Some facts you should know that would be convenient when evaluating a limit:

When evaluating a limit of fraction of two functions,

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

If f and g are both continuous inside an open interval that contains a (but not necessarily at a), we can catagorize the nature of the limit by what the numerator and denominator approaches as $x \to a$:

Division by a Non-zero number: $\boldsymbol{\frac{f(x)}{c}}$

If $g(x) \not\to 0$ as $x \to a$, then $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

Example:

$$\lim_{x\to 3}\frac{x+4}{x-5} = -\frac{7}{2}$$

In this example, $f(x) = x+4$, $g(x) = x-5$, since f and g are both continuous functions, and $g(x) \not\to 0$ as $x \to 3$, we can evaluate the limit by simply plug in the value:

$$\lim_{x\to 3}\frac{x+4}{x-5} = \frac{3+4}{3-5} = \frac{7}{-2} = -\frac{7}{2}$$

This result simply means that we can always perform a division by a non-zero real number.

Non-zero divided by zero: $\boldsymbol{\frac{c}{0}}$

If $f(x) \not\to 0$ but $g(x) \to 0$ as $x \to a$, then the limit will be either ∞ or $-\infty$, or undefined.

Example: $\lim\limits_{x\to 1}\frac{x+3}{x-1}$

In this example, the denominator is $g(x) = x-1$ while the numerator is $f(x) = x+3$. $g(x) \to 0$ while $f(x) \not\to 0$ as $x \to 1$, therefore the limit will either be ∞ or $-\infty$. In fact,

$$\lim_{x\to 1^-}\frac{x+3}{x-1} = -\infty$$

$$\lim_{x \to 1^+} \frac{x+3}{x-1} = \infty$$

This is the result of when a non-zero number is divided by a very small number, the result will be a very large number. To put it simply, **non-zero divided by zero is infinity.**

Indeterminate Form: Zero divided by Zero: $\frac{\mathbf{0}}{\mathbf{0}}$

This is the more interesting case and usually requires more algebra. What this means is that, as $x \to a$, both $f(x) \to 0$ and $g(x) \to 0$.

Understand that a fraction will be a small number if the numerator is small (close to zero), but a fraction will be a large number if the denominator is small. In the indeterminate case of $\frac{0}{0}$, we have a *competition* between the numerator and the denominator. Since both the numerator and denominator will approach 0 as $x \to a$, We are taking a small number divided by another small number, and the result will depend on which function (numerator or denominator) goes to zero faster. If the numerator *wins*, meaning that $f(x)$ goes to 0 faster than $g(x)$, the resulting limit will be 0. If the denominator *wins*, meaning that $g(x)$ goes to 0 faster than $f(x)$, the resulting limit will be ∞ or $-\infty$. If the two functions go to 0 at about the same rate, the resulting limit could be any non-zero real number depending on the function.

Example:

Consider the following limits:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

$$\lim_{x \to 1^+} \frac{\ln x}{x^2 - 2x + 1} = \infty$$

In all three of the above fractions, both the numerator and denominator approaches 0 as x approaches a, but these three limits have different values.

A similar argument can used to approach limit problems where $f(x) \to \infty$ or

$g(x) \to \infty$ as $x \to a$. In this argument the result will still work if we replace any real number a by the ∞ or $-\infty$ symbol, assuming that f and g are continuous in the appropriate region.

Infinity divided by Finite: $\boldsymbol{\frac{\infty}{c}}$

If $f(x) \to \pm\infty$ and $g(x) \not\to \pm\infty$ as $x \to a$ or as $x \to \infty$, then the limit will be ∞ or $-\infty$ or undefined.

Example:

$$\lim_{x\to\infty} \frac{\ln x}{5} = \infty$$

In this example, the numerator is $f(x) = \ln x$, and the denominator is $g(x) = 5$. $f(x) \to \infty$ as $x \to \infty$, while $g(x) \to 5$, the result of this limit is ∞.

The above means that a number that is extremely large divided by a number that is not as large, the result will still be a large number. Intuitively, **infinity divided by finite is still infinite.**

Finite divided by Infinity: $\boldsymbol{\frac{c}{\infty}}$

If $f(x) \not\to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$ or as $x \to \infty$, then the limit will be 0

Example: $\lim_{x\to\infty} \frac{\sin x}{x} = 0$

In this example, the numerator is $f(x) = \sin x$, and the denominator is $g(x) = x$. $f(x) \not\to \infty$ as $x \to \infty$, but $g(x) \to \infty$, so this limit is 0.

In other words, **finite divided by infinite is 0**

textbfIndeterminate form: Infinity divided by Infinity: $\frac{\infty}{\infty}$

This is another case where additional algebra will be needed. If both $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$ or $x \to \infty$, then the limit could be any real number, or $\pm\infty$, or undefined.

Example:

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0$$

$$\lim_{x\to\infty} \frac{x^2-3x+4}{2x^2+5} = \frac{1}{2}$$

$$\lim_{x\to\infty} \frac{e^x}{x} = \infty$$

The situation in this indeterminate case is similar to the $\frac{0}{0}$ indeterminate case. A large denominator will make the fraction small, while a large numerator will make the fraction big, so a $\frac{\infty}{\infty}$ case will depend on which function *wins*. If the numerator goes to infinity faster than the denominator, the result will be infinity. If the denominator goes to infinity faster than the numerator, the result will be 0. If the numerator and denominator goes to infinity at about the same rate, the result could be any non-zero real number depending on the function.

The following limits use some of the facts we just mentioned:

Let c be any constant,

$$\lim_{x\to\infty} \frac{c}{\ln x} = 0$$

if $p > 0$, $\lim\limits_{x\to\infty} \frac{c}{x^p} = 0$

if $b > 1$ $\lim\limits_{x\to\infty} \frac{c}{b^x} = 0$

if $0 \le r < 1$, $\lim\limits_{x\to\infty} r^x = 0$

There are other types of indeterminate forms where we have a *competition* between two functions where the value of one function would cause the limit to behave one way, but the value of the other function would cause the limit to behave another way. The result will depend on which function *wins*, or which function approaches its limit value at a faster rate. The following are all indeterminate forms:

$$\mathbf{\frac{0}{0}}, \quad \mathbf{\frac{\infty}{\infty}}, \quad \mathbf{0\cdot\infty}, \quad \mathbf{0^0}, \quad \mathbf{\infty^0}, \quad \mathbf{1^\infty},$$

We now introduce a useful theorem that helps us to evaluate the limit of an indeterminate form:

L'Hospital's Rule:

Suppose f and g are differentiable functions and $g'(x) \neq 0$ for some open interval that contains a (except possibly at a). If any one of the following two condition is satisfied:

i) $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$

ii) $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

In other words, we have a limit of the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

The above formula is still valid if a is replaced by ∞ or $-\infty$.

Before trying to use L'Hospital's Rule to evaluate a limit, you must first make sure that the limit satisfy the criterion for L'Hospital's rule. L'Hospital's rule can be applied **only** when we have an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. You may **not** apply L'Hospital's rule if you have something like $\frac{c}{0}$ or $\frac{c}{\infty}$

Example: Do not try to apply L'Hospital's rule for the following limit:

$$\lim_{x \to \infty} \frac{\sin x}{x^2}$$

We cannot apply L'Hospital's rule for this limit because, while the denominator, $g(x) = x^2$, approaches ∞ as $x \to \infty$, the numerator, $f(x) = \sin x$, does not approach ∞ or $-\infty$ as $x \to \infty$, so this does not fit the hypothesis of L'Hospital's rule, hence the rule cannot be applied.

Another thing to note is that, when applying L'Hospital's rule, you will try to evaluate the limit by taking the derivative of the numerator and the denominator as individual functions. Do **not** try to differentiate the function as a single fraction.

Example:

$$\lim_{x\to 0}\frac{\sin x}{x}$$

For this limit, since $\sin x \to 0$ and $x \to 0$ as $x \to 0$, this is an indeterminate form $\frac{0}{0}$. We may apply L'Hospital's rule. Since the derivative of $\sin x$ is $\cos x$ and the derivative of x is 1, we have:

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$$

It will be *incorrect* if you try to differential the function $\frac{\sin x}{x}$ as a single fraction when trying to apply L'Hospital's rule.

$$\lim_{x\to 0}\frac{\sin x}{x} \neq \lim_{x\to 0}\frac{-x\cos x - \sin x}{x^2}$$

Example:

Evaluate: $\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$

Ans: As $x \to \infty$, $\ln x \to \infty$, and $\sqrt{x} \to \infty$, so we may apply L'Hospital's rule:

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty}\frac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty}\frac{2\sqrt{x}}{x} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0$$

When we apply L'Hospital's rule, we are trying to turn an indeterminate form into a form that is no longer indeterminate. Once we are able to do that, we will hopefully be able to find out the behavior of the limit.

Example:

Evaluate the limit: $\lim_{x\to 0^+} x\ln x$

Ans: This limit is one of the indeterminate form, $0 \cdot \infty$, that we mentioned before. However, it is not in one of the two indeterminate form that we can apply L'Hospital's rule to. In order to turn this into a form that we can apply L'Hospital's rule, we use the fact that for any real number x, $x = \frac{1}{x^{-1}} = \frac{1}{1/x}$

In order to apply L'Hospital's Rule, we first rewrite $x\ln x = \frac{\ln x}{1/x}$

As $x \to 0^+$, $\ln x \to -\infty$ and $\frac{1}{x} \to \infty$, now we may apply L'Hospital's Rule:

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -\frac{x^2}{x} = \lim_{x\to 0^+} -x = 0$$

Example:

Evaluate the limit: $\lim_{x\to\infty} \left(3x^2+1\right)^{1/x^2}$

Ans: This expression is another indeterminate form: ∞^0. In order to apply L'Hospital's rule on this, we use the fact that:

$\ln\left(a^b\right) = b\ln a.$

Let $y = \left(3x^2+1\right)^{1/x^2}$, then $\ln(y) = \ln\left[\left(3x^2+1\right)^{1/x^2}\right] = \frac{1}{x^2}\cdot\ln\left(3x^2+1\right).$

We first evaluate: $\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{1}{x^2}\cdot\ln\left(3x^2+1\right)$

$$\lim_{x\to\infty} \frac{1}{x^2}\cdot\ln\left(3x^2+1\right) = \lim_{x\to\infty} \frac{\ln\left(3x^2+1\right)}{x^2}$$

As $x\to\infty$, $\ln(3x^2+1)\to\infty$, and $x^2\to\infty$, so we may apply L'Hospital's rule:

$$\lim_{x\to\infty} \frac{\ln\left(3x^2+1\right)}{x^2} = \lim_{x\to\infty} \frac{6x/(3x^2+1)}{2x} = \lim_{x\to\infty} \frac{6x}{2x\left(3x^2+1\right)} = \lim_{x\to\infty} \frac{6x}{6x^3+2x}$$

We apply L'Hospital's rule one more time:

$$\lim_{x\to\infty} \frac{6x}{6x^3+2x} = \lim_{x\to\infty} \frac{6}{18x^2+2} = 0$$

So we have:

$$\lim_{x\to\infty} \frac{1}{x^2}\left(3x^2+1\right) = \lim_{x\to\infty} \ln y = 0$$

Using the fact the $\ln x$ is a continuous function, we have:

$$\lim_{x\to\infty} \ln(y) = 0 \Rightarrow \ln\left(\lim_{x\to\infty} y\right) = 0 \Rightarrow \lim_{x\to\infty} y = e^0 = 1 \Rightarrow \lim_{x\to\infty} \left(3x^2+1\right)^{1/x^2} = 1$$

Understand that L'Hospital's rule *usually* does not allow you to find the limit of a function directly. Instead, it allows you to change an indeterminate form to a format that is no longer indeterminate, at which point you may be able to find the value of the limit using other knowledge.

Example: Evaluate: $\lim_{x\to\infty} \frac{2x+3}{e^x}$

Answer: As $x\to\infty$, both $2x+3\to\infty$ and $e^x\to\infty$, so this is an indeterminate form $\dfrac{\infty}{\infty}$, we may apply L'Hospital's Rule:

$$\lim_{x\to\infty} \frac{2x+3}{e^x} = \lim_{x\to\infty} \frac{2}{e^x}$$

At this point, the limit is no longer indeterminate, since the numerator does not approach ∞ as $x \to \infty$. We may no longer use L'Hospital's rule, but we do not need to. As $x \to \infty$, the denominator $e^x \to \infty$ while the numerator is a constant, this is the finite divided by infinite case, the result is 0:

$$\lim_{x\to\infty} \frac{2}{e^x} = 0$$