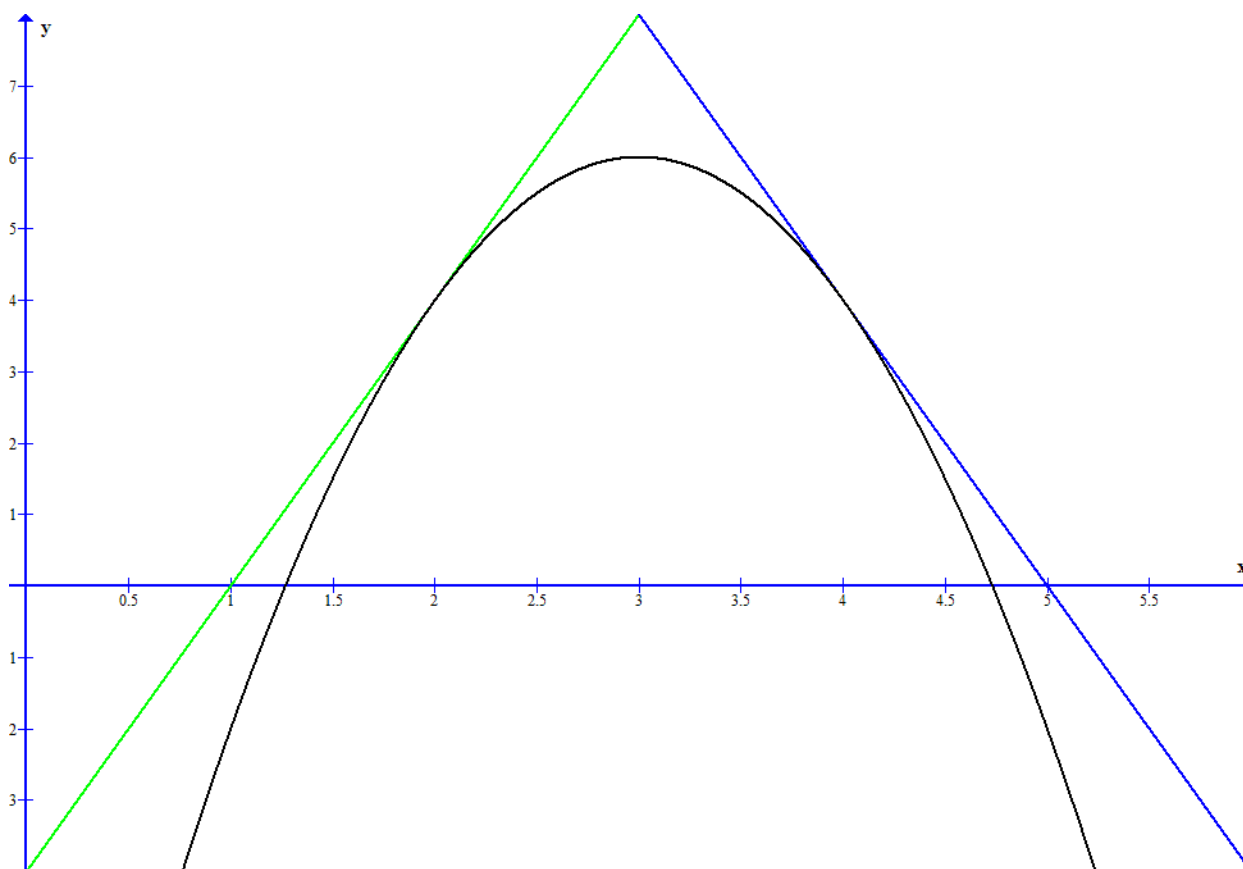
Observe from the graph that f must have a local maximum at that point. If you were climbing a mountain before reaching some point, then you start descending after reaching that point, then you must have reached the peak of the mountain. Therefore, if f' changes sign from $+$ to $-$ upon crossing the critical number c, then f must have a local maximum at c.
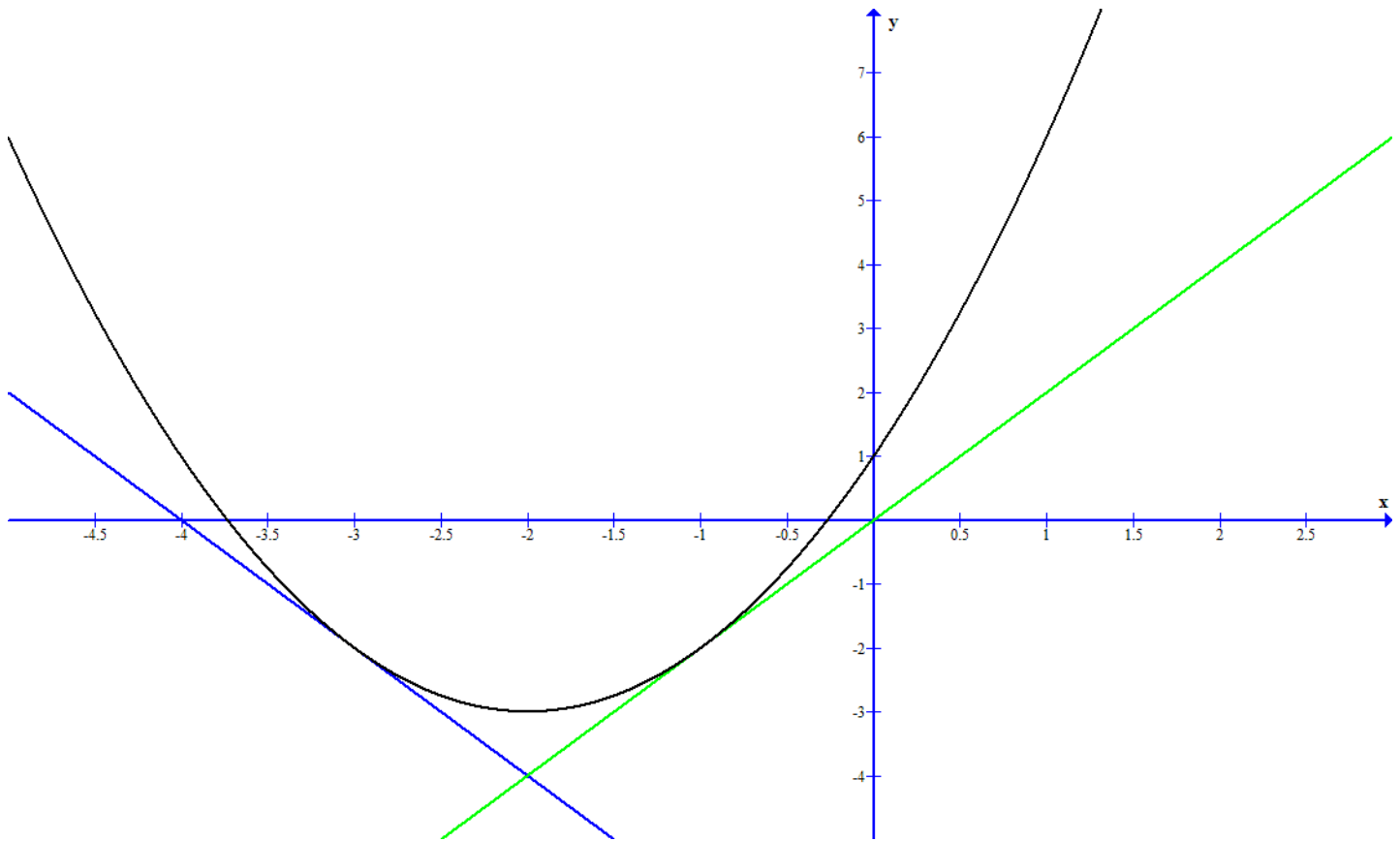
Using the same logic we see that if $'$ changes sign from $-$ to $+$ upon crossing the critical number c, then f must have local minimum at c.

What if f' does not change sign upon crossing the critical number c? Let's say f' is positive before c, and f' is also positive after c. If you are climbing a mountain before reaching some point on a mountain, and you are still climbing after reaching that point, then that point *cannot* be the peak of the mountain (otherwise you will have to start descend). So if f' does not change sign at the critical point, then the critical point cannot be a local maximum or minimum. The function simply has a horizontal or vertical tangent at that point.
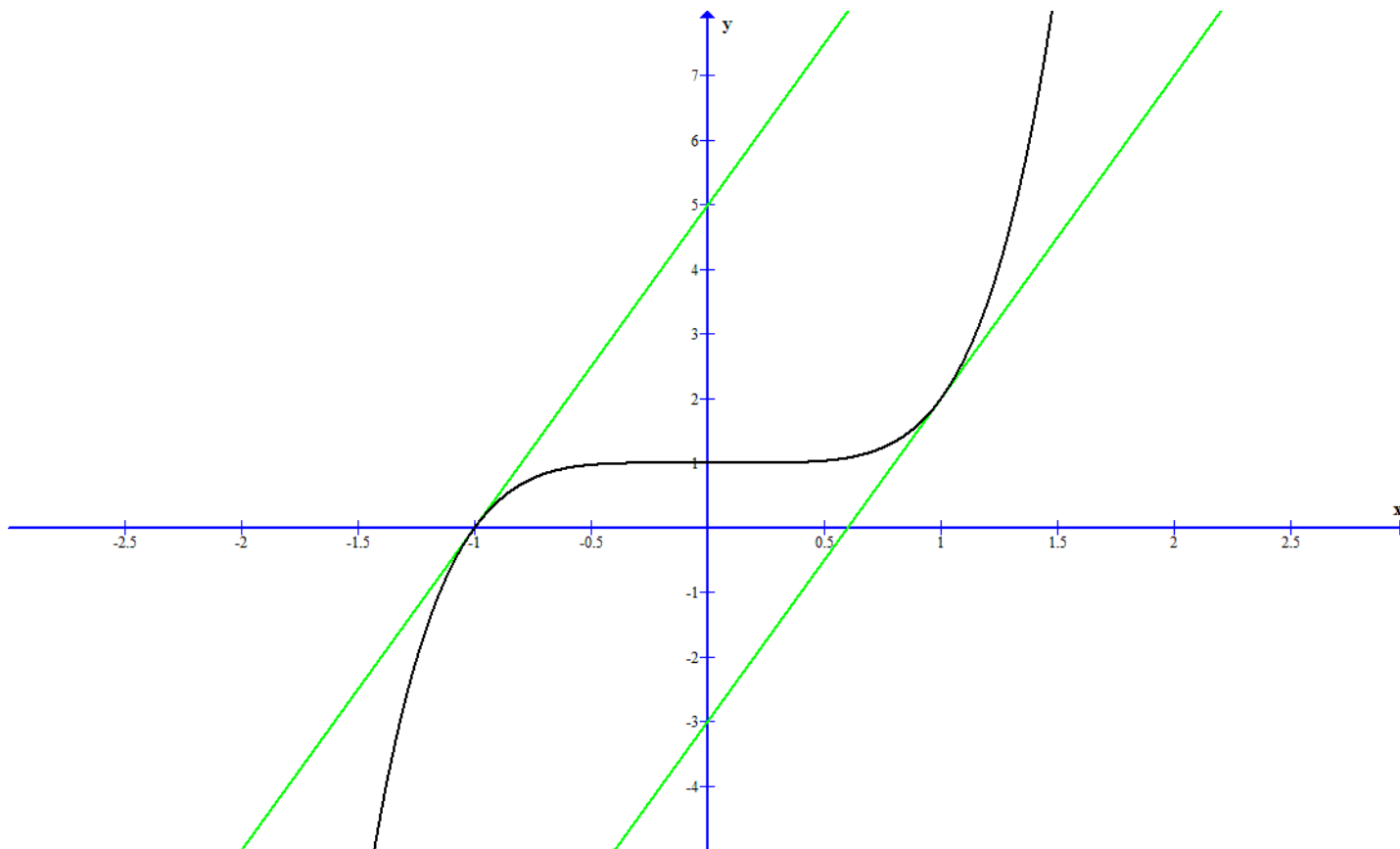
The above discussion gives us the **First Derivative Test**:

Let c be a critical number of a continuous function f,

if f' changes sign from positive to negative at c, then f has a local maximum at c.

if f' changes sign from negative to positive at c, then f has a local minimum at c.

if f' does not change sign at c, then $f(c)$ is neither a local maximum nor minimum.

Example:

Let $f(x) = x^3 + 4x^2 - 3x + 1$. Find all the critical numbers of f and classify them as local maximum, local minimum, or neither. Also determine the intervals where f is increasing and the intervals where f is decreasing.

To find the critical numbers of f, we must first take the derivative and set it equal to zero:

$$f'(x) = 3x^2 + 8x - 3$$

$$3x^2 + 8x - 3 = 0 \Rightarrow (3x - 1)(x + 3) = 0 \Rightarrow x = \frac{1}{3} \text{ or } x = -3$$

We have two critical numbers, $x = -3$ and $x = \frac{1}{3}$. What we need to do is we

want to consider the value of f' in the intervals separated by these two numbers. In particular, we need to consider the intervals $(-\infty, -3)$, $(-3, \frac{1}{3})$, and $(\frac{1}{3}, \infty)$. In order to find out how f' changes sign at the critical numbers, we pick any point inside the intervals to determine the sign of f' in that interval:

-4 is inside the interval $(-\infty, -3)$, $f'(-4) = 13 > 0$, so f' is *positive* before -3, which means f is increasing in the interval $(-\infty, -3)$.

0 is inside the interval $(-3, \frac{1}{3})$, $f'(0) = -3 < 0$, so f' is *negative* in the interval, which means f is decreasing in the interval $(-3, \frac{1}{3})$.

Using the first derivative test, we know that f has a local *maximum* at $x = -3$.

1 is inside the interval $(\frac{1}{3}, \infty)$, $f'(1) = 8 > 0$, so f' is it positive after $\frac{1}{3}$, which means f is increasing in the interval $(\frac{1}{3}, \infty)$.

Using the first derivative test, we know that f has a local *minimum* at $x = \frac{1}{3}$.

We can summarize the information in a table like this:

Intervals	$(-\infty, -3)$	-3	$(-3, \frac{1}{3})$	$\frac{1}{3}$	$(\frac{1}{3}, \infty)$
f	↗	local max	↘	local min	↗
f'	$+$	0	$-$	0	$+$
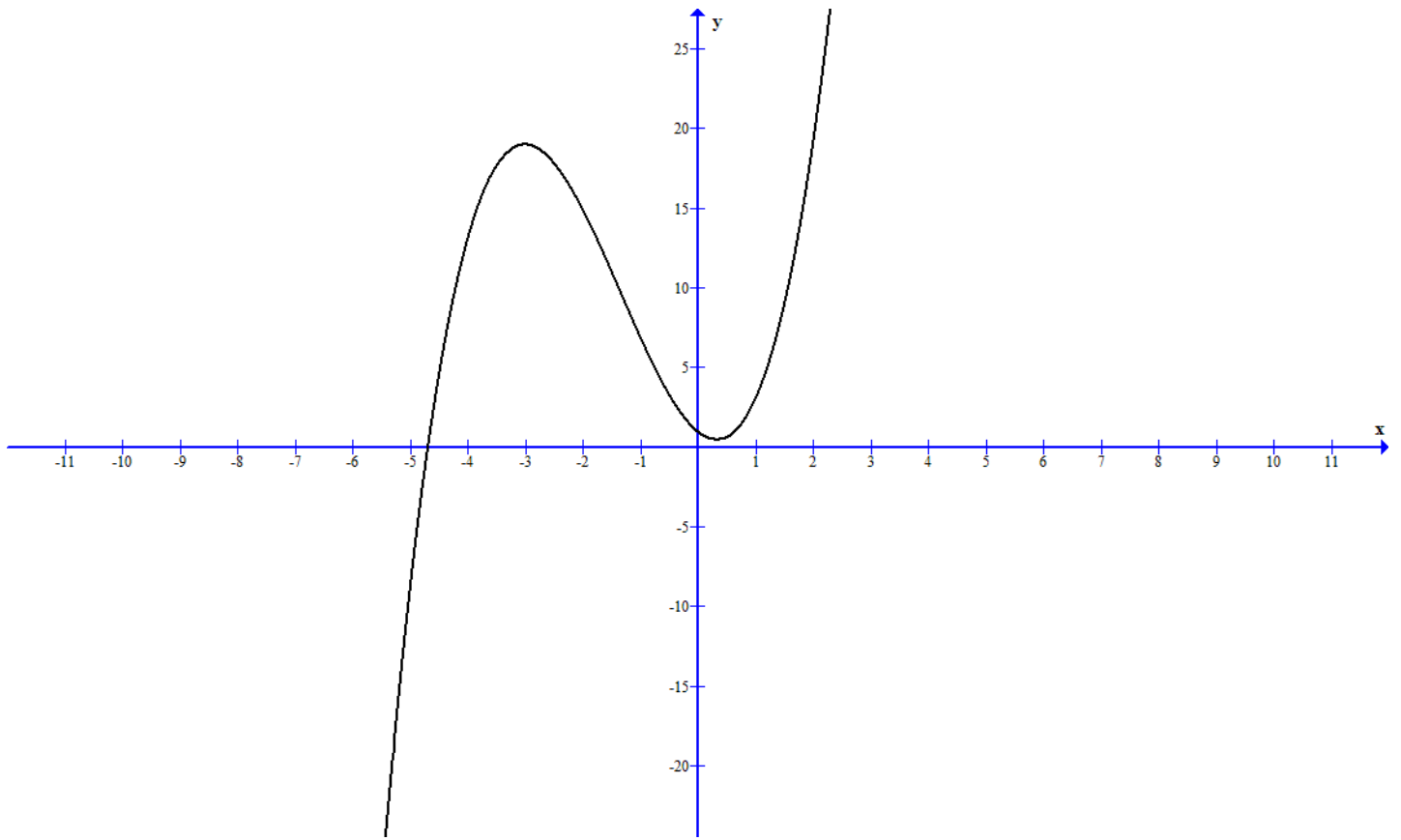
Example:

Find the critical numbers of $f(x) = x^3 - 2$ and determine if they are local minimum, local maximum, or neither. Also determine the intervals where f is increasing and the intervals where f is decreasing.

As usual we find the derivative of f and set it equal to 0 to determine the critical numbers.

$$f'(x) = 3x^2$$

Setting the derivative equal to zero we get:

$$3x^2 = 0 \Rightarrow x = 0$$

So f have a critical number at $x = 0$. This divides the real line into two intervals $(-\infty, 0)$ and $(0, \infty)$. $f'(-1) = 1 > 0$, so f is increasing on the interval $(-\infty, 0)$, and $f'(1) = 1 > 0$, so f is also increasing on the interval $(0, \infty)$. Using the first derivative test, we see that $x = 0$ is neither a local maximum or minimum.

Intervals	$(-\infty, 0)$	0	$(0, \infty)$
f	↗	horizontal tangent	↗
f'	$+$	0	$+$
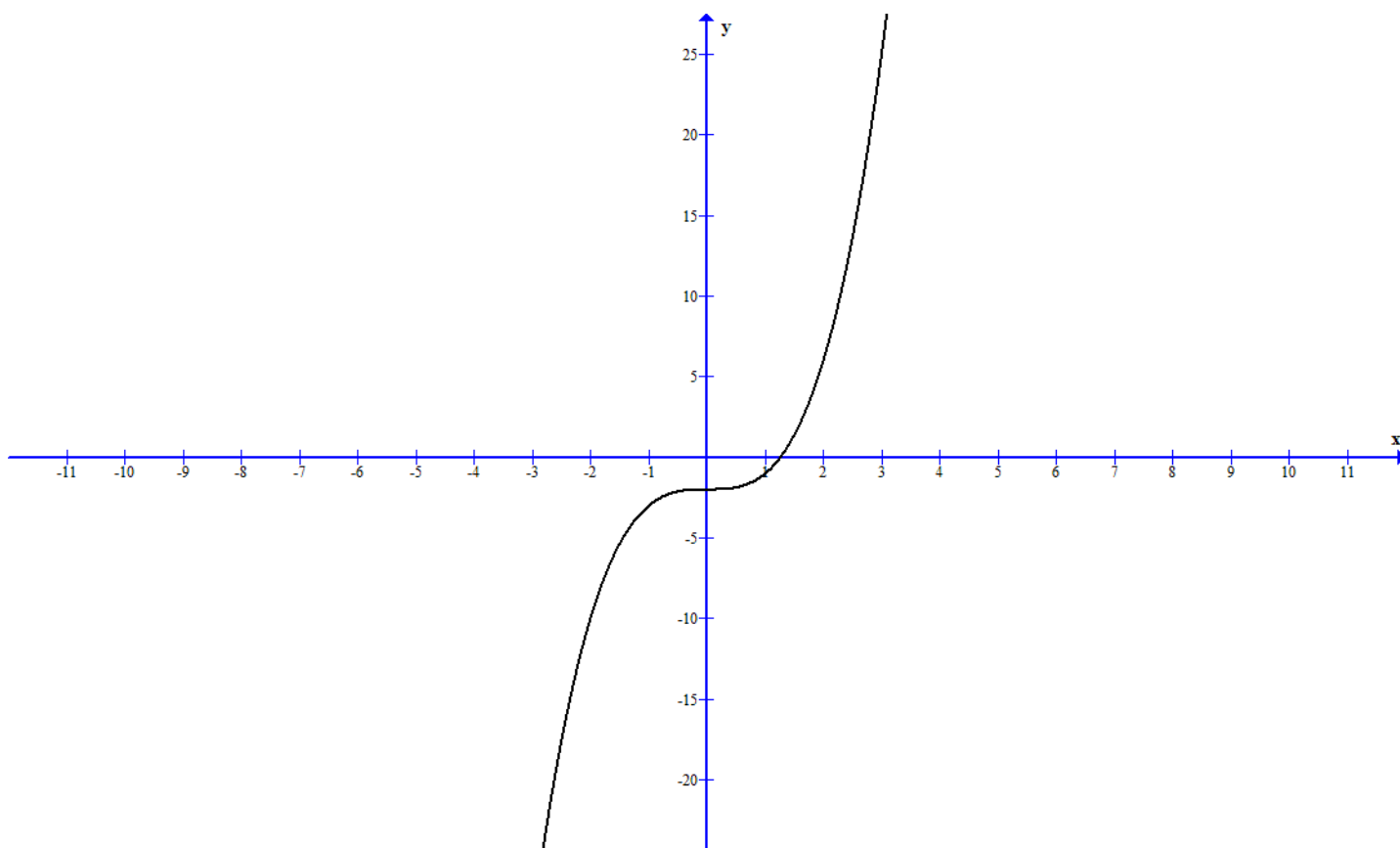
Example:

Find the critical numbers of $f(x) = x^{2/3}$ and determine if they are local minimum, local maximum, or neither. Also determine the intervals where f is increasing and the intervals where f is decreasing.

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

Notice that $f'(0)$ is undefined, but $f(0) = 0$ is defined, so $x = 0$ is a critical number of f. We consider the intervals $(-\infty, 0)$, $(0, \infty)$. Since f' is negative in $(-\infty, 0)$ and positive in $(0, \infty)$, we know that f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. The first derivative test tells us that f has a local minimum at 0.

Intervals	$(-\infty, 0)$	0	$(0, \infty)$
f	↘	local min	↗
f'	$-$	undefined (sharp corner)	$+$
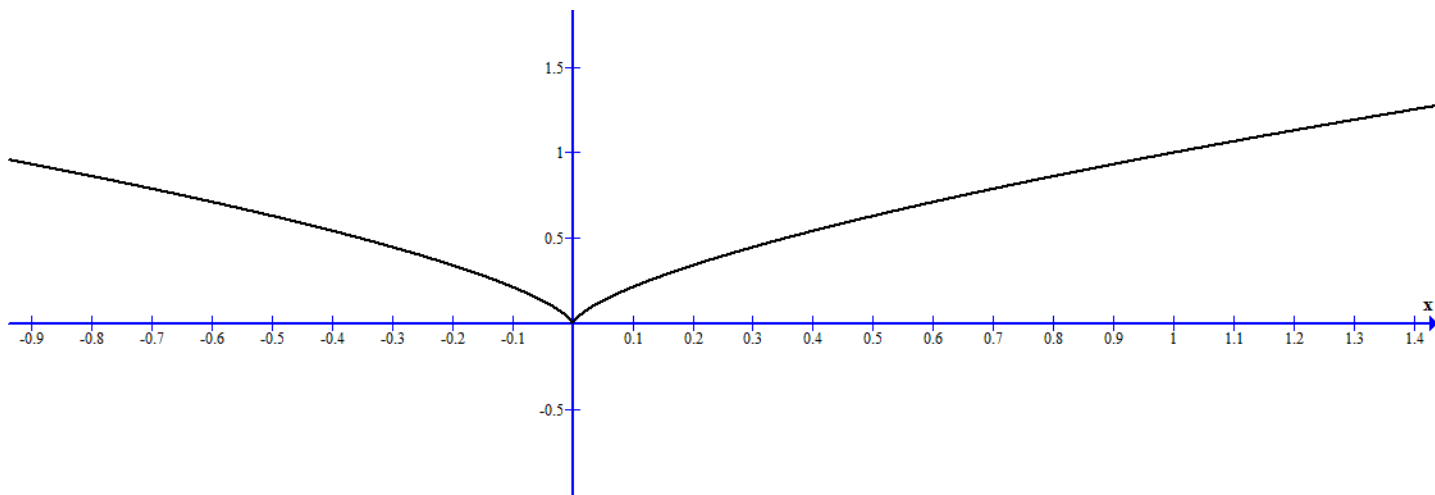
How about the second derivative? What can we tell about a function f from its second derivative?

Suppose you are driving in one direction on the highway. Let $s(t)$ represent the distance you are from your destination as a function of time t. Let's consider two different situations:

Situation 1:

Suppose you have travelled 30 miles for the first hour. So $s(1) = 30$. And let's say you are driving at 40 mph at $t = 1$. This means $s'(t) = 40$. Suppose that at the instance $t = 1$ you are *accelerating*. Remember that acceleration is the *change in velocity*. Since you are accelerating this means $s''(t) > 0$. Since you accelerate, you will increase in velocity. Let's say you increased your velocity all the way to 50 mph. Then in the next hour you will be able to cover more grounds than you did in the first hour. Let's say you travelled 45 miles in the second hour. So for the first two hours you travelled a total of 75 miles. Therefore, $s(2) = 75$.

Situation 2:

Suppose you have travelled 30 miles for the first hour. So $s(1) = 30$. And let's say you are driving at 40 mph at $t = 1$. This means $s'(t) = 40$. Suppose that at the instance $t = 1$ you are *decelerating*. Remember that acceleration is the *change in velocity*. Since you are decelerating this means $s''(t) < 0$. Since you decelerate, you will decrease in velocity. Let's say you decreased your velocity all the way down to 20 mph. Then in the next hour you will not be able to cover as much ground as you did in the first hour. Let's say you travelled 25 miles in the second hour. So for the first two hours you travelled a total of 55 miles. Therefore, $s(2) = 55$.

Notice that in both of the two situations, your velocity at time $t = 1$ is the same at 40 mph. However, whether you decelerate or accelerate (whether $s''(t)$ is positive or negative) makes a difference in the shape of the graph of s. In the case when you accelerate, the graph of s lies *above* the tangent line at $t = 1$. In

the case when you decelerate, the graph of s lies *below* the tangent line at $t = 1$. This tells us something about the graph of the function: whether the graph of the function lies above or below its tangent lines is determined by the *second derivative* of the function.

Definition:

If the graph of a function f lies *above* all of its tangent lines on an interval I, then f is said to be **concave up** on I. If the graph of f lies *below* all of its tangent lines, then f is said to be **concave down** on I.
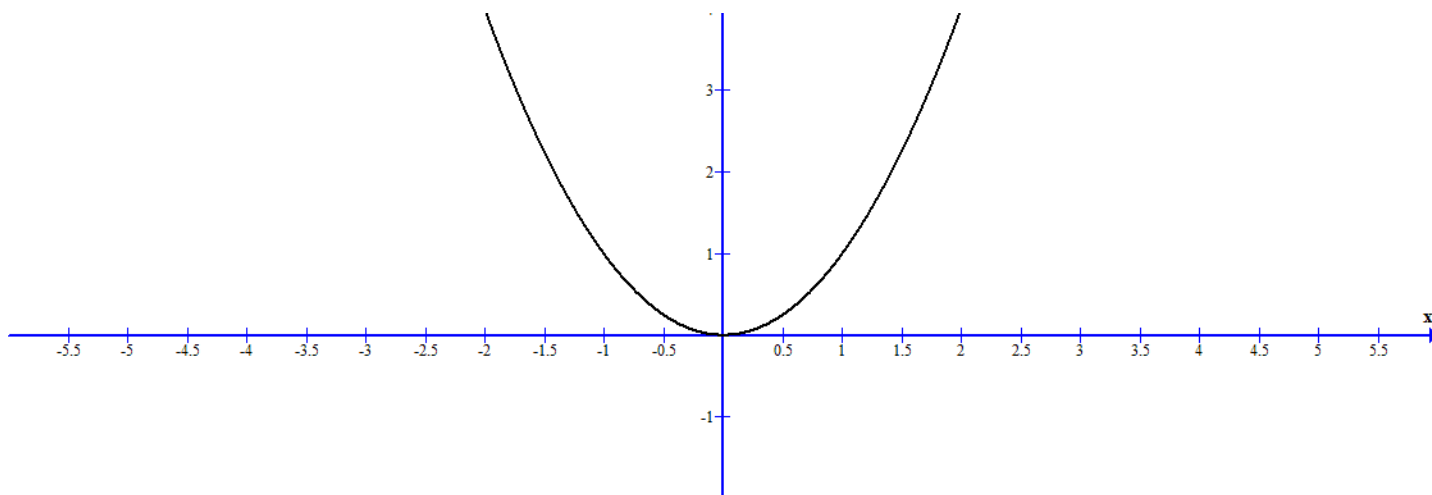
The above is a function that concaves ups
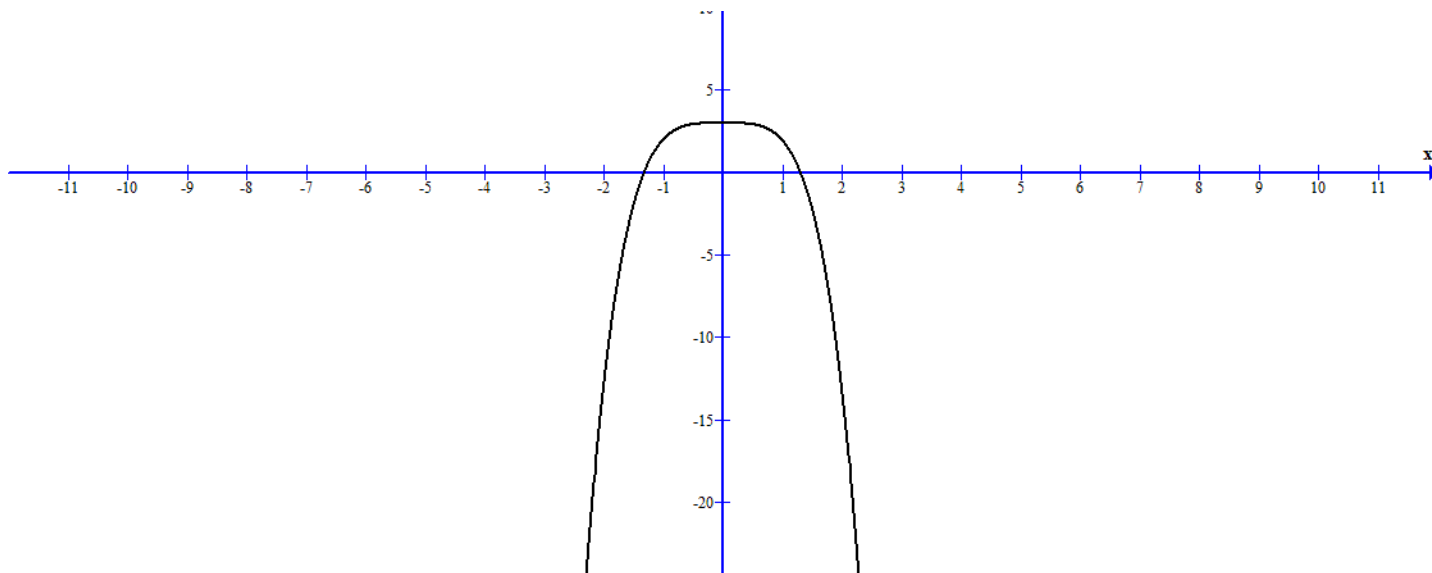
The above is a function that concaves down

Graphically, a function concave up will shape like a **U** shape, while a concave down function shape like a **N** shape. Notice that just like a function might be increasing at some interval and decreasing at another interval, the function might also be concave up at some interval while concave down at other intervals.

Definition:

A point P on a function f is called a **point of inflection** of f if the concavity of f changes at P. That is, if f goes from concave up to concave down or if f goes from concave down to concave up upon crossing P.
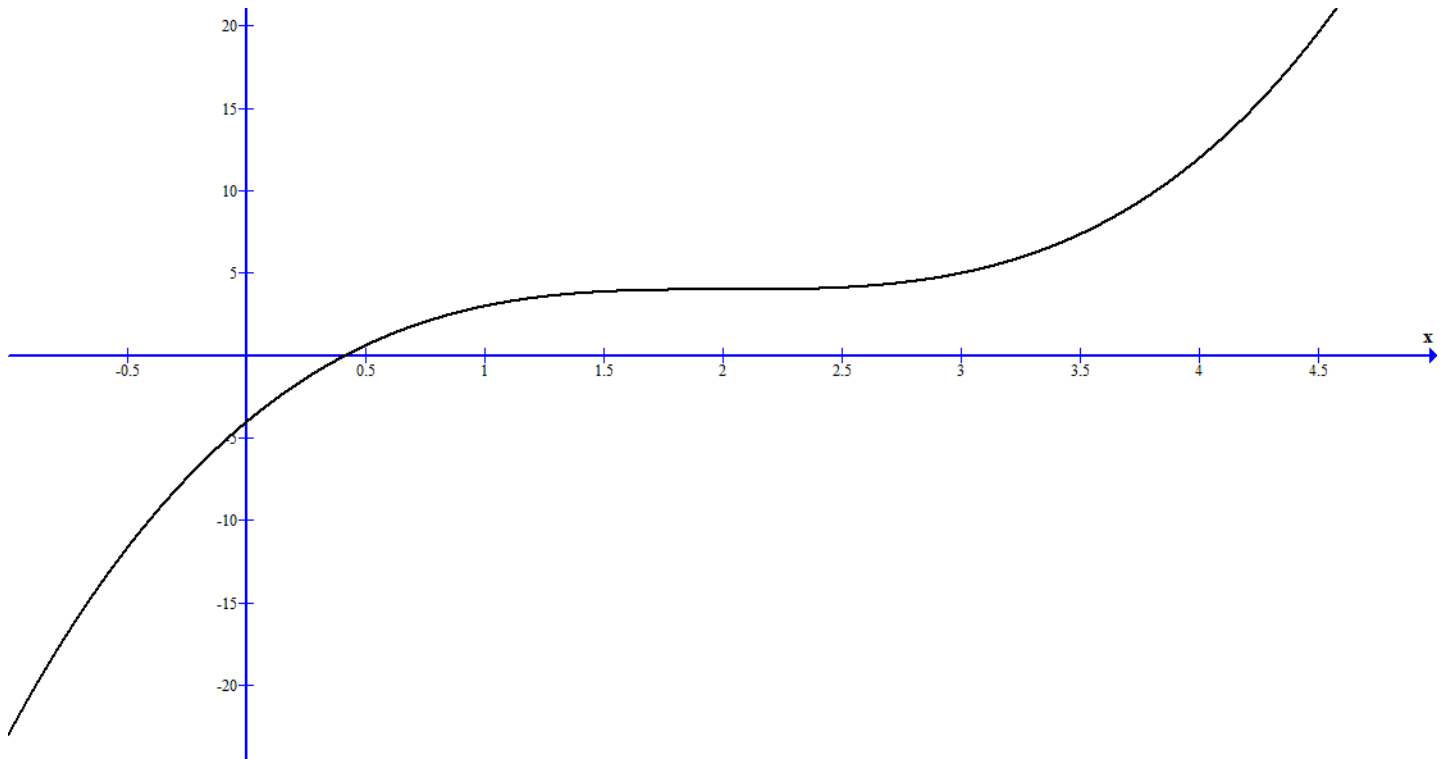
To find the inflection points of a function, we will need to know at which values of x does the second derivative changes sign. We find the *critical number of the first derivative.* That is, we find the points where the *second derivative of the function is equal to 0 or undefined.* These points give us the *candidate* for the inflection points of f. We will see examples shortly.

As we saw from our analysis, a function concaves up because *its second derivative is positive*, a function concaves down because *its second derivative is negative.* When a functions has a positive second derivative, that means the function's *first derivative is increasing.* If a function has a negative second derivative, that means the function's *first derivative is decreasing.* Using the driving as an example, if you are accelerating, then the graph of your distance verses time will be concave up. If you are decelerating, then the graph of your distance verses time will be concave down.

Concavity Test If $f''(x) > 0$ inside an interval I, then f is concave up on I.

If $f''(x) < 0$ inside an interval I, then f is concave down on I.

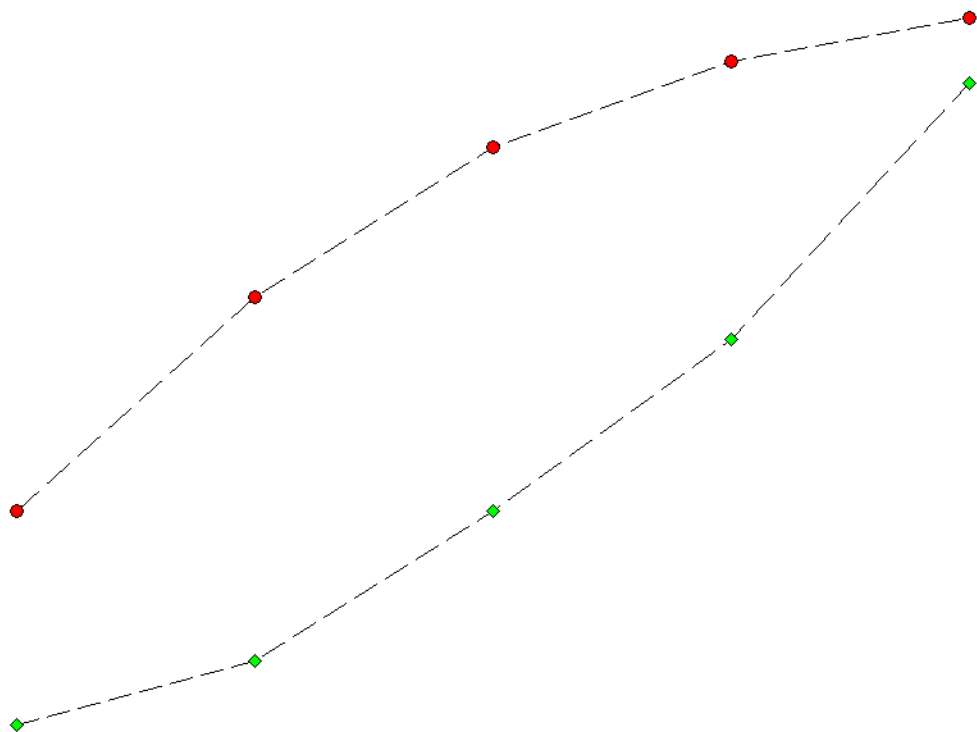E.g. In a research study, it is shown that in High School A, the average scores of the student body over the past 5 years are:

Year	score
1999	60
2000	70
2001	77
2002	81
2003	83

The average score of High School B over the same five years are given by:

Year	score
1999	50
2000	53
2001	60
2002	68
2003	80

Score Verses Years

If you are looking at the results of these test scores, which school is doing better? Both schools' scores are increasing over time (positive derivative), but school A has higher scores than school B every year. Does this mean that school A is doing better than school B? Notice that even though school A has higher scores over the years, the scores are increasing slower and slower, which means the scores has a *negative second derivative*. On the other hand, even though school B has lower scores over the years, the scores are increasing faster and faster, which means school B has a *positive second derivative*. If the trend continues, school B will out-score school A very soon.

What else does the second derivative tell us about the function f? Suppose f has a critical number at a point c, then we know that c is a candidate for a local maximum or local minimum of f. If $f''(c) < 0$. This means f is concave down at the critical number c. We have a **U**-shaped graph at c. From the shape of the graph we can see that f must have a local maximum at c.

If c is a critical number of f and $f''(c) > 0$, then this means that f is concave up at the critical number c. We have a **N**-shaped graph at c. From the shape of the graph we see that f must have a local mimimum at c.

The above discussion give us the **Second Derivative Test:**

If c is a critical number of f and $f''(c) > 0$, then f has a local *minimum* at c.

If c is a critical number of f and $f''(c) < 0$, then f has a local *maximum* at c.

If c is a critical number of f and $f''(c) = 0$ or $f''(c)$ is undefined, then the test fails.

Personally I prefer the first derivative test over the second derivative test to test the critical points. The reason being that the first derivative test is always conclusive. i.e the first derivative test can always tell us if a critical point is a local max, a local min, or neither. The second derivative test, on the other hand, fails when the second derivative is equal to 0 or undefined at the critical number.

Example:

Consider $f(x) = x^3$. $f'(x) = 3x^2$, so f has a critical number at $x = 0$. Since $f''(x) = 6x$, $f''(0) = 0$ so we cannot use the second derivative test to tell us anything about the critical number 0. Using the first derivative test, though, we see that f' is positive before 0 and positive after 0, so 0 is neither a maximum nor minimum.
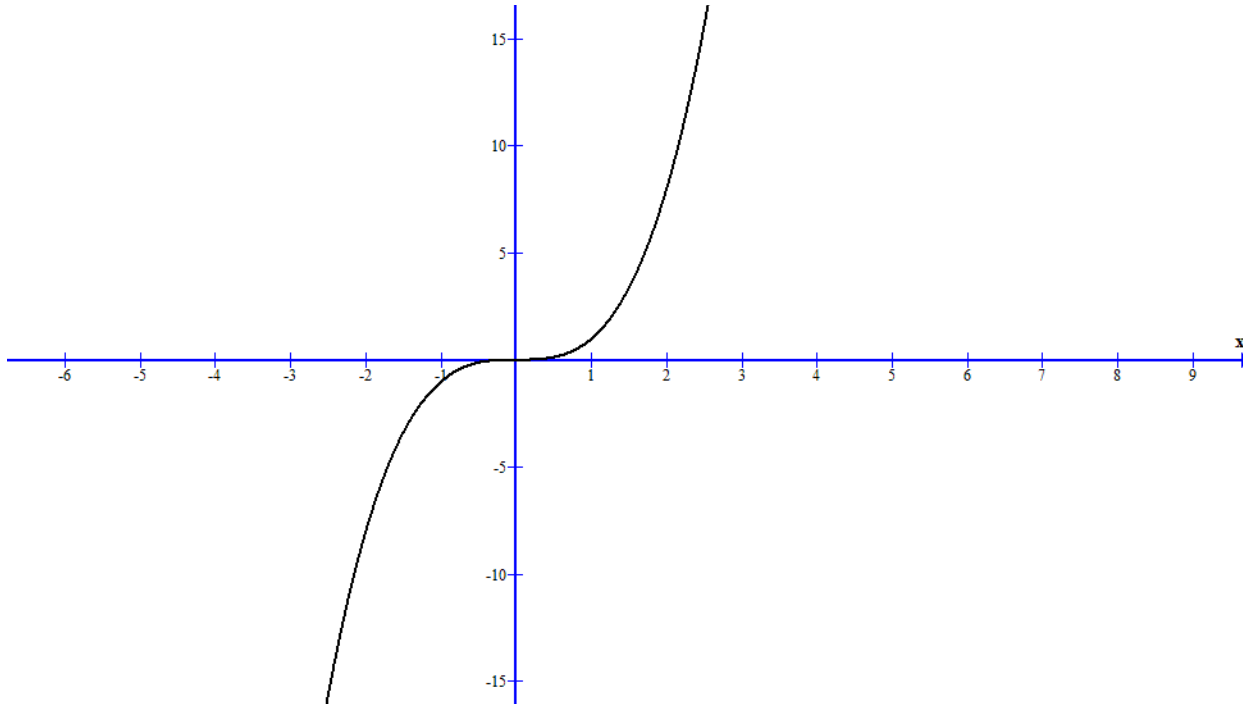
Now consider $f(x) = x^4$. $f'(x) = 4x^3$, so f has a critical number at $x = 0$. Since $f''(x) = 12x^2$, $f''(0) = 0$ so we cannot use the second derivative test to tell us anything about the critical number 0. Using the first derivative test, though, we see that f' is negative before 0 and positive after 0, so f has a local minimum at 0 (it turns out that this is also a global minimum).
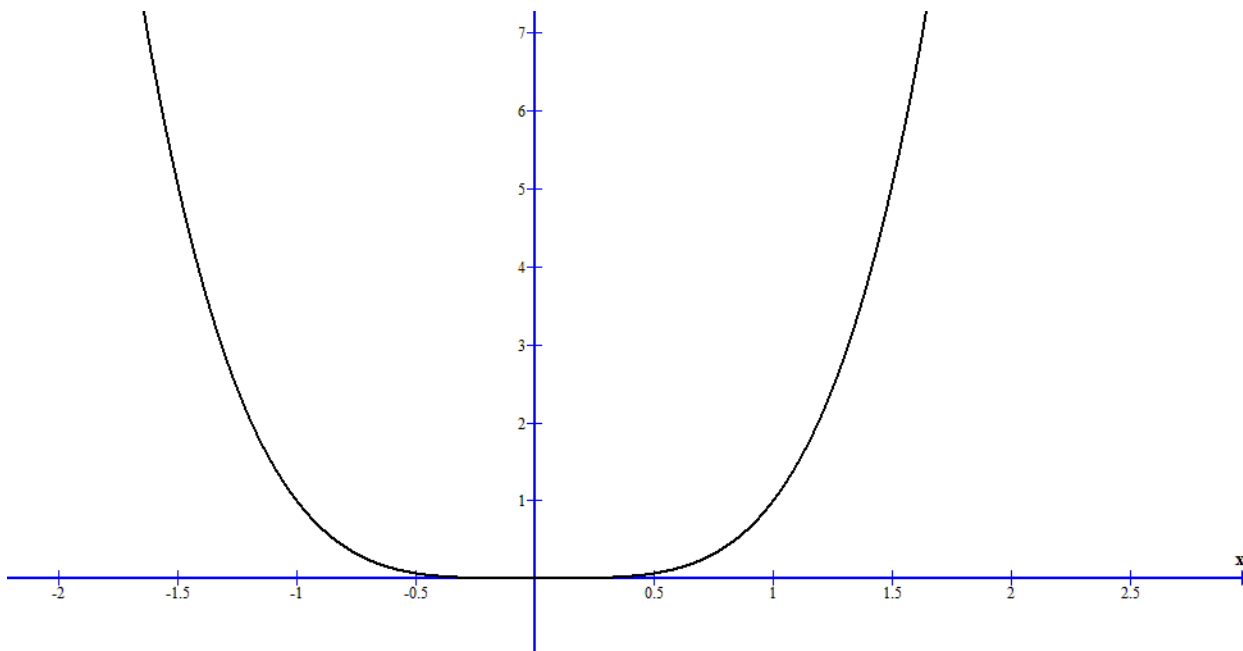
Example:

Let $f(x) = x^3$. Find the inflection points of f if there's any, and determine on which intervals is f concave up and on which intervals is f concave down.

To determine the points of inflection of f, we must find the critical numbers of the first derivative (the numbers where the second derivative is undefined or equal to 0) to find the *candidate* for inflection points:

Since $f'(x) = 3x^2$, so $f''(x) = 6x$, which means $x = 0$ is a critical number of the *first derivative* and $f''(0) = 0$. Notice that on the interval $(-\infty, 0)$ f'' is negative, while f'' is positive on the interval $(0, \infty)$. Therefore, f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. Since the concavity of f changes at 0, $(0, 0)$ is an *inflection point* of f.

We use a table similar to the ones we used to indicate the relationship:

Intervals	$(-\infty, 0)$	0	$(0, \infty)$
f	concave down	0	concave up
f''	$-$	inflection point	$+$

Example:

Let $f(x) = x^{5/3}$. Find the inflection points of f if there's any, and determine on which intervals is f concave up and on which intervals is f concave down.

We find the critical numbers of the first derivative of f.

$f'(x) = \frac{5}{3}x^{2/3}$, so $f''(x) = \frac{10}{9}x^{-1/3} = \frac{10}{9\sqrt[3]{x}}$. $x = 0$ is a critical number of the first derivative because $f''(0)$ is undefined. Since f'' is negative in the interval $(-\infty, 0)$ and f'' is positive in the interval $(0, \infty)$, we know that f concaves down on $(-\infty, 0)$ and f concaves up on $(0, \infty)$. Since f changes concavity at 0, $(0, 0)$ is an inflection point.

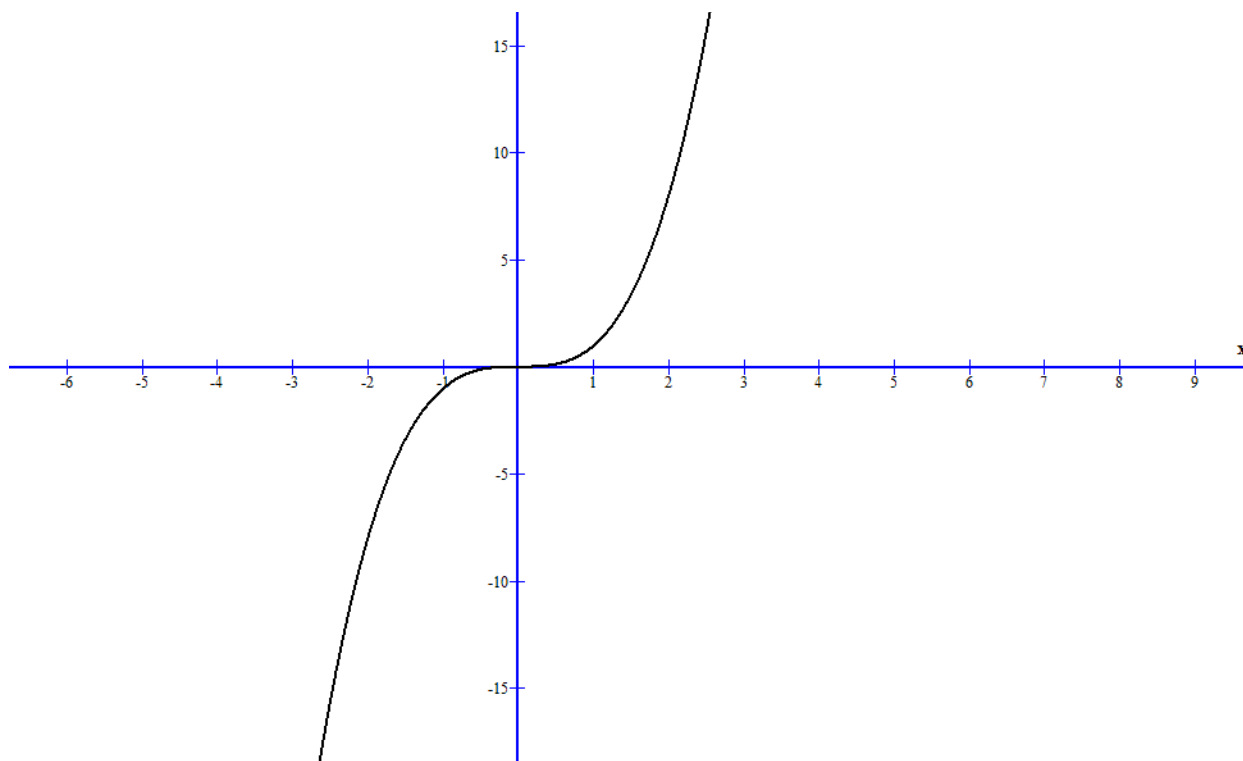| Intervals	$(-\infty, 0)$	0	$(0, \infty)$
f	concave down	0	concave up
f''	$-$	inflection point	$+$
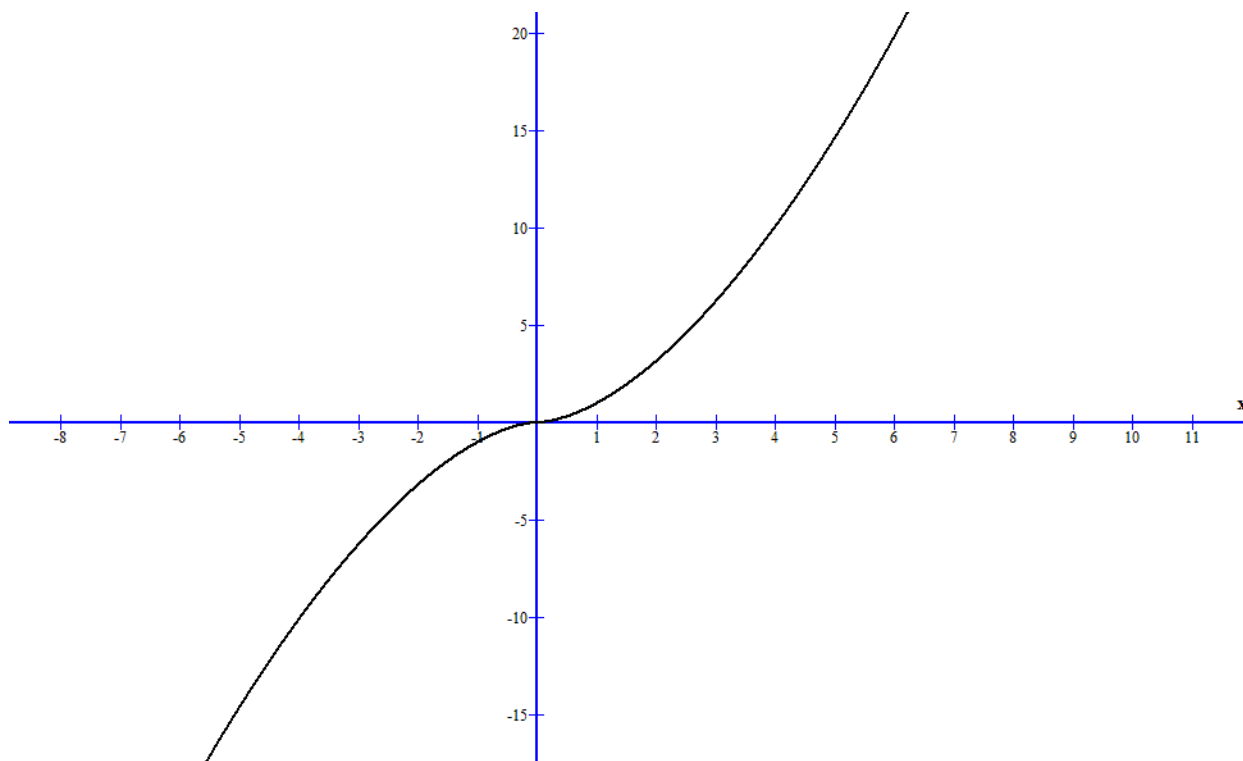
E.g.

Let $f(x) = x^4$. Find the inflection points of f if there's any, and determine on which intervals is f concave up and on which intervals is f concave down.

$f'(x) = 4x^2$ and $f''(x) = 8x$, so $x = 0$ is a critical number of the first derivative since $f''(0) = 0$. Notice that f'' is positive on the whole real line, which means f is always concave up. So even though $x = 0$ is a critical number of the first derivative, $x = 0$ is *not* an inflection point of f.

Intervals	$(-\infty, 0)$	0	$(0, \infty)$
f	concave up	0	concave up
f''	$+$	0	$+$
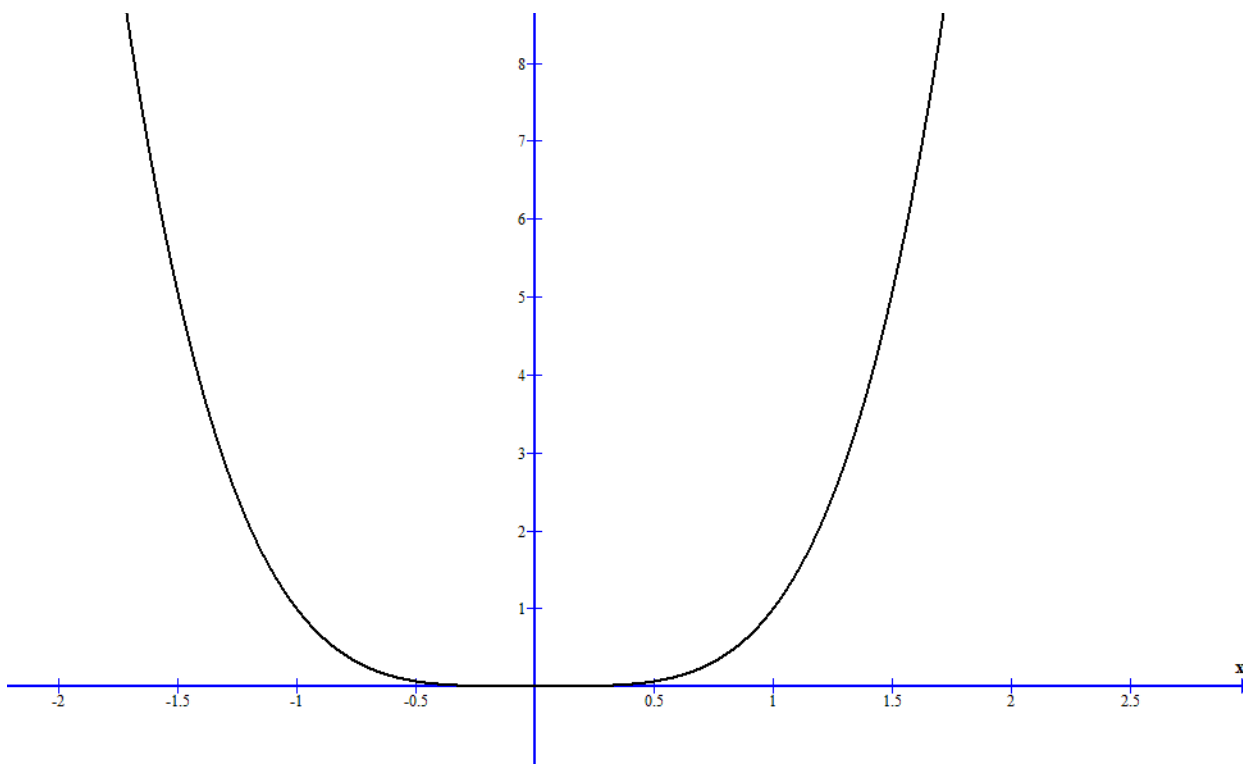
Example: $f(x) = \frac{x^2-1}{x^2+1}$

Find the critical numbers of f and determine if they are local maximum or local minimum, or neither. Determine the intervals where f is increasing and the intervals where f is decreasing. Find the inflection points of f if there's any. Determine the intervals where f concaves up and the intervals where f concaves down. Sketch the graph of f using these information.

To find the critical numbers of f we must take the first derivative:

$$f'(x) = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

We see that f' is zero only if $x = 0$, so $x = 0$ is a critical number.

f' is negative on the interval $(-\infty, 0)$ and positive on the interval $(0, \infty)$, so f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. Using the first derivative test we know that f has a local minimum at 0.

To find the intervals where f concaves up or down and to find the inflection points, we must take the second derivative:

$$f''(x) = \frac{(x^2+1)^2(4) - 4x(2(x^2+1)(2x))}{(x^2+1)^4}$$

$$= \frac{(x^2+1)(4(x^2+1) - 16x^2)}{(x^2+1)^4}$$

$$= \frac{4(x^2+1) - 16x^2}{(x^2+1)^3}$$

$$= \frac{4x^2 + 4 - 16x^2}{(x^2+1)^3}$$

$$= \frac{-12x^2 + 4}{(x^2+1)^3}$$

We set the second derivative equal to zero to find the candidate for inflection points:

$$\frac{-12x^2 + 4}{(x^2+1)^3} = 0 \Leftrightarrow -12x^2 + 4 = 0 \Leftrightarrow x = \pm\frac{1}{\sqrt{3}}$$

So $x = -\frac{1}{\sqrt{3}}$ and $x = \frac{1}{\sqrt{3}}$ are the critical numbers of the first derivative, which means they are candidate for inflection points.

$f'' < 0$ in the interval $(-\infty, -\frac{1}{\sqrt{3}})$, so f concaves down on this interval.

$f'' > 0$ in the interval $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, so f concaves up on this interval.
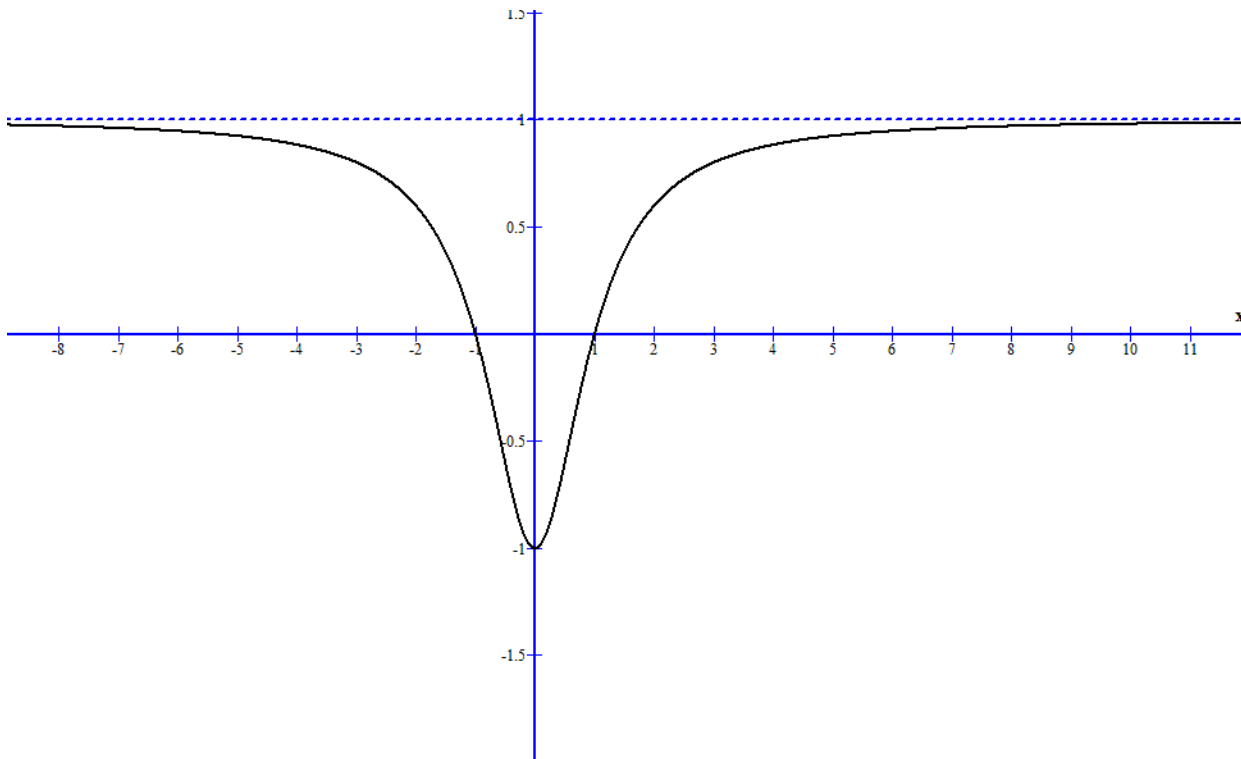
This also tells us that f has an inflection point at $-\frac{1}{\sqrt{3}}$ since it changes concavity at this point

$f'' < 0$ in the interval $(\frac{1}{\sqrt{3}}, \infty)$, so f concaves down on this interval.

This tells us that f has an inflection point at $\frac{1}{\sqrt{3}}$ since it changes concavity at this point

We can summarize all the information in a table like this:

Intervals	$(-\infty, -\frac{1}{\sqrt{3}})$	$-\frac{1}{\sqrt{3}}$	$(-\frac{1}{\sqrt{3}}, 0)$	0	$(0, \frac{1}{\sqrt{3}})$	$\frac{1}{\sqrt{3}}$	$(\frac{1}{\sqrt{3}}, \infty)$
f	$\searrow$ & C.D.	I.P.	$\searrow$ & C.U.	L Min	$\nearrow$ & C.U	I.P.	$\nearrow$ & C.D
f'	$-$	$-$	$-$	0	$+$	$+$	$+$
f''	$-$	0	$+$	$+$	$+$	0	$-$

Let us summarize the facts we learned about first and second derivatives so far:

If f' is positive on an interval I, f is increasing on I. If f' is negative on an interval I, f is decreasing on I.

If f'' is positive on an interval I, f is concave up on I. If f' is negative on an interval I, f is concave down on I.

To find the local maxima and local minima of f, find the critical numbers of f. That is, find the values c where $f'(c)$ is zero or undefined. Use the first or second derivative test to test if the critical numbers are local maximum, local minimum, or neither.

To find the inflection points of f, find the critical numbers of f'. That is, find the values c where $f''(c)$ is zero or undefined. If f'' changes sign at c, then f has an inflection point at c.

Limits at Infinity

So far we have studied the limit of a function f only at a particular number c. What if we want to study the *long term behavior* of a function f? That if, what can we tell about a function $f(x)$ when the values of x become tremendously large $(x \to \infty)$ or tremendously negative $(x \to -\infty)$?

Let's look at the behavior of the function

$$f(x) = \frac{x-1}{x+1}$$

when x becomes very large:

x	$f(x)$
1000	0.998
10000	0.9998
100000	0.99998
1000000	0.999998
10000000	0.9999998
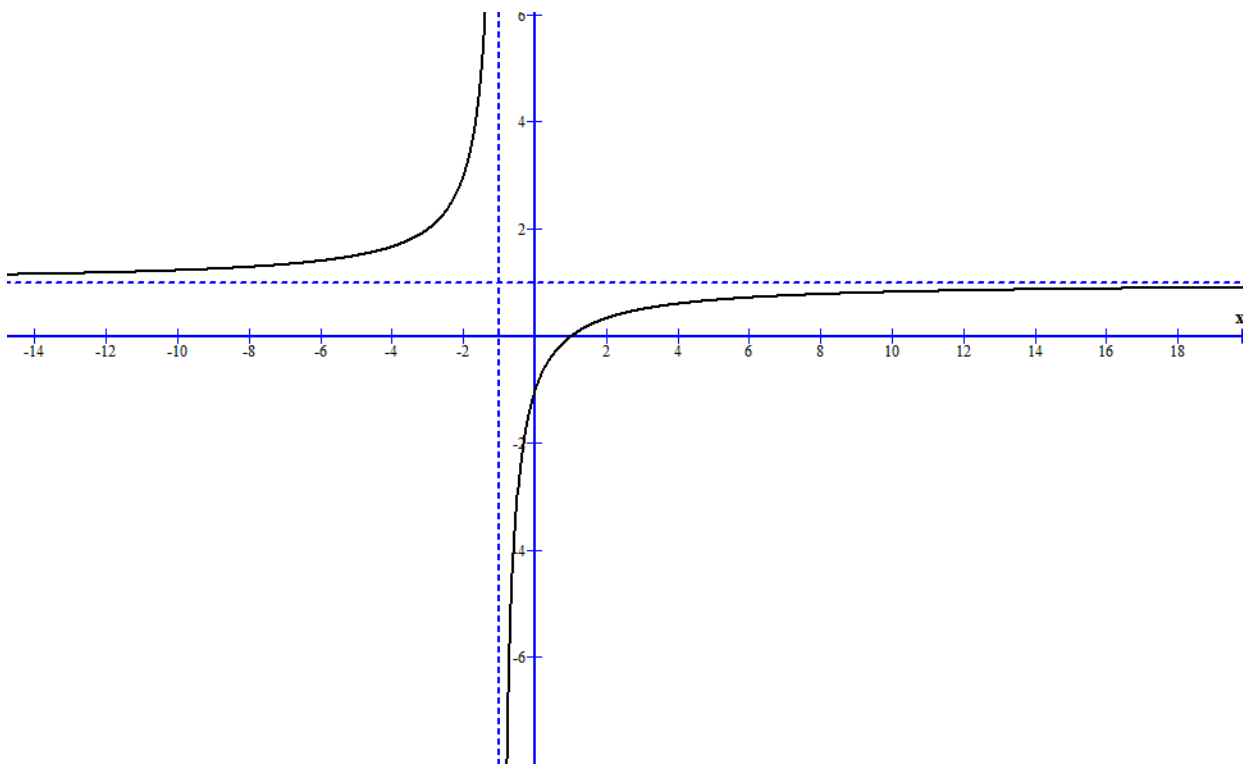
As we can see, the value of f approaches 1 as the value of x gets larger and larger. We use the notation:

$$\lim_{x \to \infty} \frac{x-1}{x+1} = 1$$

to denote this fact.

The meaning of the limit says that, the value of f approaches 1 as x approaches

infinity. As we already said, infinity is *not* a number. So when we say x approaches infinity, we are *not* saying that x approaches any particular number. When we say x approaches infinity, we are saying that the value of x increases without bound.

What happens to the function f if x approaches negative infinity? That is, what happens when the values of x decreases without bound? We make a similar table like the one above:

x	$f(x)$
-1000	1.002
-10000	1.0002
-100000	1.00002
-1000000	1.000002
-10000000	1.0000002

So f approaches 1 as x decreases without bound. We use the notation:

$$\lim_{x \to -\infty} \frac{x-1}{x+1} = 1$$

to denote this fact.

Once again, when we say x approaches negative infinity, we meant x decreases without bound. We are not suggesting that x approaches any particular number.

Definition:

We use the notation

$$\lim_{x \to \infty} f(x) = L$$

to mean that, when the value of x grows without bound, the value of $f(x)$ approaches the number L.

We use similar notation for negative infinity:

$$\lim_{x \to -\infty} f(x) = L$$

to mean that, when the value of x decreases without bound, the value of $f(x)$ approaches the number L.

If the value of f approaches a particular number L when x approaches infinity (or negative infinity), the graph of f looks like a horizontal line.

Definition:

The horizontal line $y = L$ is called a **horizontal asymptote** of f if

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$

So the function we just considered, $f(x) = \frac{x-1}{x+1}$, has a horizontal asymptote $y = 1$.

If a function has a horizontal asymptote $y = L$ at infinity, that means f remains pretty much at a constant value of L.

E.g. Suppose you are a biologist, and you predict that the population P of a particular species will increase or decrease as a function of time according to the following formula:

$$P(t) = \frac{100t^2 - t + 1}{t^2 + 1}$$

P has a horizontal asymptote of $y = 100$. This means that, in the long run, the population of the species will remain at about 100.

How do we find the limit of a function at infinity? Let us see some simple cases:

$$\lim_{x \to \infty} \frac{1}{x}$$

As x gets tremendously large, we are dividing 1 by a tremendously large number, the result is a number very close to zero.

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

In general, whenever we have a constant divided by an expression that keeps on growing to infinity, then the result is always zero. We state a slightly simpler version here:

If $r > 0$ and c is a constant then

$$\lim_{x \to \infty} \frac{c}{x^r} = 0$$

This makes sense because, since $r >$, as x approaches infinity, x^r approaches infinity, and when you divide a constant by something that is tremendously large, the result is a number that is very close to 0.

If $r > 0$, and r is a rational number, and c is a constant then

$$\lim_{x \to \infty} \frac{c}{x^r} = 0$$

if x^r is defined for negative numbers x

E.g

Evaluate

$$\lim_{x \to \infty} \frac{x^3 - 4x^2 + x - 1}{-2x^3 + x + 5}$$

In this case here, both the numerator and the denominator goes to infinity as x grows without bound. In order to evaluate this limit formally, we need to express the fraction in terms that we can evaluate the value when x approaches infinity.

The trick here is to divide the numerator and denominator by the highest power of x. In this example here, we divide the numerator and denominator by x^3:

$$\lim_{x \to \infty} \frac{x^3 - 4x^2 + x - 1}{-2x^3 + x + 5}$$

$$= \lim_{x \to \infty} \frac{\frac{x^3 - 4x^2 + x - 1}{x^3}}{\frac{-2x^3 + x + 5}{x^3}}$$

$$= \lim_{x \to \infty} \frac{1 - \frac{4}{x} + \frac{1}{x^2} - \frac{1}{x^3}}{-2 + \frac{1}{x^2} + \frac{5}{x^3}}$$

As $x \to \infty$, all the expressions that have a constant on the numerator and x to a power in the denominator will become 0, and this allows us to evaluate the limit:

$$= \lim_{x \to \infty} \frac{1 - \frac{4}{x} + \frac{1}{x^2} - \frac{1}{x^3}}{-2 + \frac{1}{x^2} + \frac{5}{x^3}}$$

$$= \frac{\lim_{x \to \infty} \left(1 - \frac{4}{x} + \frac{1}{x^2} - \frac{1}{x^3} \right)}{\lim_{x \to \infty} \left(-2 + \frac{1}{x^2} + \frac{5}{x^3} \right)}$$

$$= \frac{\lim_{x \to \infty} 1 - \lim_{x \to \infty} \frac{4}{x} + \lim_{x \to \infty} \frac{1}{x^2} - \lim_{x \to \infty} \frac{1}{x^3}}{\lim_{x \to \infty} -2 + \lim_{x \to \infty} \frac{1}{x^2} + \lim_{x \to \infty} \frac{5}{x^3}}$$

$$= \frac{1 - 0 + 0 - 0}{-2 + 0 + 0}$$

$$= -\frac{1}{2}$$

There is a easier, though less formal method to solve a limit problem of this sort. Notice that in the above limit, both the numerator and denominator are polynomials. We can argue that when x becomes tremendously large, *the highest power of x in the polynomial will dominate.* Consider the polynomial $f(x) = x^3 - 4x^2 + x - 1$. When the value of x becomes tremendously large, the x^3 term is going to be much larger than the $-4x^2$ term and the x term. i.e,

$$x^3 - 4x^2 + x - 1 \approx x^3 \text{ as } x \to \infty$$

Similarly, as x becomes tremendously large,

$$-2x^3 + x + 5 \approx x^3 \text{ as } -2x^3 \to \infty$$

Therefore, the limit:
$$\lim_{x\to\infty} \frac{x^3 - 4x^2 + x - 1}{-2x^3 + x + 5}$$
may be evaluated like this:

$$\lim_{x\to\infty} \frac{x^3 - 4x^2 + x - 1}{-2x^3 + x + 5} = \lim_{x\to\infty} \frac{x^3}{-2x^3} = \lim_{x\to\infty} \frac{1}{-2} = -\frac{1}{2}$$

It is important to note that the above argument works for polynomials (and power functions in general) only if x approaches infinity or negative infinity. If we are to evaluate
$$\lim_{x\to 5} \frac{x^3 - 4x^2 + x - 1}{-2x^3 + x + 5}$$
then the previous argument does not work, since x does not approach infinity, so we may *not* ignore the other terms.

E.g.

Evaluate
$$\lim_{x\to -\infty} \frac{\sqrt{x^2 + 1}}{x - 1}$$

To evaluate this formally, we divide by the highest power of x. In this case here, the highest power of the denominator is x. How about the numerator? Since we are taking the square root of x^2, the highest power of x in the numerator is also x. (This is generally true. If we have $\sqrt{x^4 + 3x + 1}$, then the highest power of x is x^2 since we are taking the square root of x^4. If we have $\sqrt[3]{x^7 - 4x^3 + 3}$ then the highest power of x is $x^{7/3}$.)

$$\lim_{x\to -\infty} \frac{\sqrt{x^2 + 1}}{x - 1}$$
$$= \lim_{x\to -\infty} \frac{\dfrac{\sqrt{x^2 + 1}}{x}}{\dfrac{x - 1}{x}}$$
$$= \lim_{x\to -\infty} \frac{\dfrac{\sqrt{x^2 + 1}}{-\sqrt{x^2}}}{\dfrac{x - 1}{x}}$$

Why does the x turned into $-\sqrt{x^2}$? The reason is that we are taking the limit as x goes to *negative infinity*. As x approaches negative infinity, x is a negative number. If we just turn x into $\sqrt{x^2}$, then this would have turned x into a positive number since $\sqrt{x^2}$ is a positive number for any value of x. Therefore, we must

put the negative sign in front of the expression if we were to take the limit as x goes to negative infinity.

$$\lim_{x \to -\infty} \frac{\dfrac{\sqrt{x^2+1}}{-\sqrt{x^2}}}{\dfrac{x-1}{x}}$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{\dfrac{x^2+1}{x^2}}}{\dfrac{x}{x} - \dfrac{1}{x}}$$

$$= \frac{\displaystyle\lim_{x \to -\infty} -\sqrt{1 + \dfrac{1}{x^2}}}{\displaystyle\lim_{x \to -\infty} 1 - \dfrac{1}{x}}$$

$$= \frac{-\sqrt{\displaystyle\lim_{x \to -\infty} 1 + \lim_{x \to -\infty} \dfrac{1}{x^2}}}{\displaystyle\lim_{x \to -\infty} 1 - \lim_{x \to -\infty} \dfrac{1}{x}}$$

$$= \frac{-\sqrt{1+0}}{1-0} = -1$$

We could have solve the problem more easily by arguing that, as x approaches $-\infty$, the highest power of x is going to dominate, so $\sqrt{x^2+1} \approx \sqrt{x^2}$ and $x-1 \approx x$ when x is very negatively large. We have

$$\lim_{x \to -\infty} \frac{\sqrt{x^2+1}}{x-1}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{x^2}}{x}$$

$$= \lim_{x \to -\infty} \frac{|x|}{x}$$

Since x goes to *negative infinity*, the numerator will be positive while the denominator will be negative, so their ratio will be -1:

$$\lim_{x \to -\infty} \frac{|x|}{x} = -1$$

E.g. Evaluate the limit:

$$\lim_{x \to \infty} \frac{x^3 - 3x^2 + 1}{x^2 - 1}$$

We divide by the highest power of x, which is x^3.

$$\lim_{x \to \infty} \frac{x^3 - 3x^2 + 1}{x^2 - 1}$$

$$= \lim_{x \to \infty} \frac{\dfrac{x^3 - 3x^2 + 1}{x^3}}{\dfrac{x^2 - 1}{x^3}}$$

$$= \frac{\lim\limits_{x \to \infty} \dfrac{x^3 - 3x^2 + 1}{x^3}}{\lim\limits_{x \to \infty} \dfrac{x^2 - 1}{x^3}}$$

$$= \frac{\lim\limits_{x \to \infty} \left(1 - \dfrac{3}{x} + \dfrac{1}{x^3}\right)}{\lim\limits_{x \to \infty} \left(\dfrac{1}{x} - \dfrac{1}{x^3}\right)}$$

$$= \frac{\lim\limits_{x \to \infty} 1 - \lim\limits_{x \to \infty} \dfrac{3}{x} + \lim\limits_{x \to \infty} \dfrac{1}{x^3}}{\lim\limits_{x \to \infty} \dfrac{1}{x} - \lim\limits_{x \to \infty} \dfrac{1}{x^3}}$$

$$= \frac{1 - 0 + 0}{0 + 0} = \frac{1}{0}$$

We have division by zero, which is undefined. So

$$\lim_{x \to \infty} \frac{x^3 - 3x^2 + 1}{x^2 - 1} \text{ does not exist}$$

What happens here is that, since the degree of numerator is higher than the degree of the denominator, the numerator increases much faster than the denominator as x goes to infinity, so the result is a tremendously large number.

Sometimes we use the notation

$$\lim_{x \to \infty} f(x) = \infty$$

to mean that as x increases without bound, the value of $f(x)$ also increases without bound. Once again remember that *infinity is not a number*, so this notation is not saying that the limit exists. It is simply a notation used to denote the behavior of the function f as x increases without bound.

Here's some general rules about limits at infinity of a rational function:

If $p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$ and $q(x) = b_m x^m + b_{m-1} x^{m-1} + b_{m-2} x^{m-2} + \cdots + b_2 x^2 + b_1 x + b_0$ are two polynomials of degree n and m. Then

$$\lim_{x \to \infty} \frac{p(x)}{q(x)} = 0 \text{ if } n < m$$

$$\lim_{x \to \infty} \frac{p(x)}{q(x)} = \frac{a_n}{b_m} \text{ if } n = m$$

$$\lim_{x \to \infty} \frac{p(x)}{q(x)} \text{ does not exist if } n > m$$

This is true because, if the numerator has a lower degree than the denominator, then as x increases without bound, the denominator increases much faster than the numerator, so the fraction approaches 0.

If the numerator and the denominator has the same degree, then they increase at about the same rate, and the limit will be doniminated by the highest power of x, which in this case is x^n (or x^m), and the result is

$$\lim_{x \to \infty} \frac{p(x)}{q(x)} = \lim_{x \to \infty} \frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m}$$

If the numerator has a higher degree than the denominator, the numerator will increase faster than the denominator as x goes to ∞, and the whole fraction will grow without bound.

E.g.

Evaluate

$$\lim_{x \to \infty} \frac{\sin x}{x}$$

We cannot evaluate this limit directly. However, notice that $-1 \le \sin x \le 1$, so

$$\lim_{x \to \infty} -\frac{1}{x} \le \lim_{x \to \infty} \frac{\sin x}{x} \le \lim_{x \to \infty} \frac{1}{x}$$

Since both limits $\lim\limits_{x \to \infty} -\frac{1}{x}$ and $\lim\limits_{x \to \infty} \frac{1}{x}$ are 0, using the squeeze theorem, we know that

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0$$
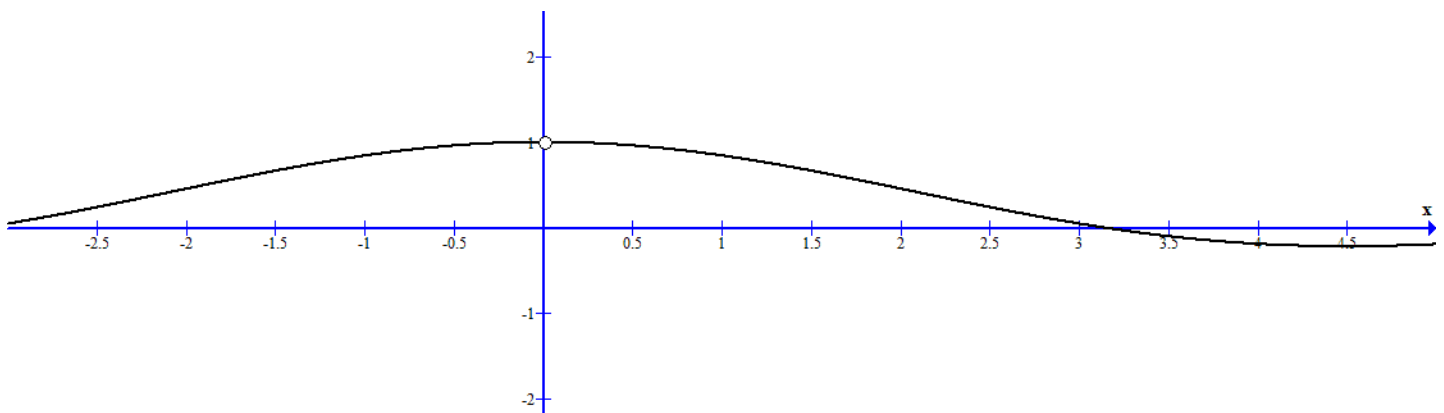
By definition, $f(x) = \dfrac{\sin x}{x}$ has a horizontal asymptote $y = 0$. Notice that the graph of f intercepts its horizontal asymptote many many times (in fact, infinitely many times). This should clarify a misconception that the graph of a function does not intercept its horizontal asymptote.

Slant Asymptote

For the function

$$f(x) = \frac{x^2}{x + 1}$$

the degree of the numerator is greater than the degree of the denominator, so we know that

$$\lim_{x \to \infty} \frac{x^2}{x + 1} \text{ does not exist}$$

However, if we perform the long division and express f as:

$$f(x) = \frac{x^2}{x + 1} = x - 1 + \frac{1}{x + 1}$$

Then we see that, as $x \to \infty$, the fraction $\frac{1}{x+1}$ approaches 0. This means that the expression $x - 1 + \frac{1}{x+1}$ approaches $x - 1$ as $x \to \infty$. What this means is that, as $x \to \infty$, $f(x)$ behaves very much like the line $y = x - 1$. i.e

$$\frac{x^2}{x + 1} \approx x - 1 \text{ as } x \to \infty$$

Since f approaches the line $y = x - 1$ as $x \to \infty$, we say that the line $y = x - 1$ is a **slant asymptote** of f. The line is not a horizontal line. Instead it is a line with a slope, and that's why the name *slant*
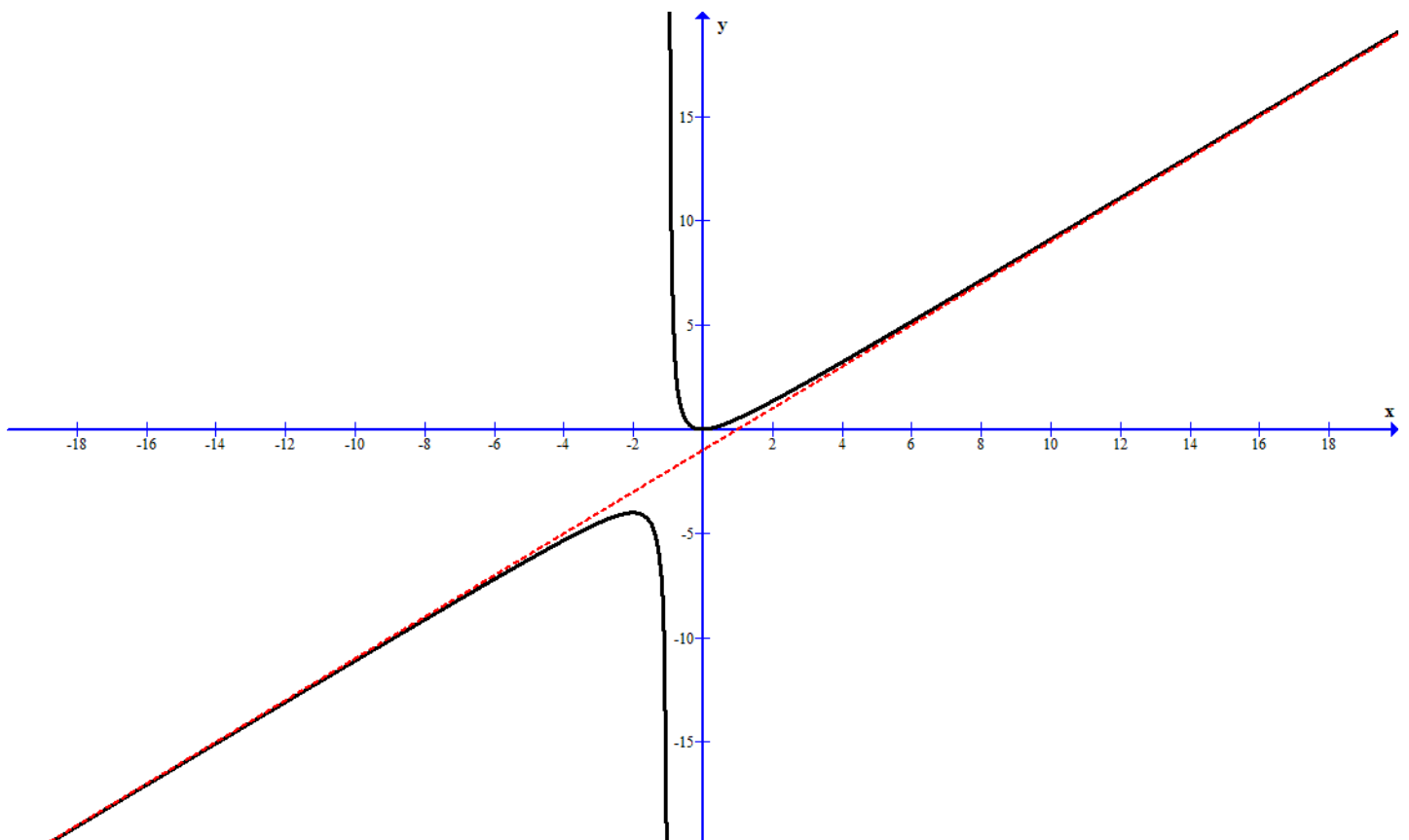
In general, a rational function will have a slant asymptote if the degree of the numerator is one greater than the degree of the denominator. In order to find the slant asymptote, we perform the long division to find the quotient. The quotient will be a line and that is the slant asymptote.

Example: Find the slant asymptote of $f(x) = \dfrac{x^3 - 3x^2 + x + 4}{x^2 + 2x - 4}$.

Ans: Performing the long division we see that

$$\frac{x^3 - 3x^2 + x + 4}{x^2 + 2x - 4} = (x - 5) + \frac{15x - 15}{x^2 + 2x - 4}$$

Therefore, the slant asymptote for f is $y = x - 5$
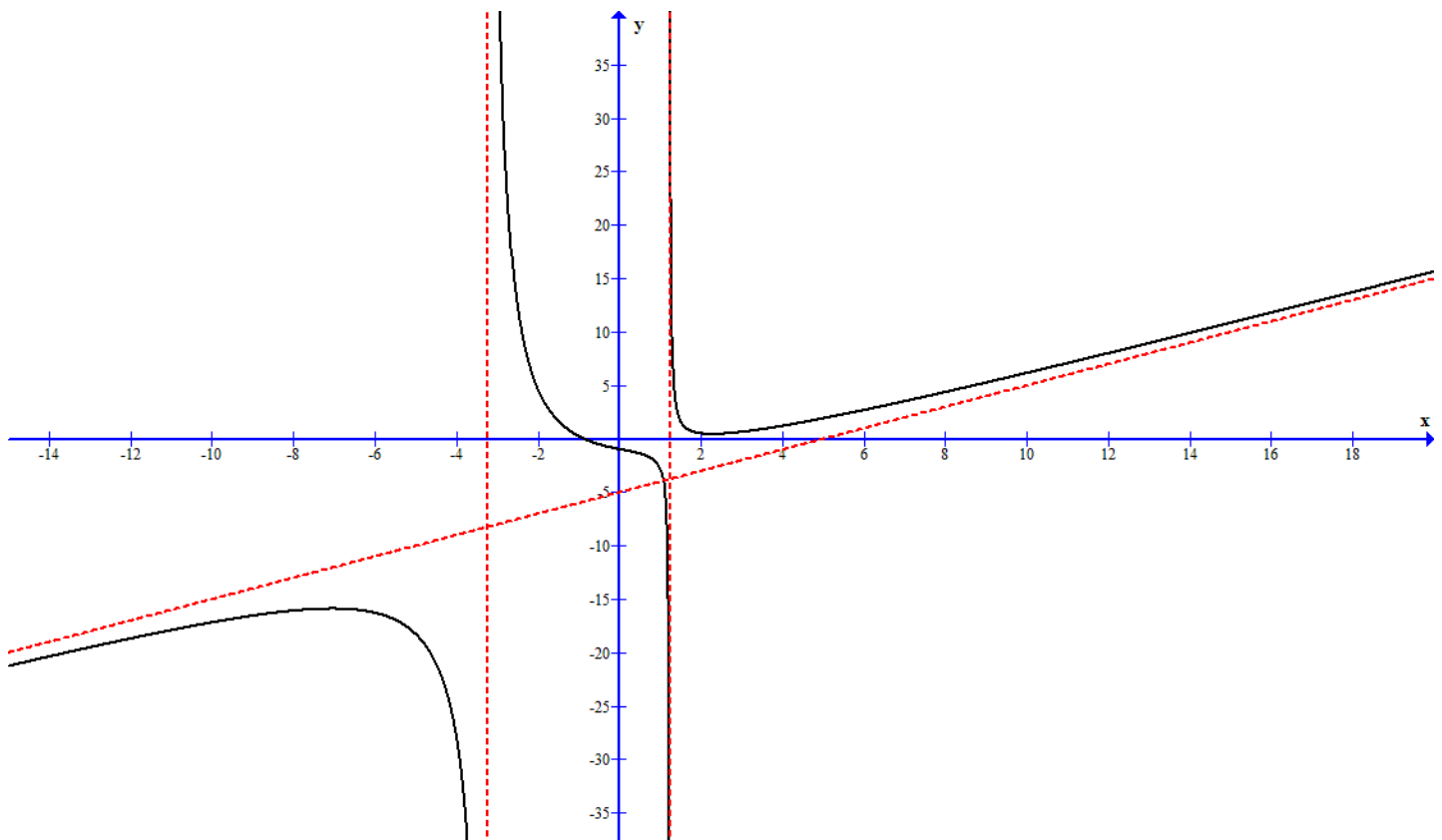
Example: Find the slant asymptote of $f(x) = \dfrac{x^2 + 2x}{x^2 + 1}$.

Ans: Since the degree of the numerator is equal to the degree of the denominator, f has *no* slant asymptote. f, though, has a horizontal asymptote of $y = 1$.
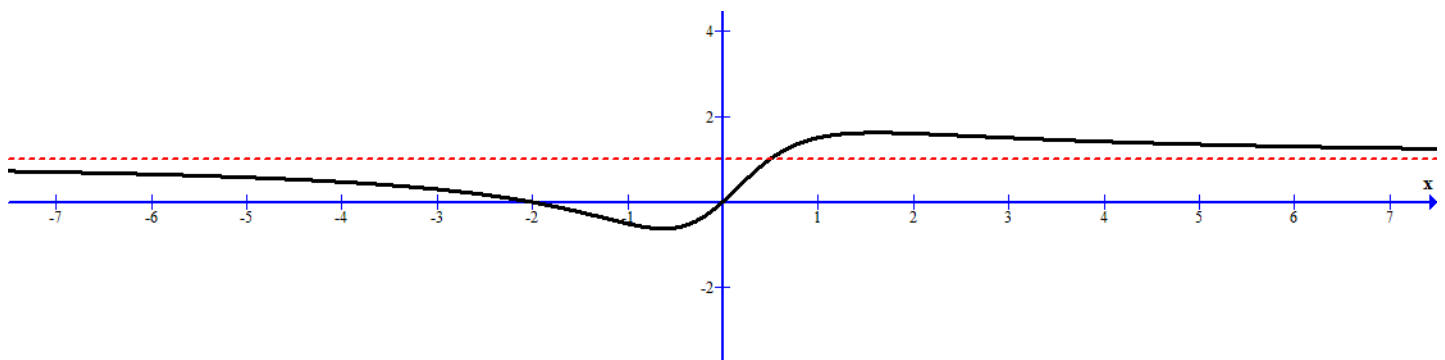
Graphing

With the knowledge of the intervals of increase and decrease, local maxima and local minima, intervals of concave up and down, inflection points, horizontal asymptote, vertical asymptote, slant asymptote, and intercepts of a function, we can draw the graphs of many functions very accurately.

Guidelines for drawing the graph of a function f:

1. Try to determine the domain of f. Once you know where is f defined you have a better idea of how to start.

2. Find the x and y intercepts of f if possible. To find the x intercept, set $f(x) = 0$ and solve the equation (if possible). To find the y intercept, evaluate $f(0)$.

3. Use symmetry if any exists. For example, $\sin x$ is an odd function, $f(x) = x^2+1$ is an even function. The graph of an odd function is symmetric with respect to the origin, and the graph of an even function is symmetric with respect to the y-axis.

4. Find asymptotes if there's any. Remember that $x = c$ is a vertical asymptote of f if $\lim_{x \to c} f(x) = \infty$ or if $\lim_{x \to c} f(x) = -\infty$. And $y = h$ is a horizontal asymptote of f if $\lim_{x \to \infty} f(x) = h$ or if $\lim_{x \to -\infty} f(x) = h$. And if f is a rational function, and the degree of the numerator is one greater than the degree of the denominator, then f has a slant asymptote, and you find the slant asymptote by performing the long division and finding the quotient.

5. Find the intervals of increase and decrease. Use the first derivative to find where f increases and where f decreases. If f' is positive in an interval I, then f is increasing on I, if f' is negative on I, then f is decreasing on I.

6. Find the local maxima and local minima. Use the first or second derivative test to test the critical points of f. Using the first derivative test, if f' changes from positive to negative at the critical number c, then f has a local maximum at c. If f' changes from negative to positive at c, then f has a local minimum at c. If f' does not change sign at c, then c is neither a local max nor local min. Using the second derivative test, if $f''(c) > 0$, then f has a local minimum at c. If $f''(c) < 0$, then f has a local maximum at c. If $f''(c) = 0$ or $f''(c)$ is undefined, the second derivative test is inconclusive.

7. Find the inflection points and the interval where f concaves up and where f concaves down. If f'' is positive in an interval I, then f concaves up in I. If f'' is negative in I, then f concaves down in I. To find the inflection points, find the critical number of the first derivative. That is, find the numbers c where $f''(c) = 0$ or $f''(c)$ is undefined. If f'' changes sign at c, then f has an inflection point at c.

8. Using all the information above, and whatever other information that you know about f to help you sketch the graph of f.

Example: Let $f(x) = \dfrac{x^2}{x^2 - 1}$. Find the domain of f. Find the horizontal, vertical, and slant asymptotes of f, if any exist. Find the x and y intercepts of f, if possible. Find the intervals of increase and decrease. Find the local maxima and local minima of f. Find the inflection points of f. Find the intervals where f concaves up and the intervals where f concaves down. Sketch the graph of f.

What is the domain of f? Notice that f is a rational function. So f is defined everywhere except when the denominator is 0. In this case here the denominator is 0 is $x = \pm 1$. So the domain of f is all real numbers except ± 1.

We first find the intercepts of f: $f(0) = 0$, so the y-intercept of f is the origin. Setting $f(x) = 0$ and solve for x we see that $\dfrac{x^2}{x^2 - 1} = 0$ if and only if $x = 0$. So the x intercept is also just the origin.

$$\lim_{x \to 1^-} \frac{x^2}{x^2 - 1} = -\infty$$

$$\lim_{x \to 1^+} \frac{x^2}{x^2 - 1} = \infty$$

$$\lim_{x \to -1^-} \frac{x^2}{x^2 - 1} = \infty$$

$$\lim_{x \to -1^+} \frac{x^2}{x^2 - 1} = -\infty$$

So $x = 1$ and $x = -1$ are vertical asymptotes of f.

$$\lim_{x \to \infty} \frac{x^2}{x^2 - 1} = 1$$

$$\lim_{x \to -\infty} \frac{x^2}{x^2 - 1} = 1$$

So $y = 1$ is a horizontal asymptote of f.

Notice that f is an even function, so its graph must be symmetric with respect to the y-axis.

We now take the first derivatives to find the critical numbers and intervals of increase and decrease:

$$f'(x) = \frac{-2x}{(x^2 - 1)^2}$$

$f'(0) = 0$, so $x = 0$ is a critical number. Since $f' > 0$ in the interval $(-\infty, -1)$ and $(-1, 0)$, so f is increasing in there, and $f' < 0$ in $(0, 1)$ and $(1, \infty)$ so f

is decreasing there. Using the first derivative test we see that f has a local maximum at 0.

We now take the second derivative to find the inflection points and the intervals of concavity:

$$f''(x) = \frac{6x^2 + 2}{(x^2 - 1)^3}$$

Since $f'' > 0$ in $(-\infty, -1)$ and in $(1, \infty)$, so f concaves up in these intervals. Since $f'' < 0$ in $(-1, 1)$, f concaves down there. Since there is no value for which f'' is equal to 0, f has no inflection point.

We summarize the information in the table below:

Intervals	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \infty)$
f	↗ & C.U.	V.A	↗ & C.D.	L Max	↘ & C.D.	V.A.	↘ & C.U.
f'	$+$	V.A	$+$	0	$-$	V.A.	$-$
f''	$+$	V.A	$-$	$-$	$-$	V.A.	$+$
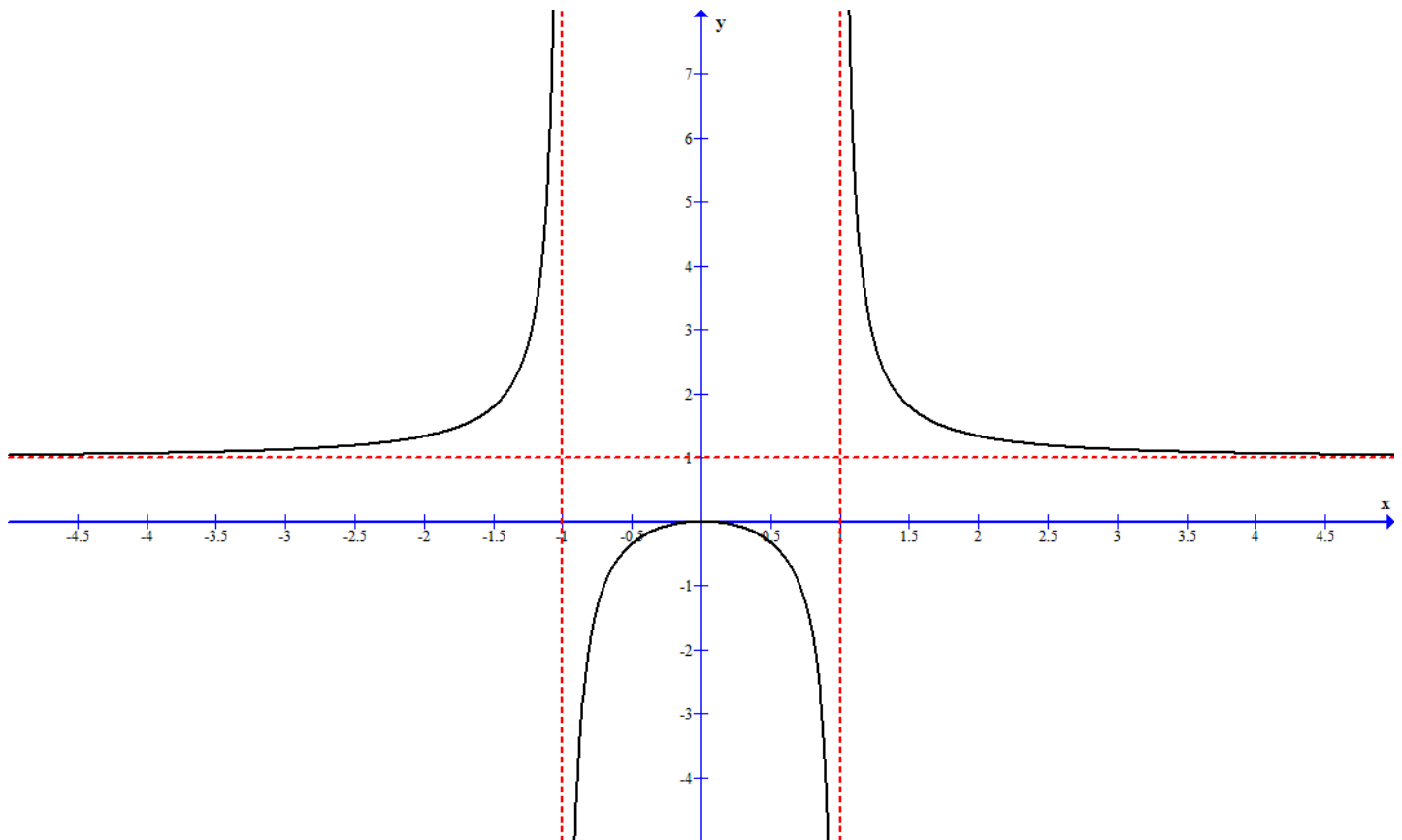
Example: Let $f(x) = x\sqrt{4 - x^2}$. Find the domain of f. Find the horizontal, vertical, and slant asymptotes of f, if any exist. Find the x and y intercepts of f, if possible. Find the intervals of increase and decrease. Find the local maxima and local minima of f. Find the inflection points of f. Find the intervals where f concaves up and the intervals where f concaves down. Sketch the graph of f.

We first consider the domain of f. We are taking the square root of an expression, so we must make sure that whatever is inside the square root must be ≥ 0. So we want

$$4 - x^2 \geq 0 \Leftrightarrow x^2 \leq 4 \Leftrightarrow -2 \leq x \leq 2$$

The domain of f is the closed interval $[-2, 2]$.

$f(0) = 0$, so the y intercept is $(0, 0)$. Setting $f(x) = 0$ we see that

$$f(x) = 0 \Leftrightarrow x\sqrt{4 - x^2} = 0$$
$$\Leftrightarrow x = 0 \text{ or } 4 - x^2 = 0$$
$$\Leftrightarrow x = 0 \text{ or } x = \pm 2$$

So the x intercepts are at $(-2, 0)$, $(0, 0)$, and $(2, 0)$.

There is not going to be any vertical asymptote of f since f is defined everywhere in the closed interval $[-2, 2]$.

There is no horizontal asymptote of f since f is not even defined for large values of x.

There is no slant asymptote of f.

Notice that f is an odd function, so its graph is symmetric with respect to the origin.

Taking the derivative of f we have:

$$f'(x) = \sqrt{4 - x^2} - \frac{x^2}{\sqrt{4 - x^2}}$$

We see that the derivative is undefined at the boundary of the domain, namely $x = \pm 2$. If we set f' equal to 0 we get:

$$\sqrt{4 - x^2} - \frac{x^2}{\sqrt{4 - x^2}} = 0 \Leftrightarrow \frac{(4 - x^2) - x^2}{\sqrt{4 - x^2}} = 0$$
$$\Leftrightarrow \frac{4 - 2x^2}{\sqrt{4 - x^2}} = 0 \Leftrightarrow 4 - 2x^2 = 0 \Leftrightarrow x = \pm\sqrt{2}$$

So we have 4 critical numbers, $x = -2$, $x = -\sqrt{2}$, $x = \sqrt{2}$, and $x = 2$. On the two critical numbers at the boundary of the domain, ± 2, f has a vertical tangent since its derivative goes to infinity.

Since f' is negative on $(-2, -\sqrt{2})$ and on $(\sqrt{2}, 2)$, f is decreasing on these intervals. Since f' is positive on $(-\sqrt{2}, \sqrt{2})$, f is increasing on this intervals.

Using the first derivative test we see that f has a local minimum at $-\sqrt{2}$ and f has a local maximum at $\sqrt{2}$

Taking the second derivative of f gives:

$$f''(x) = \frac{\sqrt{4-x^2}(-4x) - (4 - 2x^2)\frac{-2x}{2\sqrt{4-x^2}}}{4 - x^2}$$

$$= \frac{-4x\sqrt{4-x^2} + \frac{x(4-2x^2)}{\sqrt{4-x^2}}}{4 - x^2}$$

$$= \frac{1}{\sqrt{4-x^2}} \cdot \frac{-4x(4-x^2) + (4 - 2x^2)x}{4 - x^2}$$

$$= \frac{-16x + 4x^3 + 4x - 2x^3}{(4 - x^2)^{3/2}}$$

$$= \frac{2x^3 - 12x}{(4 - x^2)^{3/2}}$$

Setting the second derivative to 0 and solve for x we get:

$$\frac{2x^3 - 12x}{(4 - x^2)^{3/2}} = 0 \Leftrightarrow 2x^3 - 12x = 0 \Leftrightarrow x = 0 \text{ or } x = \pm\sqrt{6}$$

Since ± 6 is outside the domain of f, we do not need to consider them.

Notice that f'' is positive on the interval $(-2, 0)$ so f concaves up on $(-2, 0)$, and f'' is negative on $(0, 2)$ so f concaves down on $(0, 2)$.

f'' changes sign at 0, so f has an inflection at 0.

We use the following table to summarize the information we have:

Intervals	$(-2, -\sqrt{2})$	$-\sqrt{2}$	$(-\sqrt{2}, 0)$	0	$(0, \sqrt{2})$	$\sqrt{2}$	$(\sqrt{2},)$
f	$\searrow$ & C.U.	L. Min	$\nearrow$ & C.U.	I.P.	$\searrow$ & C.D	L. Max	$\searrow$ & C.D
f'	$-$	0	$+$	$+$	$+$	0	$-$
f''	$+$	$+$	$+$	0	$-$	$-$	$-$
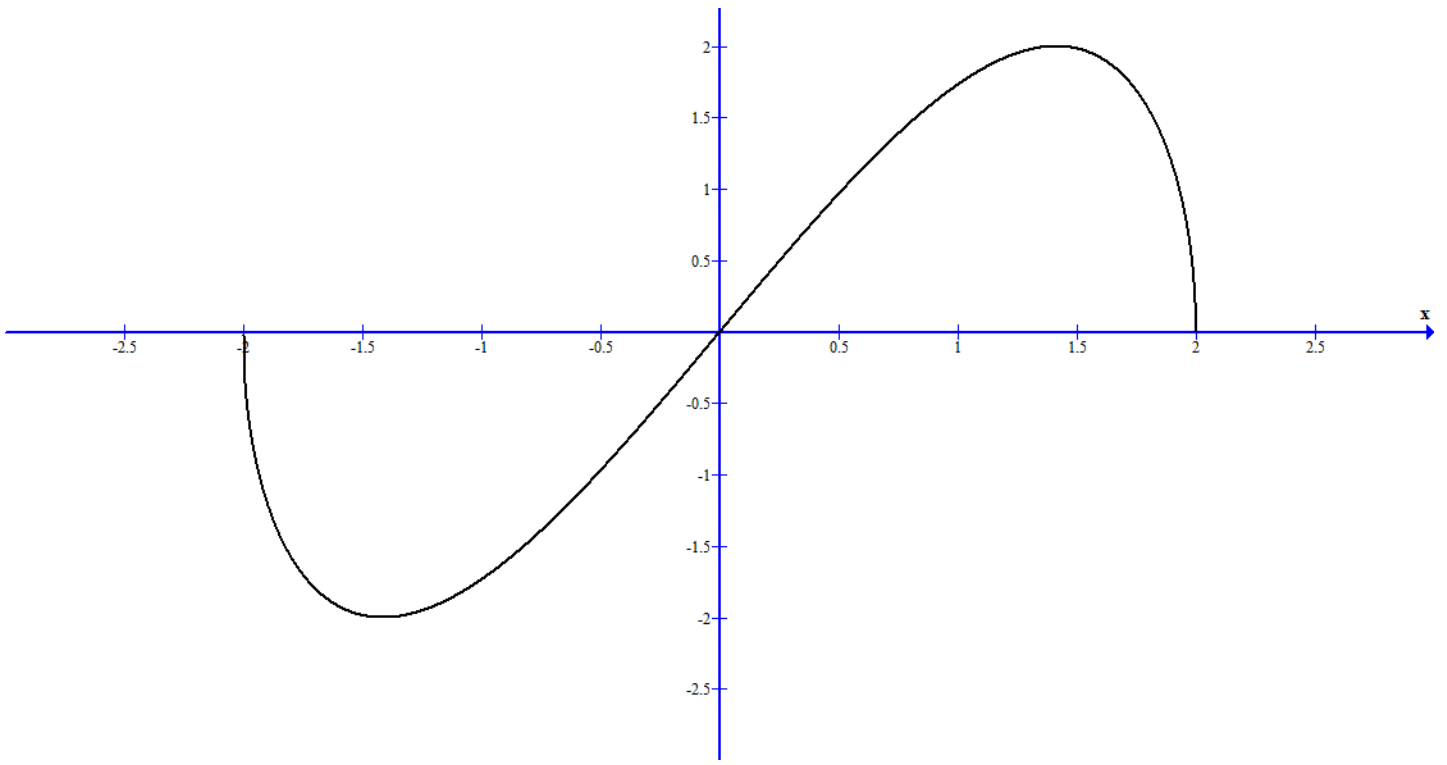
Optimization

We saw how we can find the local maximum and local minimum of a function. In the case when the domain of the function is confined to a closed interval and the function is continuous, we can even find the absolute maximum and minimum and use these to solve real world application problems. For example, we can find the production level that would maximize profit. We can find the dimension of a box that would minimize the raw materials used. We may find the path that would minimize the time that takes us going from one point to another. All of these are applications of the derivative that fall under the catagory of **optimization**.

In most optimization problems, we will have a quantity that needs to be optimized. It may be the cost that needs to be minimized, the area that needs to be maximized, or the time that needs to be minimized. In all these cases, the quantity that you want to optimize will be expressible as a function of some other variables, and you want to find the values of those other variables that would optimize the quantity in question.

For example, you may be asked to find the dimension of a cylindrical can that would minimize the materials used (which means minimizing the surface area). The dimension of a cylinder, of course, will be its radius r and height h.

And knowing the formula for the surface area S of a cylinder allows you to express their relation as $S = 2\pi r^2 + 2\pi rh$. This equation expresses the relationship of the quantity that you want to optimize s, in terms of other variables (r and h). After that, you would be given the conditions that gives you the relationship between the variables, which allows you to express the variables in terms of one another. In this example with the can, you may be told that the volume of the can must be 1000 cubic meters, so you have $\pi r^2 h = 1000$. These additional limitations on the variables are called **constrains**. They allow you to express the variables in terms of the other. In this example, you may solve for h in terms of r to get:

$$h = \frac{1000}{\pi r^2}$$

Plugging h in terms of r back to the formula for surface area gives:

$$S = 2\pi r^2 + 2\pi r \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}$$

We now have the quantity we want to minimize, S, in terms of just one variable, r. To find the dimension that would minimize the surface area, we take the derivative of S with respect to r to find the critical numbers.

$$\frac{dS}{dr} = 4\pi r - \frac{2000}{r^2}$$

To find the critical numbers, we set the derivative equal to 0.

$$4\pi r - \frac{2000}{r^2} = 0 \Leftrightarrow \frac{4(\pi r^3 - 500)}{r^2} = 0 \Leftrightarrow$$

$$\pi r^3 - 500 = 0 \Leftrightarrow r^3 = \frac{500}{\pi} \Leftrightarrow r = \sqrt[3]{500/\pi}$$

So $r = \sqrt[3]{500/\pi}$ is a critical number. Using the first derivative test we can see that r is a local minimum. In most applications there would be some kind of boundary points which we can test also. In this case here, $r > 0$, but seemingly r can go as large as we like, so we don't have an implied boundary. However, looking at the fact $\frac{dA}{dr}$ is always negative before the critical number $\sqrt[3]{500/\pi}$ and that $\frac{dA}{dr}$ is always positive after the critical number. This means that A is always decreasing before $\sqrt[3]{500/\pi}$ and always increasing after $\sqrt[3]{500/\pi}$. This means that the local minimum at $\sqrt[3]{500/\pi}$ must in fact be a global minimum, which is what we want. After we found the value of r that would give a global minimum, we can plug it back into the equation
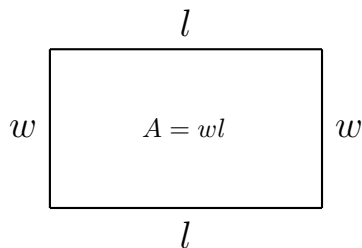
$$h = \frac{1000}{\pi r^2}$$

to solve for h. Solving for h we see that

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi (\sqrt[3]{500/\pi})^2}$$

Example:

Find the dimension of the largest rectangle that can be formed using a fixed perimeter p.

Ans:

The area of the rectangle, A, is what we want to maximize. The area of a rectangle is the product of its width w and length l, so we have $A = wl$.

The *constrain* on the variables is the fact that the perimeter must be p. So $2w + 2l = p$. Solving for l in terms of w we get $l = \frac{p-2w}{2}$. Plug the value of l in terms of w into the equation for A we get

$$A = w \left(\frac{p - 2w}{2} \right) = \frac{1}{2}(pw - 2w^2)$$

We have expressed the area of the rectangle A in terms of just w. We now want to differentiate A with respect to w to find the critical numbers.

$$\frac{dA}{dw} = \frac{1}{2}(p - 4w)$$

Setting the derivative to 0 and solving for w we get:

$$\frac{1}{2}(p - 4w) = 0 \Leftrightarrow p - 4w = 0 \Leftrightarrow w = \frac{p}{4}$$

So $w = \frac{p}{4}$ is a critical number. Using the first derivative test we see that this is a local maximum of A. However, notice that A is a quadratic function in w, with the coefficient for w^2 being negative, so the graph of A (as a function of w) is a porabola that opens down, which means the local maximum must be a global maximum.

So $w = \frac{p}{4}$ is the width of the rectangle that would maximize the area. Plug the value of w to solve for l we get:

$$l = \frac{p - 2(\frac{p}{4})}{2} = \frac{p - \frac{p}{2}}{2} = \frac{p}{4}$$

So w and l both equals to $\frac{p}{4}$, which means the rectangle is a square. This is to be expected since we know that for a fixed perimeter, the rectangle with the largest area is a square.

Example: A cable is to be installed from the cable company to a house. The house is located across a river from the cable company and 10 km to the west. The river is 6 km wide.

It costs 40 dollars per km to install the cable under water, and 25 dollars per km to install the cable on land. Find the location of the point from the house where the cable should be installed to minimize cost.

Ans:
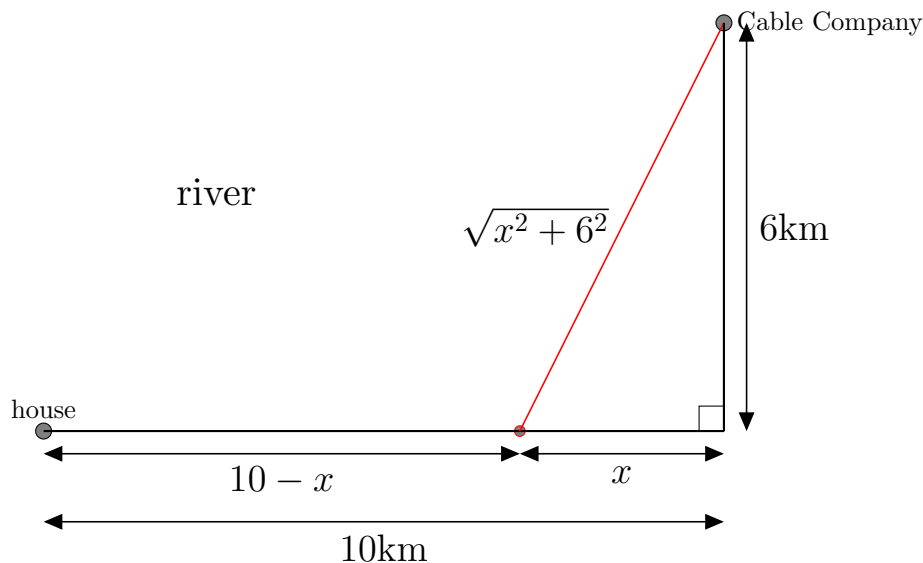
We let the distance of the point of installation from the point directly across the river from the cable company be x, then the distance from the point of installation to house will be $10-x$. Using the Pythgorean theorem we know that the distance from the cable company to the point of installation, s_{water}, which is always under water, is given by:

$$s_{\text{water}} = \sqrt{x^2 + 36}$$

The cost to install the cable over this distance is

$$c_{\text{water}} = 40 \cdot s_{\text{water}} = 40\sqrt{x^2 + 36}$$

The cost to install the cables from the point of installation to the house is

$$c_{\text{land}} = 25(10 - x) = 250 - 25x$$

The total cost, as a function of x, is given by

$$C_{\text{total}}(x) = c_{\text{land}} + c_{\text{water}} = (250 - 25x) + (40\sqrt{x^2 + 36})$$

Before we proceed any further, notice that x must be between 0 and 10. So this time, we do have an implied boundary for the function, which is the closed interval $[0, 10]$.

In order to minimize the cost, we take C' to try to find any critical numbers.

$$C'(x) = -25 + 40\left(\frac{2x}{2\sqrt{x^2 + 36}}\right) = -25 + \frac{40x}{\sqrt{x^2 + 36}}$$

Setting C' to 0 to try to find the critical number gives:

$$-25 + \frac{40x}{\sqrt{x^2 + 36}} = 0$$

$$\frac{40x}{\sqrt{x^2 + 36}} = 25$$

$$40x = 25\sqrt{x^2 + 36}$$
$$1600x^2 = 625(x^2 + 36)$$
$$1600x^2 = 625x^2 + 22500$$
$$975x^2 = 22500$$

$$x^2 = \frac{22500}{975} = \frac{4500}{195} = \frac{900}{39}$$
$$x = \pm\sqrt{\frac{900}{39}}$$

One of the solution, $x = -\sqrt{\dfrac{900}{39}}$, is outside of the range of x, so we have only one critical number,

$$x = \sqrt{\frac{900}{39}} \approx 4.804$$

If we compare the values of C at the boundary points and at the critical number $x = 4.804$, we see that $C(0) = 490$, $C(4.804) = 437.3$, and $C(10) \approx 466$. Since $C(4.804) = 437.3$ is the smallest among the three, it is the absolute minimum. This means if the point of installation is about 5.196 km from the house the installation cost will be minimized.

Example: A piece of board is to be carried from a hallway a meters wide and make a right-angled turn into another hallway that is b meters wide. Find the length of the longest board that can be carried around the corner.

Ans:
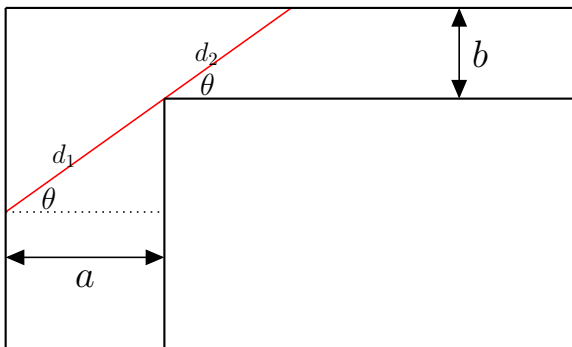
Let d be the distance in red as shown in the figure. Notice that we want to find the *shortest* value of d because, in order to pass the corner, the piece of board must be able to pass through the shortest distance allowed by the corner, so this distance is the longest the piece of board can be.

Notice that $d_1 = \dfrac{a}{\cos\theta}$ and $d_2 = \dfrac{b}{\sin\theta}$

Since $d = d_1 + d_2$ so we have

$$d = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}$$

Taking derivative of d with respect to θ we get:

$$d'(\theta) = a\sec\theta\tan\theta - b\csc\theta\cot\theta$$

Setting $d' = 0$ gives

$$a\sec\theta\tan\theta - b\csc\theta\cot\theta = 0 \Leftrightarrow$$

$$a\sec\theta\tan\theta = b\csc\theta\cot\theta \Leftrightarrow$$

$$a \cdot \frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta} = b \cdot \frac{1}{\sin\theta} \cdot \frac{\cos\theta}{\sin\theta}$$

Cross mulitply gives:

$$a\sin^3\theta = b\cos^3\theta \Leftrightarrow$$

$$\frac{\sin^3\theta}{\cos^3\theta} = \frac{b}{a} \Leftrightarrow$$

$$\tan^3\theta = \frac{b}{a} \Leftrightarrow$$

$$\tan\theta = \left(\frac{b}{a}\right)^{1/3} \Leftrightarrow$$

$$\theta = \tan^{-1}\left[\left(\frac{b}{a}\right)^{1/3}\right]$$

A little algebra shows that if

$$\tan\theta = \left(\frac{b}{a}\right)^{1/3}$$

then

$$\sin\theta = \frac{b^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}$$

and

$$\cos\theta = \frac{a^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}$$

So

$$d = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}$$

$$= \frac{a}{\frac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}} + \frac{b}{\frac{b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}}$$

$$= (a^{2/3})\sqrt{a^{2/3} + b^{2/3}} + (b^{2/3})\sqrt{a^{2/3} + b^{2/3}}$$

Observe that θ must be a number between 0 and $\dfrac{\pi}{2}$, non-inclusive. If we look at the graph of d as a function of θ or do a little analysis (by looking at the derivative of d) we can see that this value is indeed the global minimum of d for θ between 0 and $\dfrac{\pi}{2}$.

Newton's Method Newton's method is a numerical method used to solve equations. For some equations, it is very difficult, or even impossible, to algebraically find solutions. For example, consider the equation

$$\sin x^2 + 2\sin(x+1) - \cos x = 1$$

How do we find the solution(s) to this equation? Or, how do we even know that this equation has a solution? It is very difficult to solve this equation algebraically. Instead, let us try it this way: If we move everything to the left hand side of the equation we have:

$$\sin x^2 + 2\sin(x+1) - \cos x - 1 = 0$$

Let

$$f(x) = \sin x^2 + 2\sin(x+1) - \cos x - 1$$

Then finding the solution of the equation becomes the problem of finding the **root** of the function f. That is, we want to find a number r such $f(r) = 0$. This is the same as finding the x-intercept of f.
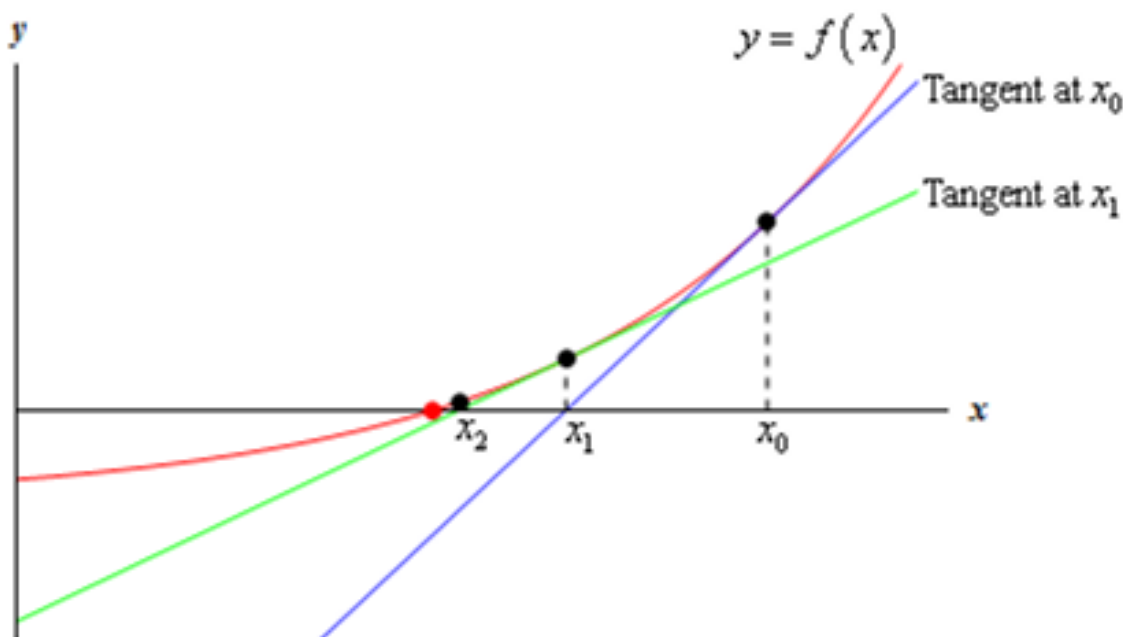
Let's consider the graph of f as illustrated on the picture. the red dot is the $x-$intercept (root) we want to find. We do not know what the $x-$ value of the red dot is. However, we can *estimate* a value. For must functions, this estimation does not have to be very good. (There are functions that you must estimate the root more carefully. We will see examples of what type of functions are). Regardless, though, you should try your best to come up a an estimate that is as good as possible in the first place. Let's call this estimate x_0.

What is the equation of the line tangent to f at x_0? Its slope is $f'(x_0)$, and it contains the point $(x_0, f(x_0))$. Using the point-slope form we know that the equation of this tangent line is:

(1) $$y - f(x_0) = f'(x_0)(x - x_0) \Rightarrow y = f(x_0) + f'(x_0)(x - x_0)$$

Notice from the picture that, the x intercept of this line is x_1. Notice also that x_1 is closer to the red dot than x_0. We want to find the x-intercept of this line. Solving for x when $y = 0$ in equation 1 we have:

$$0 = f(x_1) + f'(x_1)(x - x_1) \Rightarrow x - x_1 = -\frac{f(x_1)}{f'(x_1)} \Rightarrow x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

So $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

Notice that x_2 is still closer to the red dot than x_1.

Repeating this process to solve for the subsequent x_{n+1} gives the following recursive formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

We start with x_1, which is our best estimate as to what is the value of the root of the function. To find x_2, we plug x_1 into the above formula to get:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Once we found x_2 we can find x_3 by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

And we can find x_4, x_5, ... x_n however large n is. In the picture, x_1 is closer to the red dot than x_0, and x_2 is closer to the red dot than x_1

Using this method, we can approximate the root r to as accurate a degree as we want.

How do we know that we have gotten the degree of accuracy we want? Since we do not know what the actual root r is, how do we know if we are already within say 0.0001 of the actual root r? Notice that, since the x_n's are going to be very close to the actual root, they must also be very close to each other. For example, suppose the actual root is $r = 1$. If x_{10} is very close to 1, say $x_{10} = 1.0098$, then x_{11} should be even closer to r, say maybe x_{11} will be equal to 1.00087. Notice that x_{10} and x_{11} are also very close to each other. The difference between x_{10} and x_{11} is less than 1 in a 100. We will know that we are close to the actual root r when the x_n and x_{n+1} are close to each other. In general, if we want an accuracy of say 1 in 1000, then if x_n and x_{n+1} differs by less than 1 in 1000, the accuracy is probably achieved. (If we want to be safe, finding n so that x_n and x_{n+1} differs by less than 1 in 2000 would almost always do the work.)

Example: Estimate the solution of the equation

$$\cos x = \sqrt{x}$$

Ans: The function we need is:

$$f(x) = \cos x - \sqrt{x}$$

Then solving the equation is the same as finding the root of f.

Since

$$f'(x) = -\sin x - \frac{1}{2\sqrt{x}}$$

We have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\cos x_n - \sqrt{x_n}}{-\sin x_n - \frac{1}{2\sqrt{x_n}}}$$

We need to come up with the first estimate, x_1. Since $\sqrt{x}$ can only take non-negative numbers, we must pick only non-negative numbers as our x_1. Notice that $\cos x$ can never be greater than 1, so we should not pick a number that is too large either. A reasonable number to pick will be a number between 0 and 1. Let's choose $x_1 = 1$. Then we have:

$$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{\cos 1 - \sqrt{1}}{-\sin 1 - \frac{1}{2\sqrt{1}}} \approx 0.657318$$

The table below lists the values of the x_n's up to x_6:

n	x_n
1	1
2	0.657318
3	0.641746
4	0.641714
5	0.641714
6	0.641714

Notice that the values of the x_n's begin to repeat around the number 0.641714. Therefore, 0.641714 is the root of the function $f(x) = \cos x - \sqrt{x}$, which means it is the solution of the equation $\cos x = \sqrt{x}$.
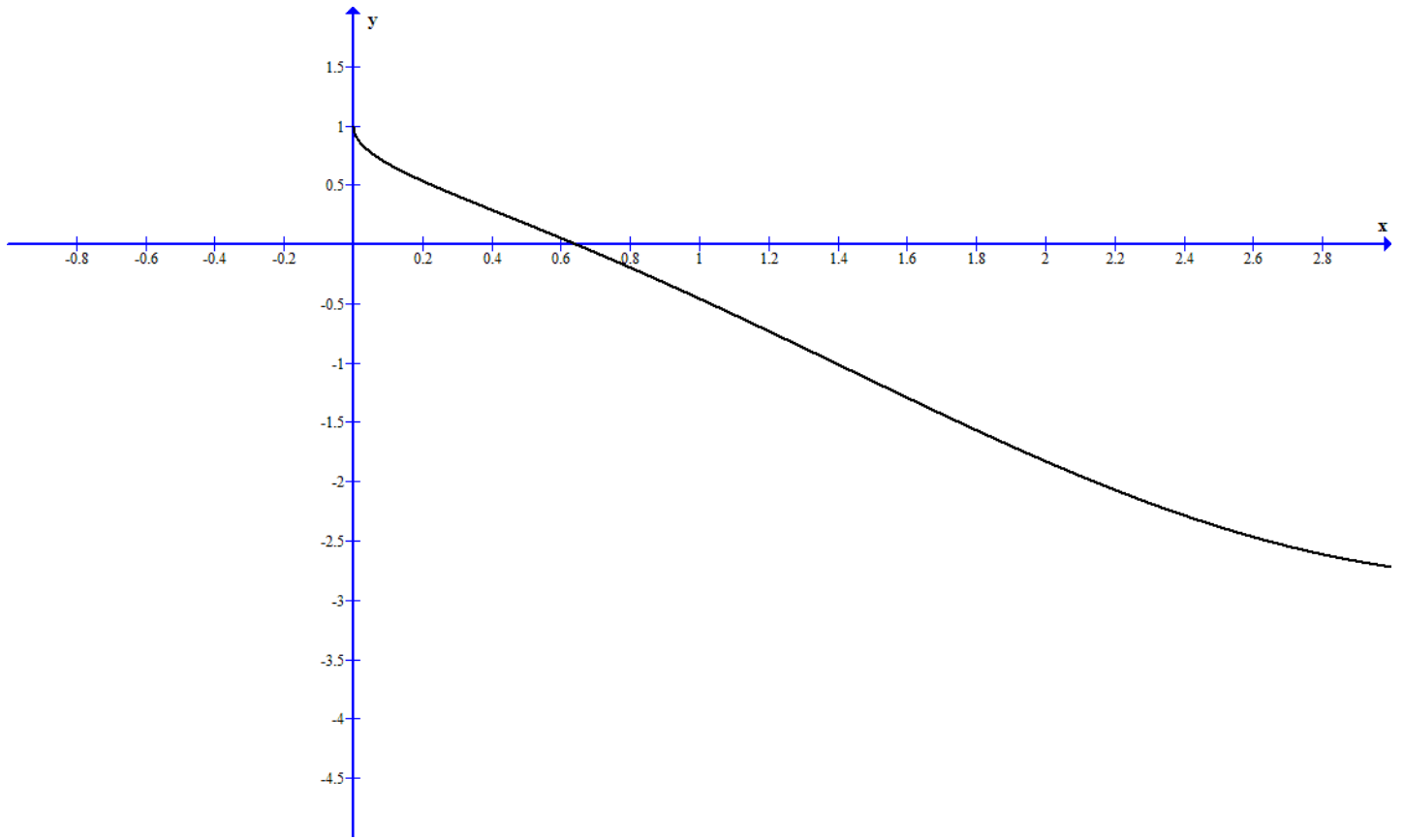
Example: Find all the roots of the function:

$$f(x) = x^5 - x^4 - 5x^3 - x^2 + 4x + 3$$
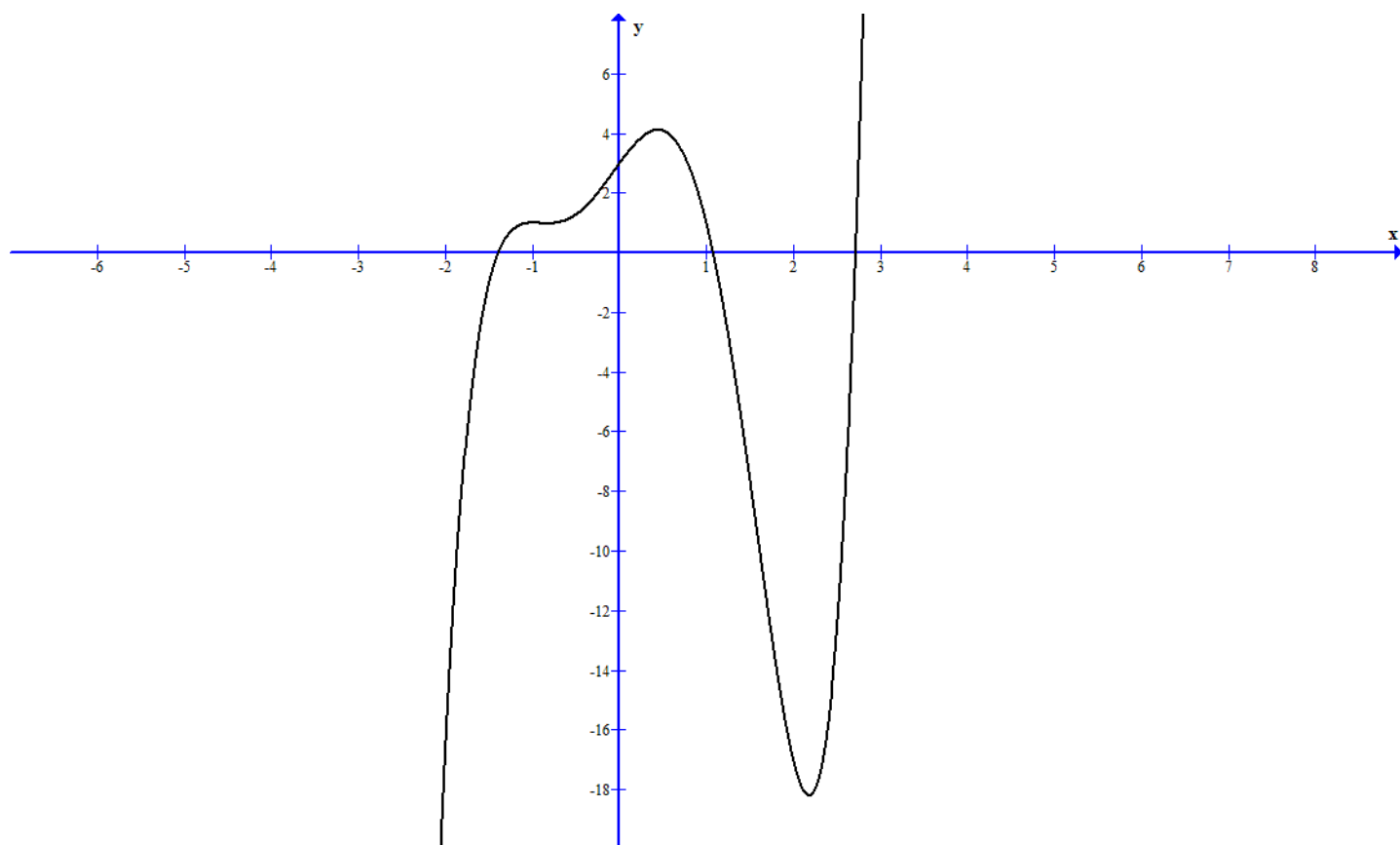
If we look at the graph of f, we see that f has 3 roots. Their values are approximately -2, 1, and 3. To find the roots of f, it makes sense to start with these numbers:

The derivative of f is $f'(x) = 5x^4 - 4x^3 - 15x^2 - 2x + 4$. So using Newton's method we have:

$$x_{n+1} = x_n - \frac{x^5 - x^4 - 5x^3 - x^2 + 4x + 3}{5x^4 - 4x^3 - 15x^2 - 2x + 4}$$

With $x_1 = -2$ we have:

n	x_n
1	-2
2	-1.716667
3	-1.526545
4	-1.424909
5	-1.394487
6	-1.391963
7	-1.391947
8	-1.391947

The values of x_n cluster around -1.391947, so we expect this to be one of the roots of the function.

In this case, we saw that the the value of the root is near -1. However, if we have started with $x_1 = -1$, Newton's method is actually going to *fail*. The reason is that $f'(-1) = 0$, so the expression $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ will be undefined if we pick -1 as our x_1. This is something you will simply have to watch out for when applying Newton's method. Newton's method is flexible enough so that your choice of x_1 is not *too* important. However, occasionally your choice of x_1 will fail to work. If this is the case you simply have to choose another x_1. In this example if you have started with $x_1 = -1$, then you will find out that x_2 is undefined, so you will choose a different x_1.

Now let's try $x_1 = 1$

n	x_n
1	1
2	1.083333
3	1.077424
4	1.077394
5	1.077394

The values of x_n cluster around 1.077394, so this is another root of the function.

Notice in this example that, depending on what the starting value x_1 you choose, you might get *different* roots. This is because f has different roots, and if you choose a value x_1 that is close to root r_1, you will most likely get r_1 as your root. What happens if you are interested in root r_2 but you happened to have started with x_1? For example, in the above case, what if you are interested in the root 1.077394, but for some reason you started with $x_1 = -2$? This is something you should be careful when applying Newton's method and can be avoided only if you have some idea of where the root lies so you can choose a value as close to the root as possible. Remember that Newton's method is *not* a method used to study the behavior of a function. It is a numerical method used to solve equations which we may not otherwise be able to solve using algebraic methods. Idealy, before applying Newton's method, we already have an approximation of what the root(s) is(are). This can be accomplished by graphing the function using the graphing methods we introduced earlier, or a graphing calculator.

We find the last root of f by starting with $x_1 = 3$.

n	x_n
1	3
2	2.793750
3	2.726723
4	2.719944
5	2.719878
6	2.719878

The values of x_n cluster around 2.719878, so this is the third (and final) root of the function.