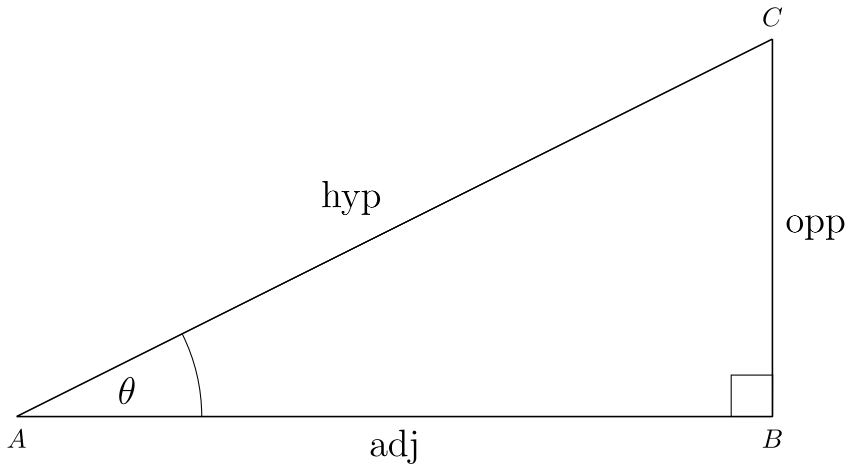
Definition of Trigonometric Functions using Right Triangle:



Given **any** right triangle ABC, suppose angle θ is an angle inside ABC, label the leg opposite θ the **opposite** side, label the leg adjacent to θ the **adjacent** side, and the side opposite the right angle is the **hopotenuse**. We define the six trigonometric functions of θ as the ratio of the lengths of two of the sides of the triangle:

(the **sine** function) $$\sin\theta = \frac{\text{opp}}{\text{hyp}}$$

(the **cosine** function) $$\cos\theta = \frac{\text{adj}}{\text{hyp}}$$

(the **tangent** function) $$\tan\theta = \frac{\text{opp}}{\text{adj}}$$

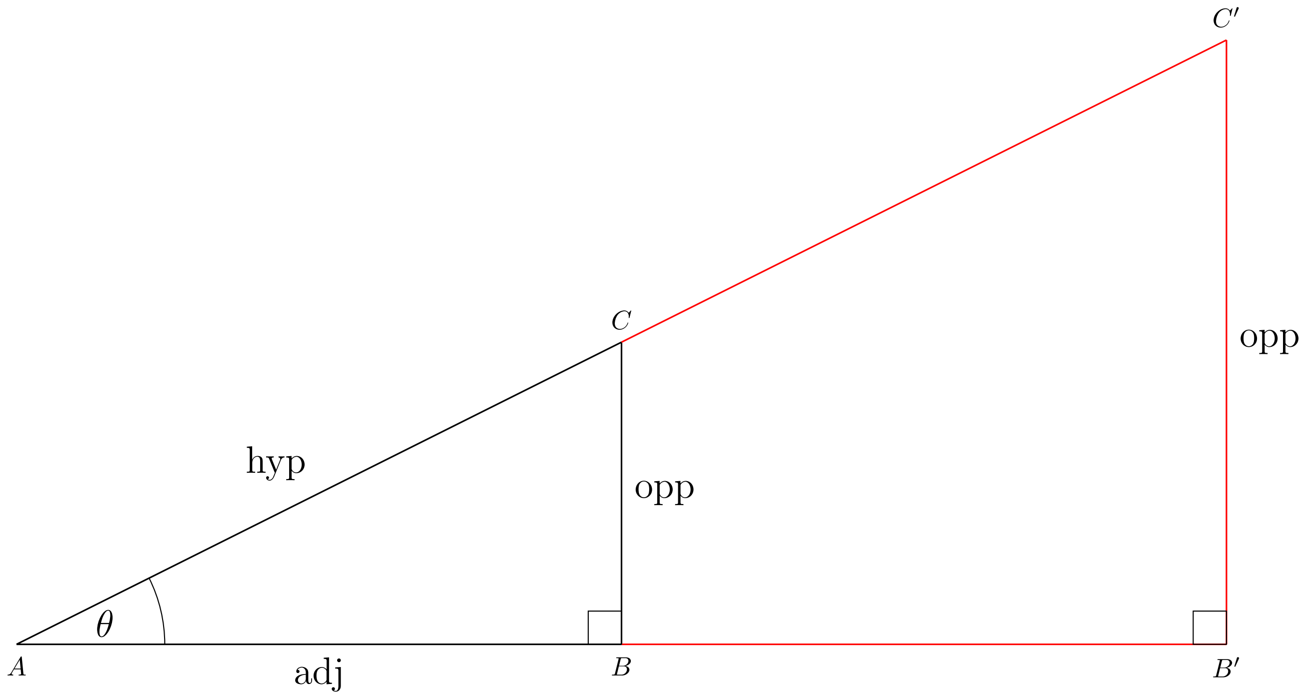
(the **secant** function) $$\sec\theta = \frac{\text{hyp}}{\text{adj}}$$

(the **cosecant** function) $$\csc\theta = \frac{\text{hyp}}{\text{opp}}$$

(the **cotangent** function) $$\cot\theta = \frac{\text{adj}}{\text{opp}}$$

Using this definition, the six trigonometric functions are well-defined for all angles θ where $0^\circ < \theta < 90^\circ$

In defining the six trigonometric functions using a right triangle, we wanted to have consistency. That is, if we used a different right triangle with the same angle θ, we wanted to know that we will get the same value for each of the functions. This will be the case as illustrated by the following:



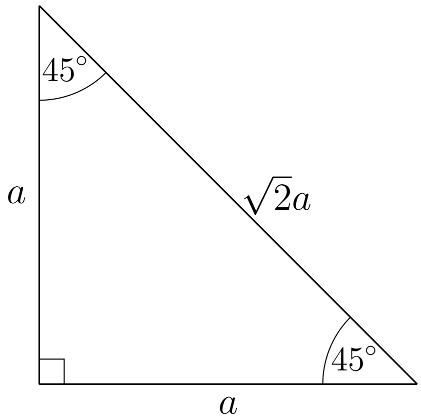
Notice in the picture above that triangle ABC is similar to triangle $AB'C'$ (why?), therefore, their corresponding sides are in proportion, in particular, using the definition of the trig functions,

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\overline{CB}}{\overline{AC}} = \frac{\overline{C'B'}}{\overline{AC'}}$$

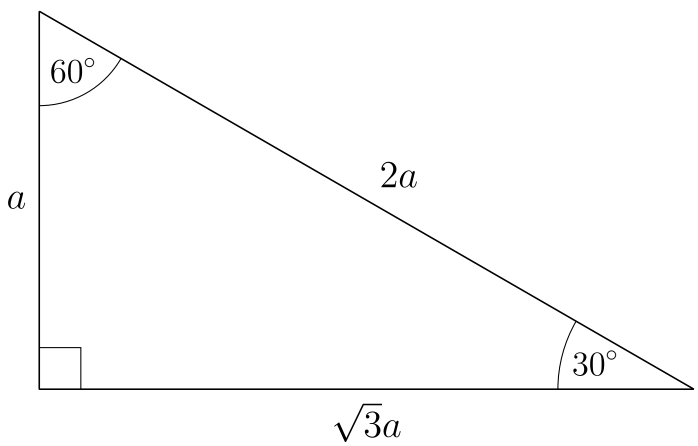
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\overline{AB}}{\overline{AC}} = \frac{\overline{AB'}}{\overline{AC'}}$$

This fact tells us that, in determining the value of a trig function on any angle θ, we may use **any right triangle**. The value of the six trig functions is determined by the angle θ, not by the triangle we use.

Special Triangles:



In a $45°-45°-90°$ triangle, the two sides have the same length and the hypotenuse is $\sqrt{2}$ times the length of the side. In other words, if any one of the side has length a, then the other side also has length a and the hypotenuse has length $\sqrt{2}a$

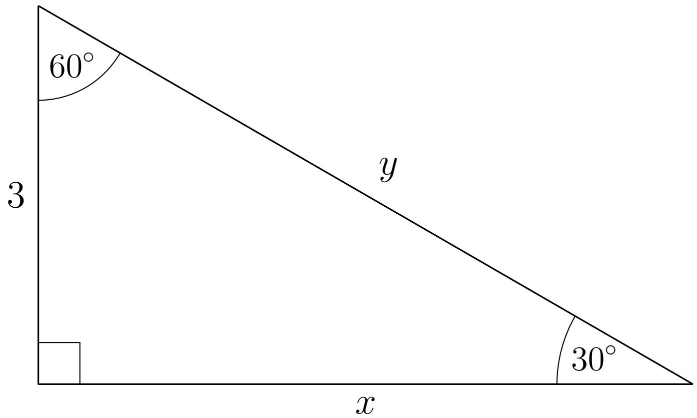


In a $30° - 60° - 90°$ triangle, if the side opposite the $30°$ angle has length a, then the side opposite the $60°$ angle has length $\sqrt{3}a$, and the hypotenuse has length $2a$. In other words, the hypotenuse is twice the length of the shorter side.

Also remember from geometry that in any triangle, the longest side is the side opposite the largest angle, and the shortest side is the side opposite the smallest angle. Using this fact and the above **special triangle**, we can find the exact value of the lengths of some triangles:

Example:

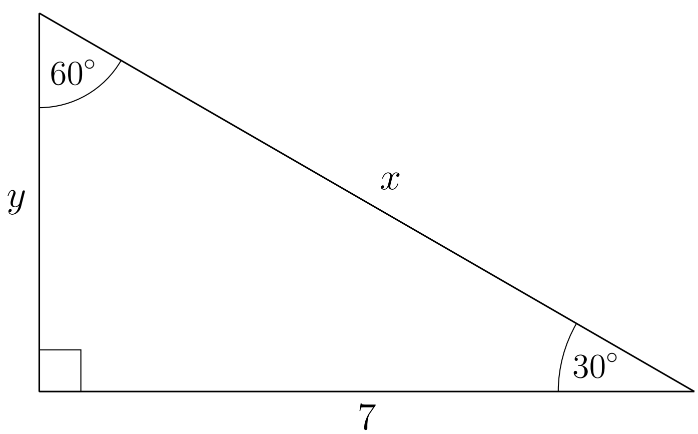
Find the value of the variables:



Ans: The shortest side in this triangle is the one opposite the 30° angle, its length is 3. This is a $30^\circ - 60^\circ - 90^\circ$ triangle, the hypotenuse is twice the shortest side, so $y = 2(3) = 6$. The side opposite the 60° angle is $\sqrt{3}$ times the length of the shortest side, so $x = \sqrt{3} \cdot 3 = 3\sqrt{3}$.

Example:

Find the value of the variables:



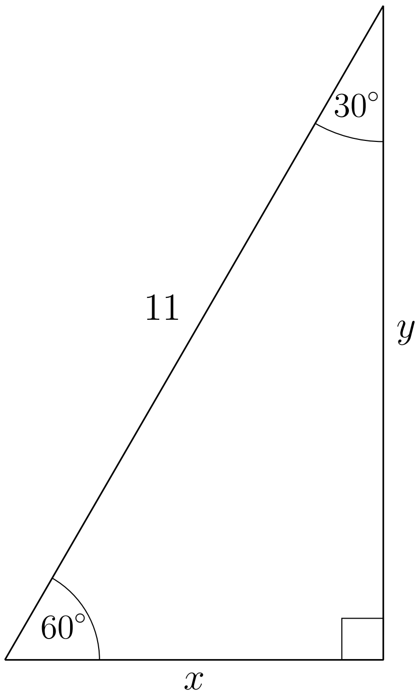
Ans: In this triangle, y is the shortest side (it is opposite the smallest angle), the side opposite the 60° angle is $\sqrt{3}y$, so we can set up the equation:

$\sqrt{3}y = 7$, solving for y gives: $y = \dfrac{7}{\sqrt{3}}$

The hypotenuse, x, is twice y, so $x = 2 \cdot \dfrac{7}{\sqrt{3}} = \dfrac{14}{\sqrt{3}}$

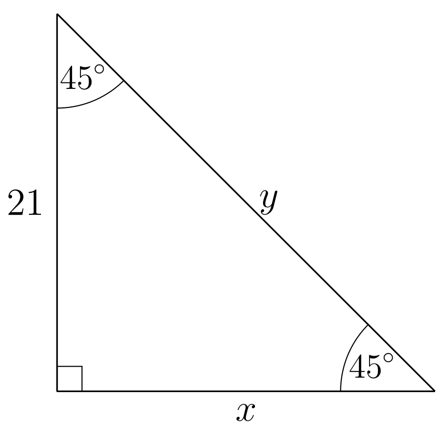
Example:

Find the value of the variables:



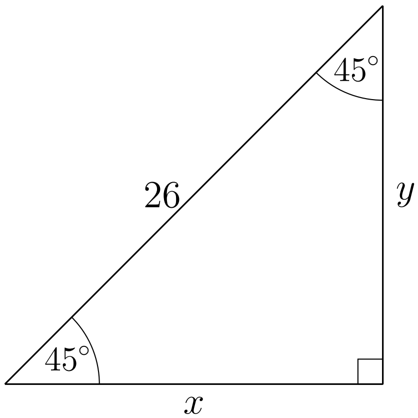
Ans: The length of the hypotenuse is twice of the shortest side, x, so $x = \dfrac{11}{2}$, y is $\sqrt{3}$ times the shorter side, so $y = \sqrt{3} \cdot \dfrac{11}{2} = \dfrac{11\sqrt{3}}{2}$

Example: Find the value of the variables:



Ans: This is a $45° - 45° - 90°$ triangle, the two sides are equal in length, so $x = 21$. The hypotenuse is $\sqrt{2}$ times the length of the side, so $y = \sqrt{2} \cdot 21 = 21\sqrt{2}$

Example: Find the value of the variables:



Ans: In this $45° - 45° - 90°$ triangle, the hypotenuse is 26. The hypotenuse is $\sqrt{2}$ times the length of the side, so we can set up the equation:

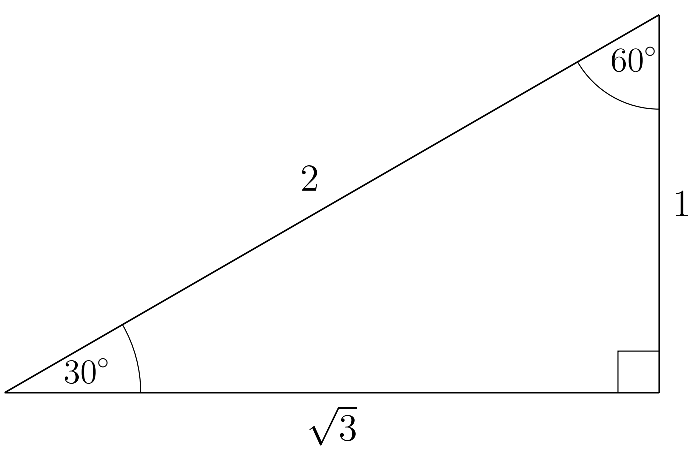
$x\sqrt{2} = 26$, solving for x gives: $x = \dfrac{26}{\sqrt{2}}$

The two sides have the same length, so $y = x = \dfrac{26}{\sqrt{2}}$

Using the special triangles, we can find the value of the trig functions on some special angles.

Example: Find $\sin(30°)$, $\cos(30°)$, $\tan(30°)$, $\cot(30°)$, $\sec(30°)$, and $\csc(30°)$

Ans:



For convenience, we can assign the side opposite the 30° angle with a length of 1, then the adjacent side to the 30° angle will have a length of $\sqrt{3}$, and the hypotenuse will have a length of 2, and we have:

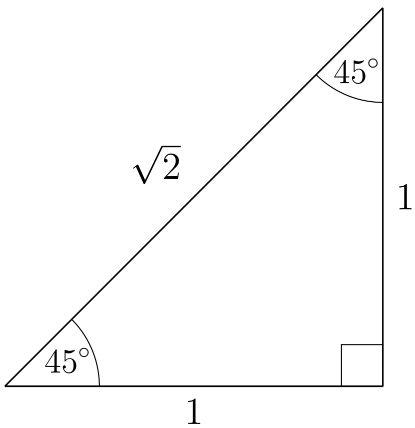
$$\sin(30°) = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2} \qquad \cos(30°) = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2} \qquad \tan(30°) = \frac{\text{opp}}{\text{adj}} = \frac{1}{\sqrt{3}}$$

$$\cot(30°) = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{3}}{1} = \sqrt{3} \qquad \sec(30°) = \frac{\text{hyp}}{\text{adj}} = \frac{2}{\sqrt{3}}$$

$$\csc(30°) = \frac{\text{hyp}}{\text{opp}} = \frac{2}{1} = 2$$

Example: Find $\sin(45^\circ)$, $\cos(45^\circ)$, $\tan(45^\circ)$, $\cot(45^\circ)$, $\sec(45^\circ)$, and $\csc(45^\circ)$

Ans:



For convenience, we can assign the side opposite the 45° angle with a length of 1, then the adjacent side to this 45° angle will also have a length of 1, and the hypotenuse will have a length of $\sqrt{2}$, and we have:

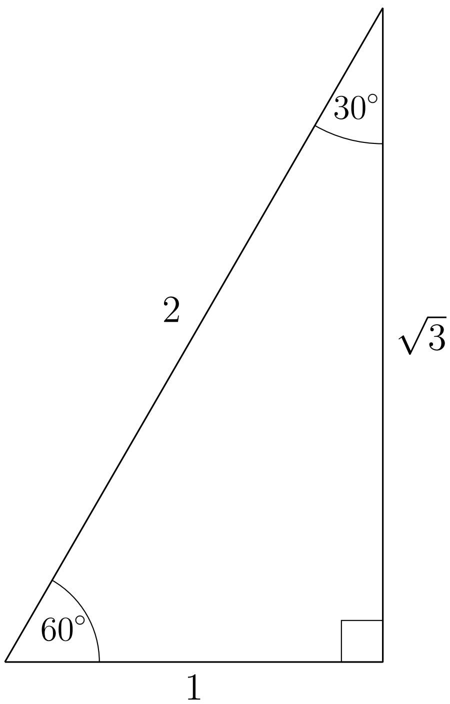
$$\sin(45^\circ) = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}} \qquad \cos(45^\circ) = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{2}} \qquad \tan(45^\circ) = \frac{\text{opp}}{\text{adj}} = \frac{1}{1} = 1$$

$$\cot(45^\circ) = \frac{\text{adj}}{\text{opp}} = \frac{1}{1} = 1 \qquad \sec(45^\circ) = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\csc(45^\circ) = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

Example: Find $\sin(60^\circ)$, $\cos(60^\circ)$, $\tan(60^\circ)$, $\cot(60^\circ)$, $\sec(60^\circ)$, and $\csc(60^\circ)$

Ans:



For convenience, we still assign the side opposite the 30° angle (the side adjacent to the 60° angle) with a length of 1, then the opposite side to the 60° angle will have a length of $\sqrt{3}$, and the hypotenuse will have a length of 2, and we have:

$$\sin(60^\circ) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2} \qquad \cos(60^\circ) = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2} \qquad \tan(60^\circ) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\cot(60^\circ) = \frac{\text{adj}}{\text{opp}} = \frac{1}{\sqrt{3}} \qquad \sec(60^\circ) = \frac{\text{hyp}}{\text{adj}} = \frac{2}{1} = 2$$

$$\csc(60^\circ) = \frac{\text{hyp}}{\text{opp}} = \frac{2}{\sqrt{3}}$$

The right triangle approach to defining trigonometric functions is easy to understand, and can be used conveniently to solve goemetric problems involving triangles. The drawback with this definition is the fact that the trig functions are defined only for angle θ between $0^\circ < \theta < 90^\circ$. We wanted the trig functions to be defined for all (or most) of the real numbers. In order to do that, we need a different approach to defining the trig functions.