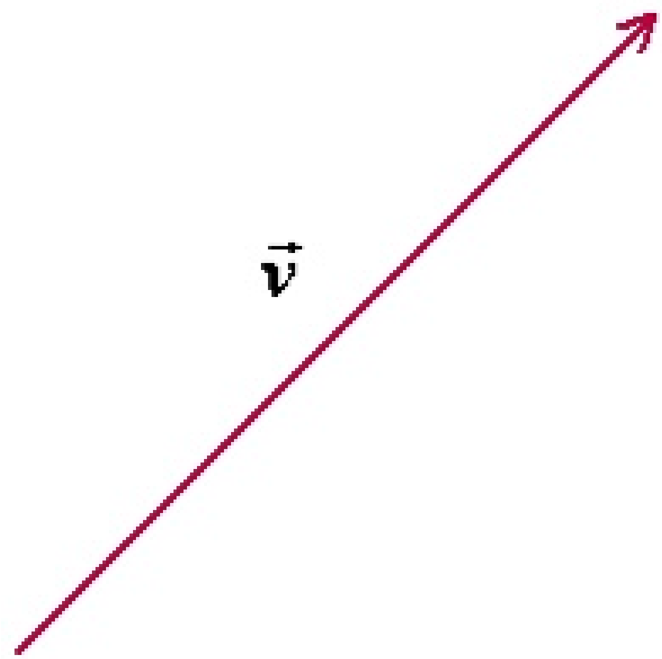
Vectors

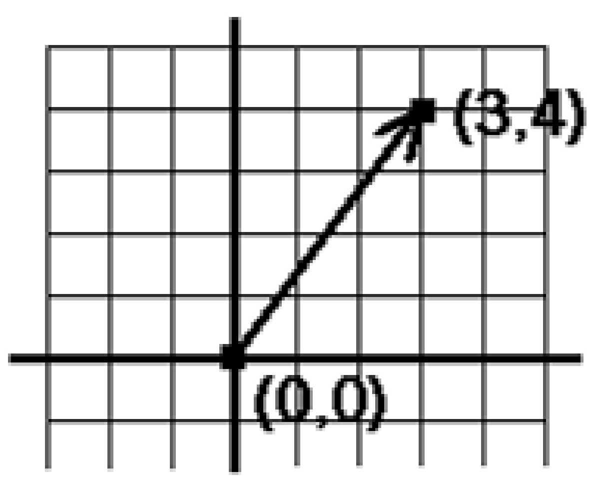
Intuitively, a **vector** is a mathematical object that has both a **magnitude** and a **direction**. An arrow is a vector. The tip of the arror points to the direction of the vector, and the length of the arror is the magnitude of the vector.



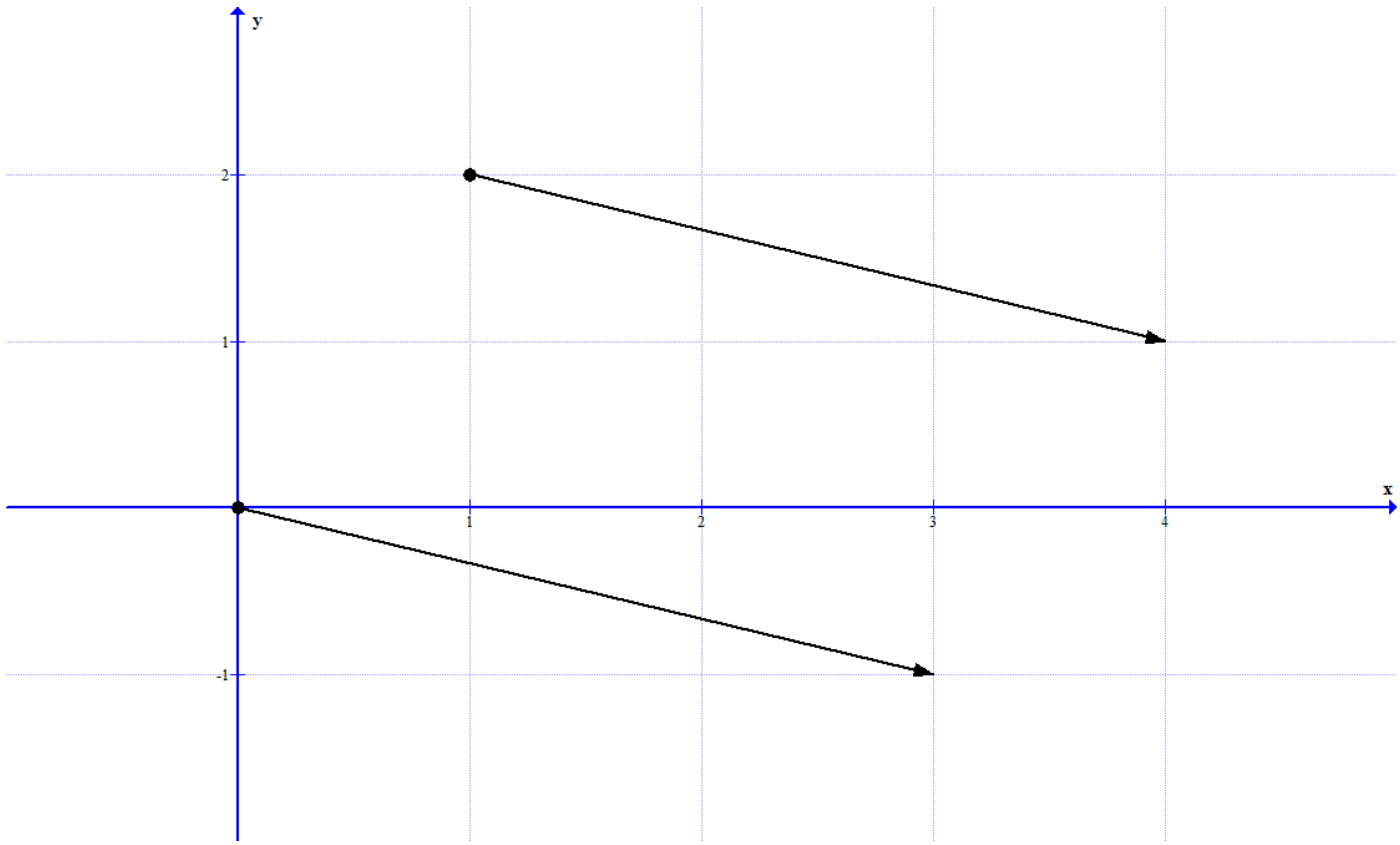
Many quantities in physics have both a direction and a magnitude. Velocity is an example. It matters not just how fast you are going, but also where you are going.

In $\mathbf{R}^2$, we use $\mathbf{v} = \langle x, y \rangle$ to represent a vector. The coordinates x, y gives the location of the *tip* of the vector, and we always assume that the *tail* of the vector is at the origin.

Example: The vector $\mathbf{v} = \langle 3, 4 \rangle$ is the vector with its tail at the origin and its tip at the point $(3, 4)$.



It is important to note that, even though by assumption we said that the vector $\mathbf{v} = \langle x, y \rangle$ always has its tail at the origin, the *location* of a vector is irrelevant. The reason we need to specify that its tail is at the origin is because we want to know to which direction that vector is pointing. Therefore, the vector $\mathbf{v} = \langle 3, -1 \rangle$ (with its tail at the origin) and the vector $\mathbf{w}$ with its tail at $(1, 2)$ and its tip at $(4, 1)$ are exactly the same vector, because they have the same length and points to the same direction.



In other words, a vector is distingushed only by its magnitude and direction, **not** by its location in space.

We distingush between a point and a vector by using bold type when printing vector. So $p = (1, 2)$ represents the point located at that particular coordinate, while $\mathbf{v} = \langle 1, 2 \rangle$ represents the vector that has its tail at the origin and its tip at the point $(1, 2)$. Sometimes we also write $\vec{v}$ to mean a vector.

For a point (x, y), we say that the vector $\mathbf{v} = \langle x, y \rangle$ is the **position vector** of the given point. In short, the position vector of a given point in space is a vector whose tip points to that point (and its tail is at the origin).

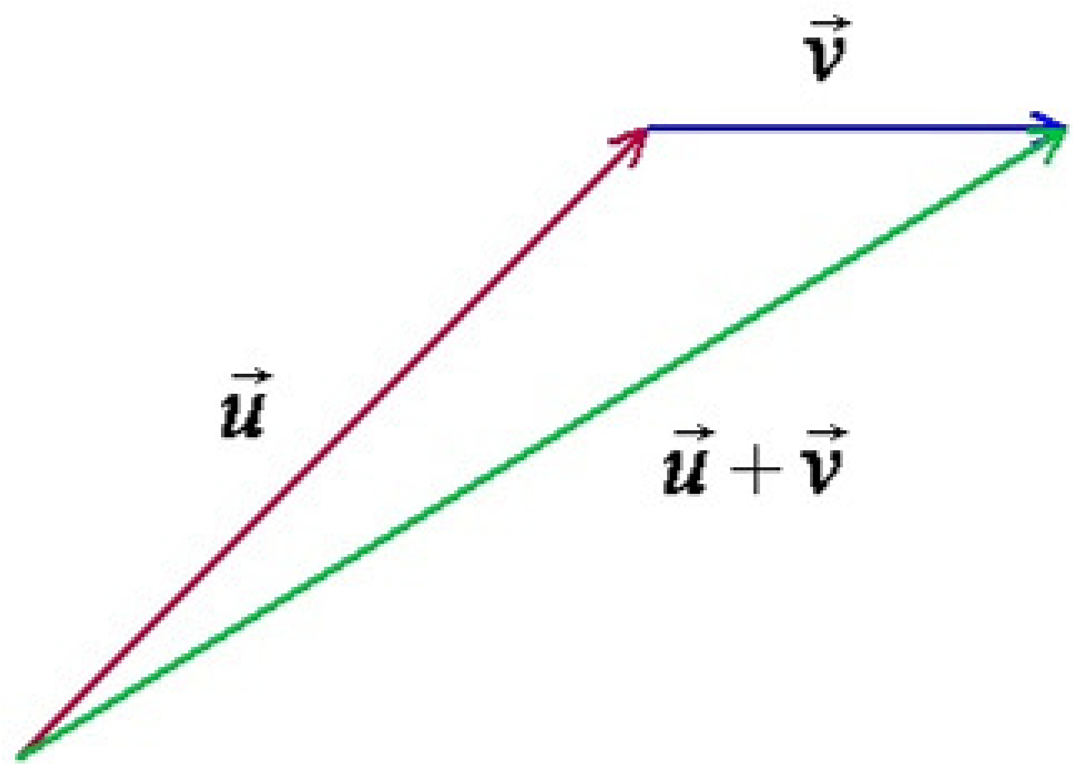
Vector Addition

Let $\mathbf{u} = \langle x_1, y_1 \rangle$ and $\mathbf{v} = \langle x_2, y_2 \rangle$, we define

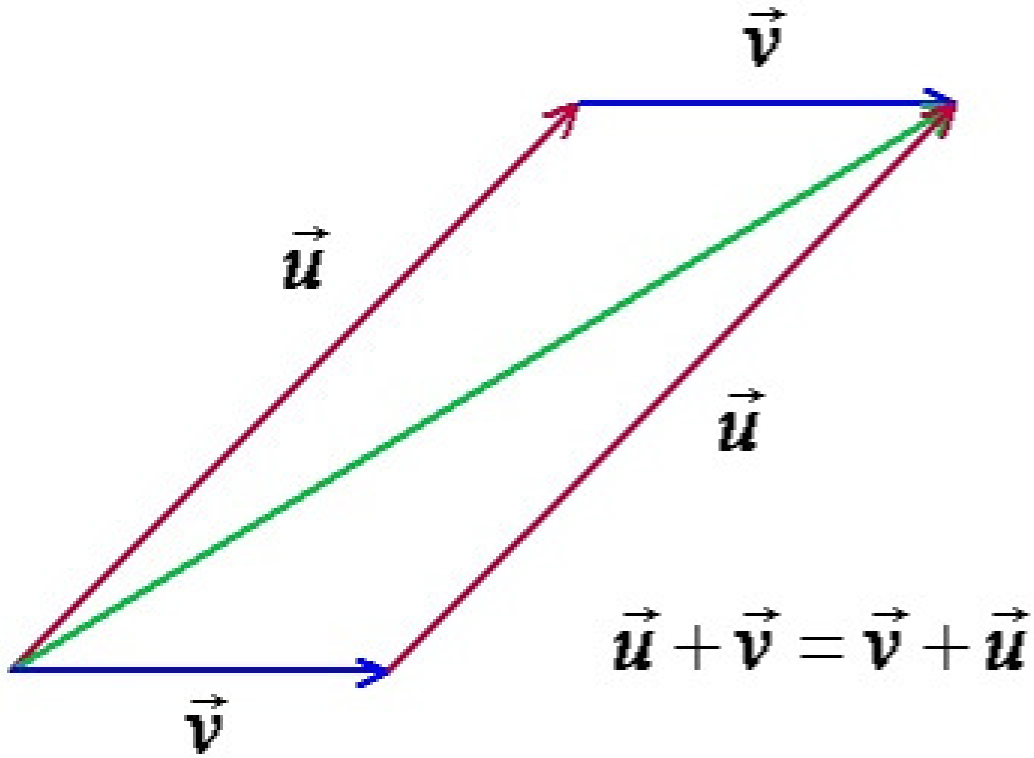
$$\mathbf{u} + \mathbf{v} = \langle x_1 + x_2, y_1 + y_2 \rangle$$

In other words, to add two vectors, add their components correspondingly.

Geometrically, when adding two vectors, the resulting vector is obtained by placing the tail of the second vector ($\mathbf{v}$) on the tip of the first vector ($\mathbf{u}$), and the result is the vector with its tail at the tail of the first vector ($\mathbf{u}$) and its tip at the tip of the second vector ($\mathbf{v}$).



Also notice that vector multiplication is **commutative**. In other words, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all vectors $\mathbf{u}$ and $\mathbf{v}$



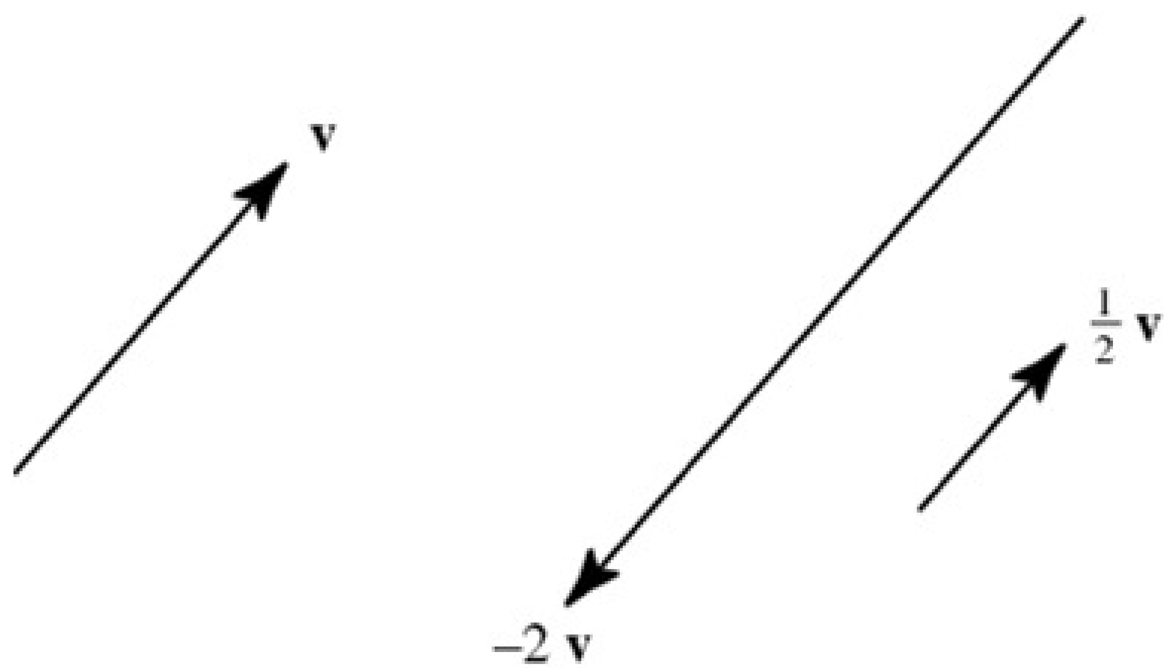
Scalar Multiplication

If $\mathbf{v} = \langle x, y \rangle$ is a vector, and c is a real number (a **scalar**), then we define

$$c\mathbf{v} = \langle cx, cy \rangle$$

To multiply a number to a vector, we multiply the number to each of the components of the vector.

Geometrically, multiplying a scalar c to a vector stretches or shrinks the vector by a factor of $|c|$ units, and reverses the direction of the vector if c is negative.

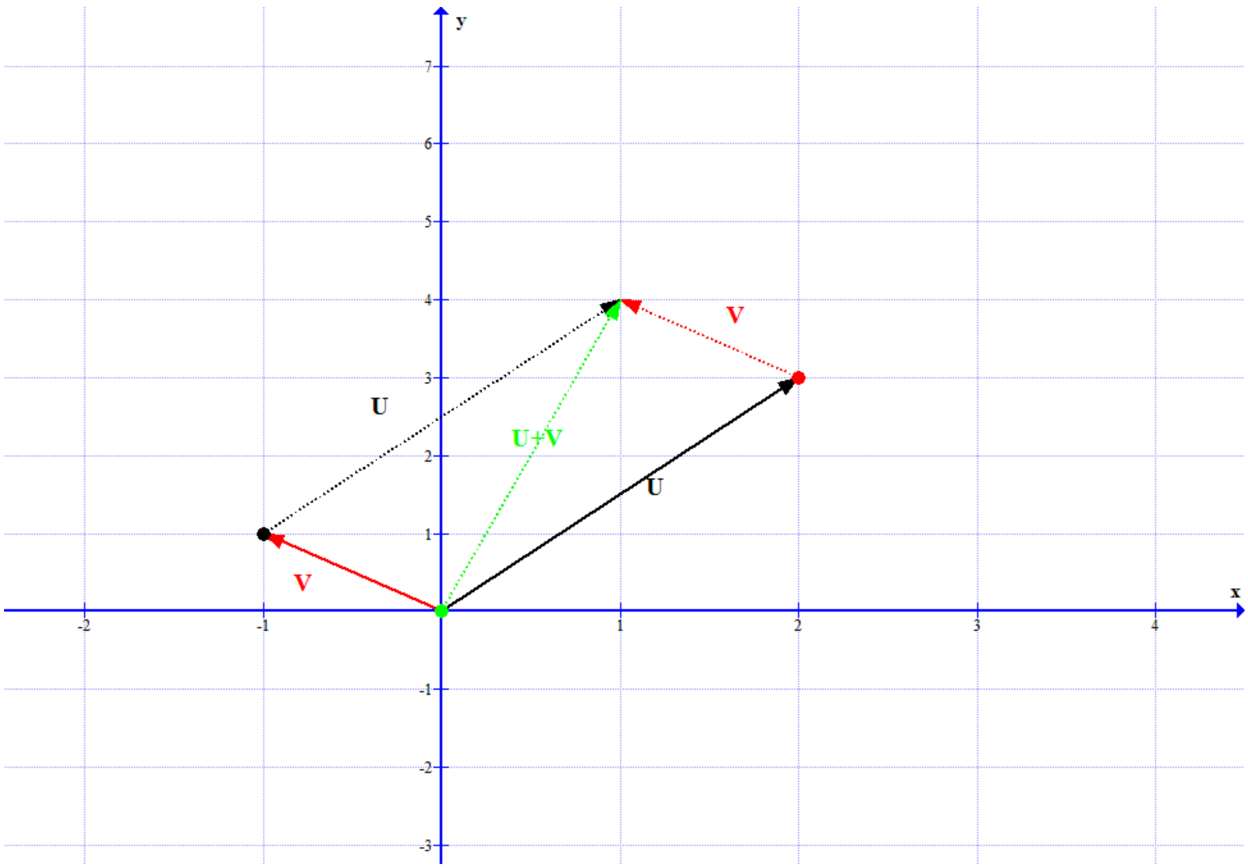


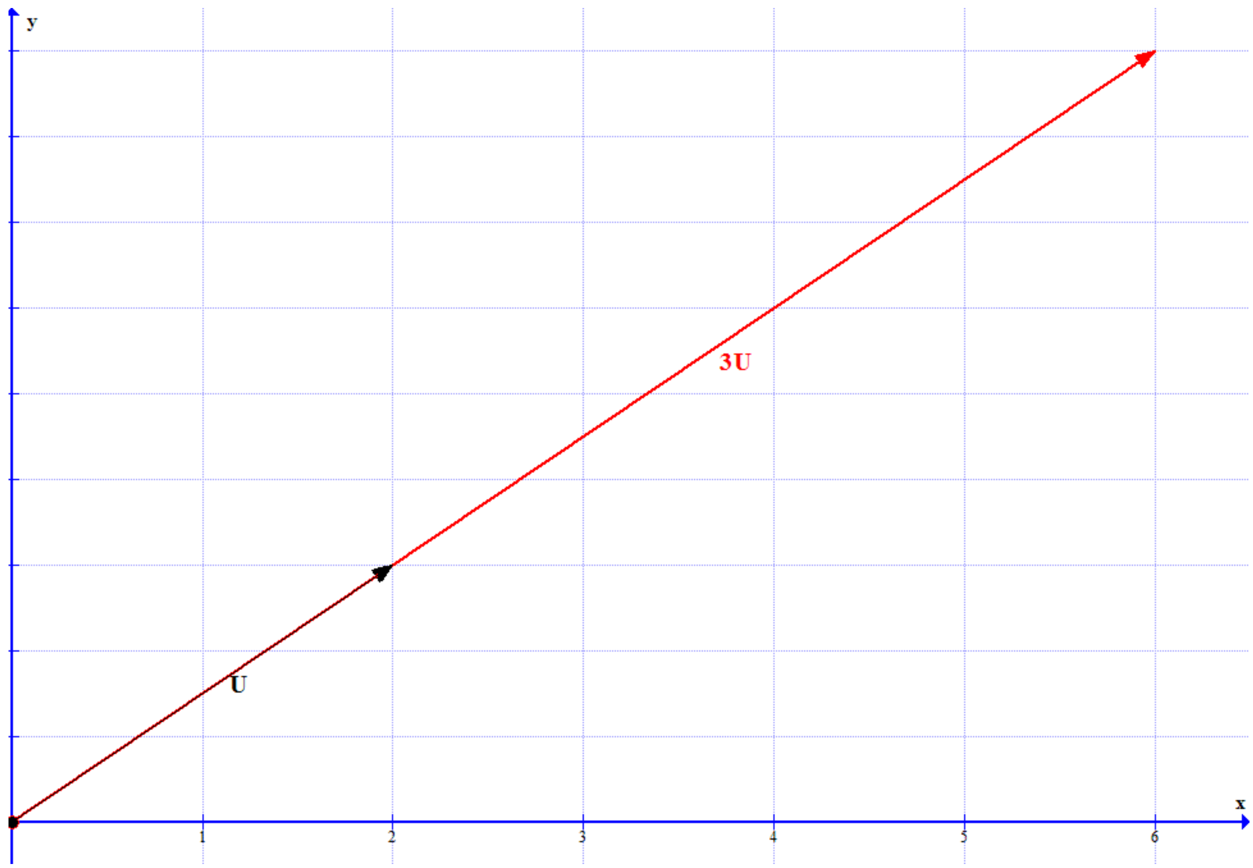
Example:

Let $\mathbf{u} = \langle 2, 3 \rangle$, $\mathbf{v} = \langle -1, 1 \rangle$

then

$$\mathbf{u} + \mathbf{v} = \langle 2 + -1, 3 + 1 \rangle = \langle 1, 4 \rangle$$
$$3\mathbf{u} = \langle 3 \cdot 2, 3 \cdot 3 \rangle = \langle 6, 9 \rangle$$

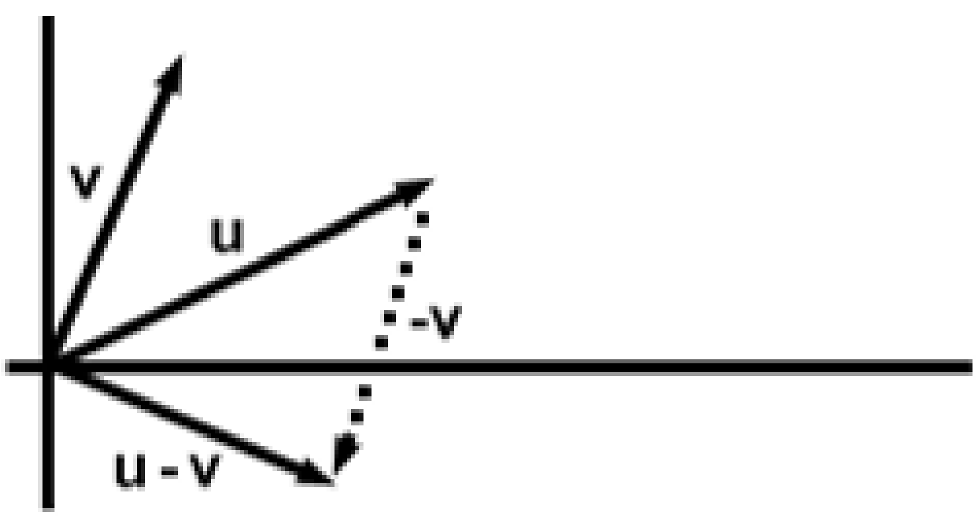




Notation: The vector $\mathbf{0} = \langle 0, 0 \rangle$ is the *zero vector*. It has both it tip and tail at the origin. The magnitude (length) of the zero vector is 0, and it is the only vector whose direction is arbitrary.

Notice that for any scalar c and any vector $\mathbf{v}$, $\mathbf{v} + \mathbf{0} = \mathbf{v}$, and $c\mathbf{0} = \mathbf{0}$. In this sense, the zero vector serves the same function as the number 0 when working with vectors.

We also write $-\mathbf{v}$ for the vector $(-1)\mathbf{v}$. When we write $\mathbf{u} - \mathbf{v}$, we mean $\mathbf{u} + -\mathbf{v}$.



We say that two vectors $\mathbf{v}$ and $\mathbf{w}$ are *parallel* to each other if there is a scalar $c > 0$ such that $\mathbf{v} = c\mathbf{w}$.

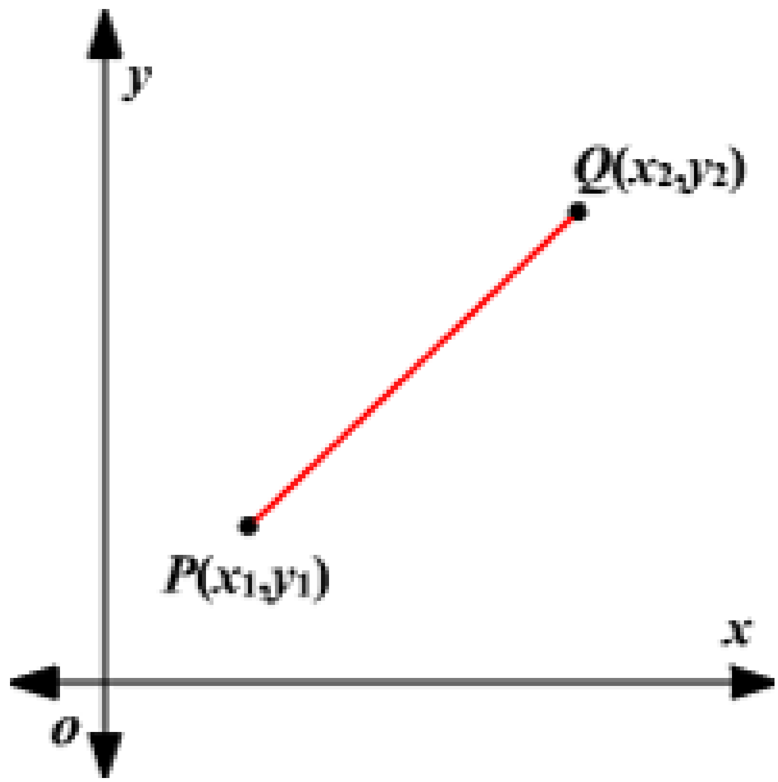
Geometrically, two vectors are parallel to each other if they point to the same direction.

Two vectors are **anti-parallel** if they point to opposite direction. In other words, $\mathbf{v}$ and $\mathbf{w}$ are anti-parallel if there exists a number $c < 0$ such that $\mathbf{v} = c\mathbf{w}$.

Notation: Sometimes we want to specify a particular vector that has its tail not at the origin. Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are two points in $\mathbf{R}^2$, then the vector represented by

$$\vec{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

is the vector with its tip at Q and its tail at P.



The **magnitude**, or **length** of a vector $\mathbf{v} = \langle x, y \rangle$ is represented by $|\mathbf{v}|$ and is given by the formula:

$$|\mathbf{v}| = \sqrt{x^2 + y^2}$$

This should be expected since it is just the distance formula restated.

A vector whose magnitude is 1 is called a **unit vector**.

Given a non-zero vector $\mathbf{v}$, we can form a unit vector that points to the same direction as $\mathbf{v}$. We do that by dividing $\mathbf{v}$ by its magnitude.

Example: $\mathbf{v} = \langle 1, 3 \rangle$, so $|\mathbf{v}| = \sqrt{1+9} = \sqrt{10}$, and the vector $\mathbf{u} = \frac{\mathbf{v}}{\sqrt{10}} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$ is a unit vector that points to the same direction as $\mathbf{v}$.

Important Note: It is very important to distingush the difference between a number (scalar) and a vector. We can add two numbers, or add two vectors, and we can multiply a number to a vector. However, we cannot add a number with a vector, and we cannot multiply or divide two vectors. It makes sense to write $\frac{\mathbf{v}}{|\mathbf{v}|}$, because the magnitude of a vector is a number, and multiplying a number to a vector is defined. However, it *never* makes sense to write $\frac{\mathbf{v}}{\mathbf{w}}$, where both $\mathbf{v}$ and $\mathbf{w}$ are vectors.

Dot Product

If $\mathbf{v} = \langle x_1, y_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2 \rangle$ are vectors, then the **dot product** of $\mathbf{v}$ and $\mathbf{w}$, denoted by $\mathbf{v} \cdot \mathbf{w}$, is given by

$$\mathbf{v} \cdot \mathbf{w} = x_1 x_2 + y_1 y_2$$

Example:

$\mathbf{v} = \langle 2, -2 \rangle$, $\mathbf{w} = \langle 1, -3 \rangle$, then

$\mathbf{v} \cdot \mathbf{w} = 2 \cdot 1 + -2 \cdot -3 = 8$

Notice that the dot product of two vectors is a scalar.

The following can be easily proved using the definition of the dot products:

If $\mathbf{v}, \mathbf{w}, \mathbf{s}$ are vectors and c is a scalar, then:

$$\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$$

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$$

$$\mathbf{v} \cdot (\mathbf{w} + \mathbf{s}) = \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{s}$$

$$(c\mathbf{v}) \cdot \mathbf{w} = c(\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (c\mathbf{w})$$

$$\mathbf{0} \cdot \mathbf{v} = 0$$

$$\mathbf{v} \cdot \mathbf{v} \geq 0, \quad \text{and} \quad \mathbf{v} \cdot \mathbf{v} = 0 \quad \text{if and only if} \quad \mathbf{v} = \mathbf{0}$$

One of the most useful feature of the dot product between two vectors is that it gives us geometry of the vectors. If θ is the angle formed between two vectors $\mathbf{v}$ and $\mathbf{v}$, then we have:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos \theta$$

This theorem tells us that the dot product of two vectors is just the product of the length of the vectors times the cosine of the angle between the vectors.

We know from geometry that two lines are perpendicular if the angles they formed is $\pi/2$, we use the same concept for vectors. We say that two vectors are perpendicular, or **orthogonal**, if the angle between them is $\pi/2$. Since $\cos(\pi/2) = 0$, we have the following:

Two vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if $\mathbf{v} \cdot \mathbf{w} = 0$