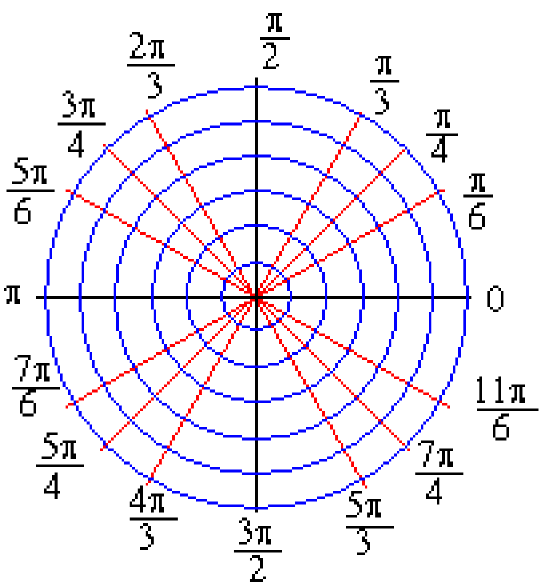
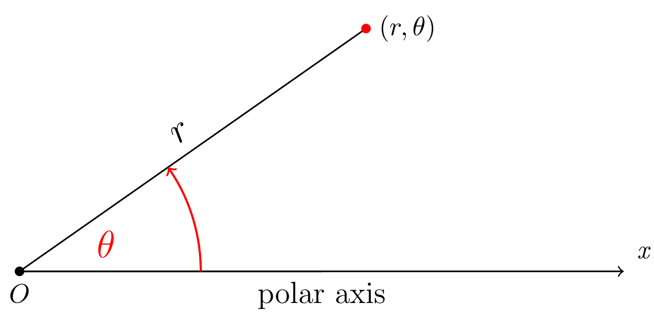
Polar Coordinates:

Instead of the *rectangular coordinate* system (x, y), which uses intersecting perpendicular lines to represent locations in a plane, sometimes it is useful to represent the location of a point in the coordinate plane using concentric circles intersecting with lines radiating from a common center. A point on this coordinate plane is formed when a circle with a certain radius r intersects a straight line radiating from the center making a certain angle θ with the $x-$axis (the polar axis). This coordinate system is the **polar coordinate system**.

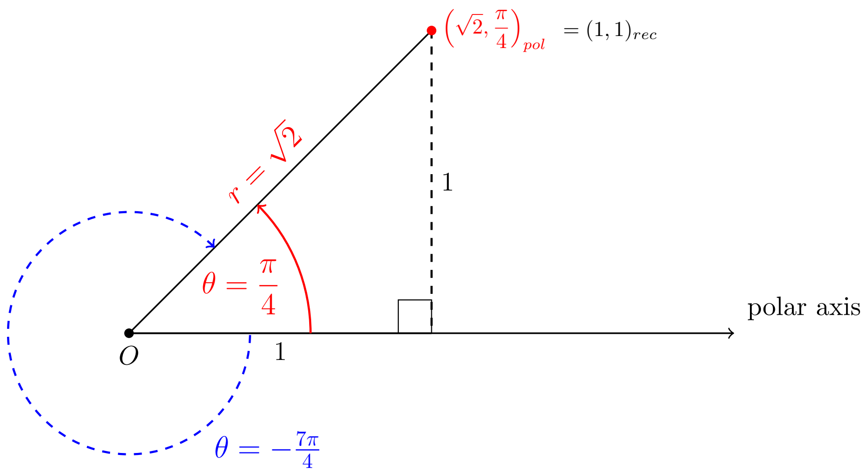


Like the rectagular coordinate system, a point in polar coordinate consists of an ordered pair of numbers, (r, θ). The first coordinate is the distance of the point from the origin $(0, 0)$, and the second coordinate is the angle, in standard position, and the point we want to locate is the terminal point of this angle.



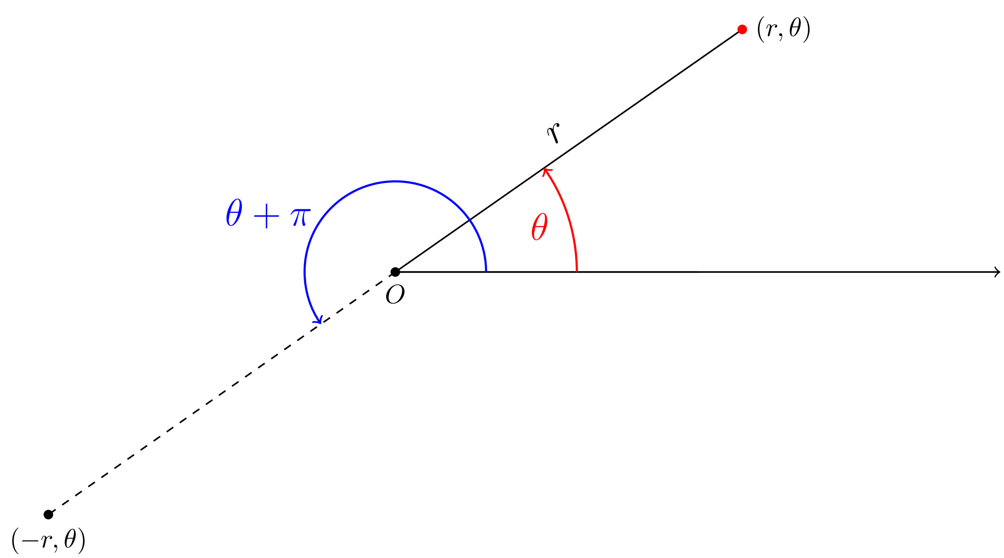
Example:

The point $P = (1,1)$ (in rectangular coordinate) has a distance of $\sqrt{2}$ from the origin, and is the terminal point of the angle $\left(\frac{\pi}{4}\right)$, therefore, the equivalent polar coordinate of the point is: $P = \left(\sqrt{2}, \frac{\pi}{4}\right)$



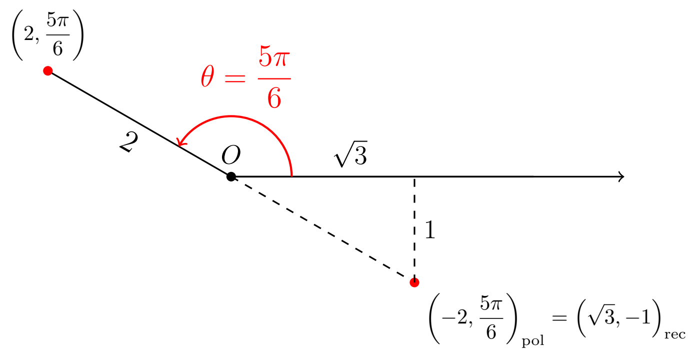
Unlike the rectangular coordinate system, the polar representation of a point is **not** unique. For example, the above point P can also be represented as $P = \left(\sqrt{2}, -\frac{7\pi}{4}\right)$ or $P = \left(\sqrt{2}, \frac{9\pi}{4}\right)$.

If r is a positive number, we define the polar point $(-r, \theta)$ to be the point with the same distance r from the origin, but is to the opposite direction of the terminal point of θ. In other words, the point $(-r, \theta)$ is the same point as $(r, \theta + \pi)$

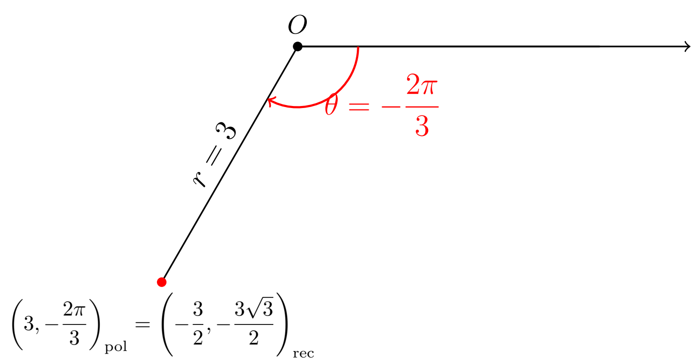


Example:

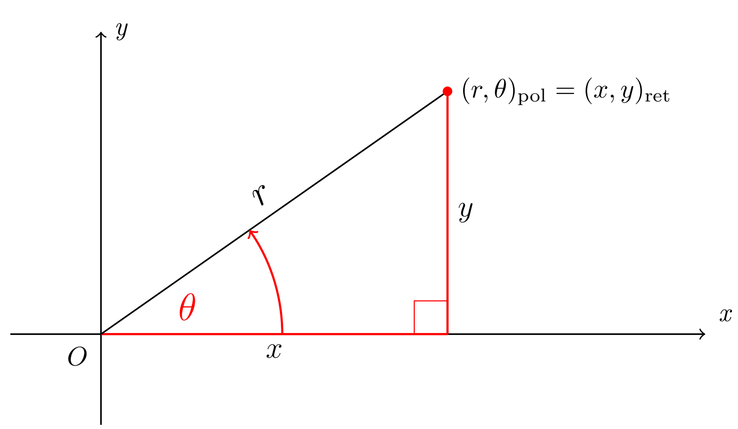
The polar point $\left(-2, \frac{5\pi}{6}\right)$ is at the opposite side of the polar point $\left(2, \frac{5\pi}{6}\right)$.
The rectangular coordinate of this point is $\left(\sqrt{3}, -1\right)$



Example: The polar point $\left(3, -\frac{2\pi}{3}\right)$ has a rectangular coordinate of $\left(-\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)$.



We want to find formula that allows us to interchage between a point in polar coordinate (r, θ) and rectangular coordinate (x, y).



From the definition of the trig functions we have:

$\cos\theta = \dfrac{x}{r} \Rightarrow x = r\cos\theta$

$\sin\theta = \dfrac{y}{r} \Rightarrow y = r\sin\theta$

The above two formula allow us to change a polar coordinate (r, θ) to rectangular coordinate, (x, y).

To change from rectangular to polar, we use that fact that:

$x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2 + y^2}$

and

$\tan\theta = \dfrac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\dfrac{y}{x}\right) \quad \text{if} \quad x > 0$

If $x < 0$, then (x, y) will be in the second or third quadrant. Since the range of arctangent is only in the first or fourth quadrant, we need to make a little adjustment to angle θ:

$\theta = \pi + \tan^{-1}\left(\dfrac{y}{x}\right) \quad \text{if} \quad x < 0$

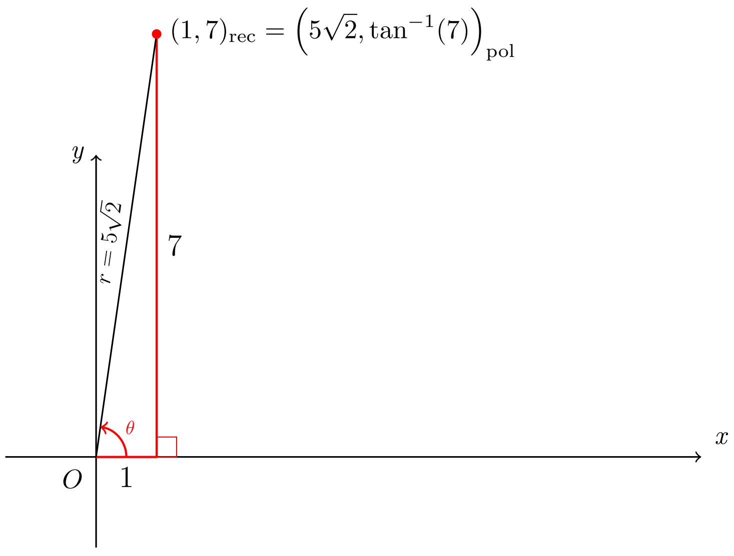
Example: Express the rectangular point $(1, 7)$ in polar coordinate:

This is a point in the first quadrant. The distance the point from the origin is $r = \sqrt{1^2 + 7^2} = \sqrt{50} = 5\sqrt{2}$

Since $x = 1 \geq 0$, we use:

$\theta = \tan^{-1}\left(\dfrac{7}{1}\right) = \tan^{-1}(7)$

The polar equivalent of the rectangular point $(1, 7)$ is $\left(5\sqrt{2}, \tan^{-1}(7)\right)$



Example: Express the rectangular point $(-2, -5)$ in polar coordinate:

$r = \sqrt{(-2)^2 + (-5)^2} = \sqrt{4+25} = \sqrt{29}$

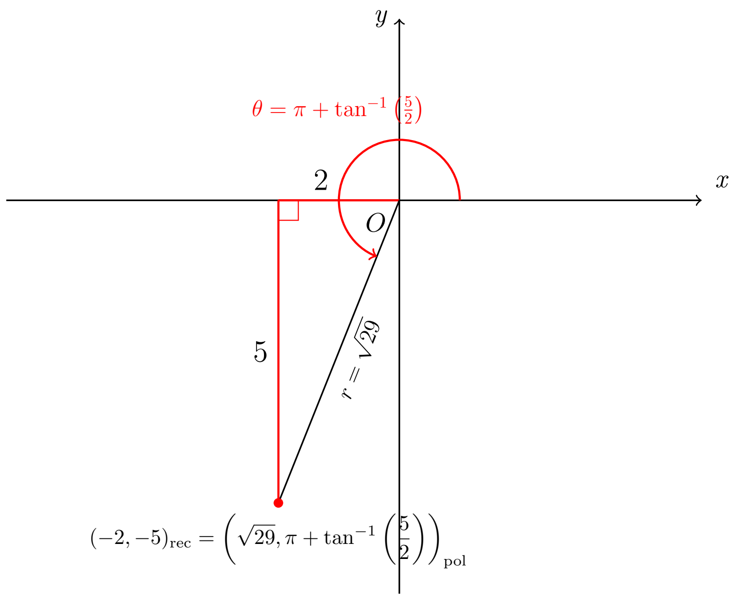
$\tan\theta = \dfrac{-5}{-2} = \dfrac{5}{2}$

This time, since $x = -2 < 0$, we need to add π to arctangent to correctly represent angle θ. We have:

$\theta = \pi + \tan^{-1}\left(\dfrac{5}{2}\right)$

The polar equivalent of the rectangular point $(-2, -5)$ is

$\left(\sqrt{29}, \pi + \tan^{-1}\left(\dfrac{5}{2}\right)\right)$



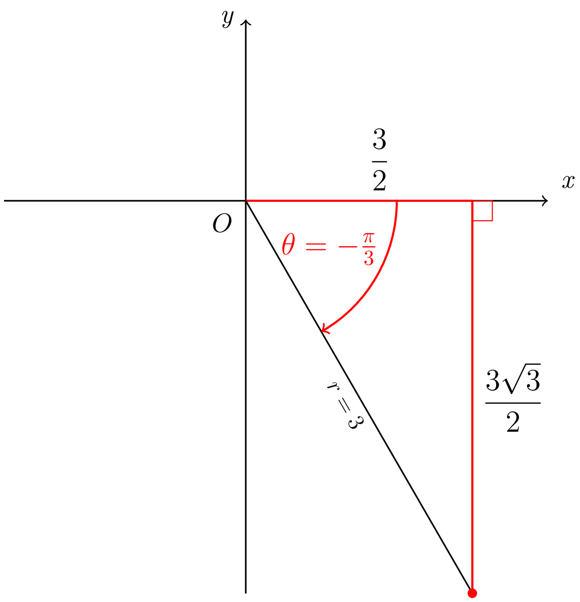
Example: Express the polar coordinate $\left(3, -\dfrac{\pi}{3}\right)$ in rectangular form:

Ans:

$x = 3\cos\left(-\dfrac{\pi}{3}\right) = 3\left(\dfrac{1}{2}\right) = \dfrac{3}{2}$

$y = 3\sin\left(-\dfrac{\pi}{3}\right) = 3\left(-\dfrac{\sqrt{3}}{2}\right) = -\dfrac{3\sqrt{3}}{2}$

The coresponding rectangular point is $\left(\dfrac{3}{2}, -\dfrac{3\sqrt{3}}{2}\right)$

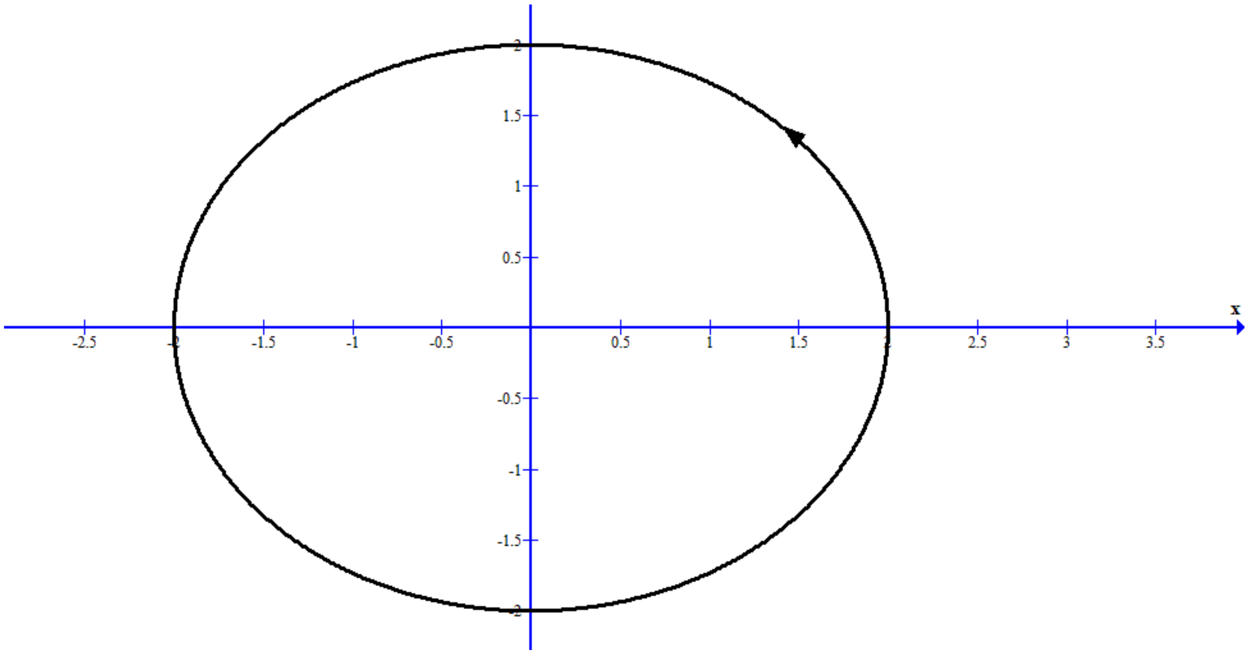


$$\left(3, -\frac{\pi}{3}\right)_{\text{pol}} = \left(\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)_{\text{rec}}$$

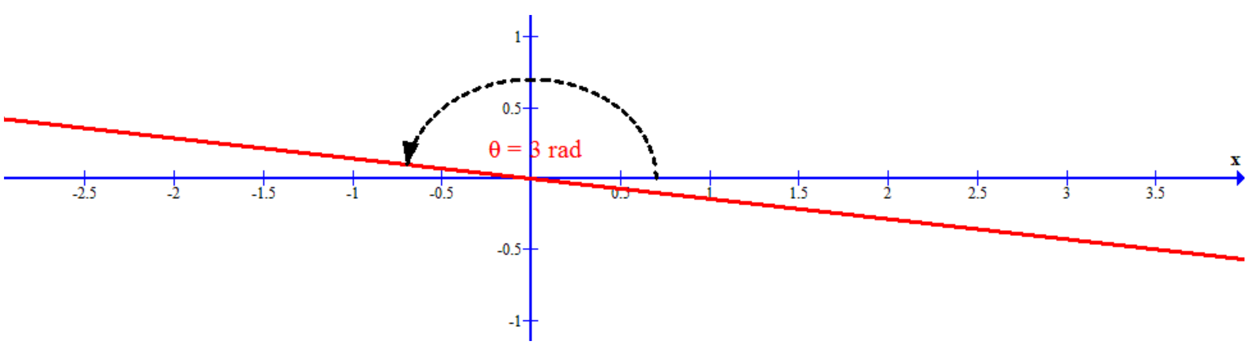
Equations in Polar Form:

In rectangular coordinate, we know that the equations $x = 2$ or $y = 3$ are equations of vertical and horizontal lines, respectively. What do the equations $r = 2$ and $\theta = 3$ in polar coordinate represent?

Note that in the rectangular equation, $x = 2$, y is a **free variable**, meaning that y can be any value. In the equation $r = 2$, θ is also a free variable and can assume any value, therefore, any point that has a distance of 2 units from the origin will be a solution to the equation $r = 2$, this is the graph of a circle with radius 2 centered at the origin.



In $\theta = 3$ (3 is 3 radian, not 3 degrees), r is the free variable, so any points, as long as they are the terminal point of the angle (in standard position) that makes 3 radian angle with the x axis, will be on the curve, regardless of its distance from the origin. This is a straight line throught the origin with slope equal to $m = \tan(3)$.



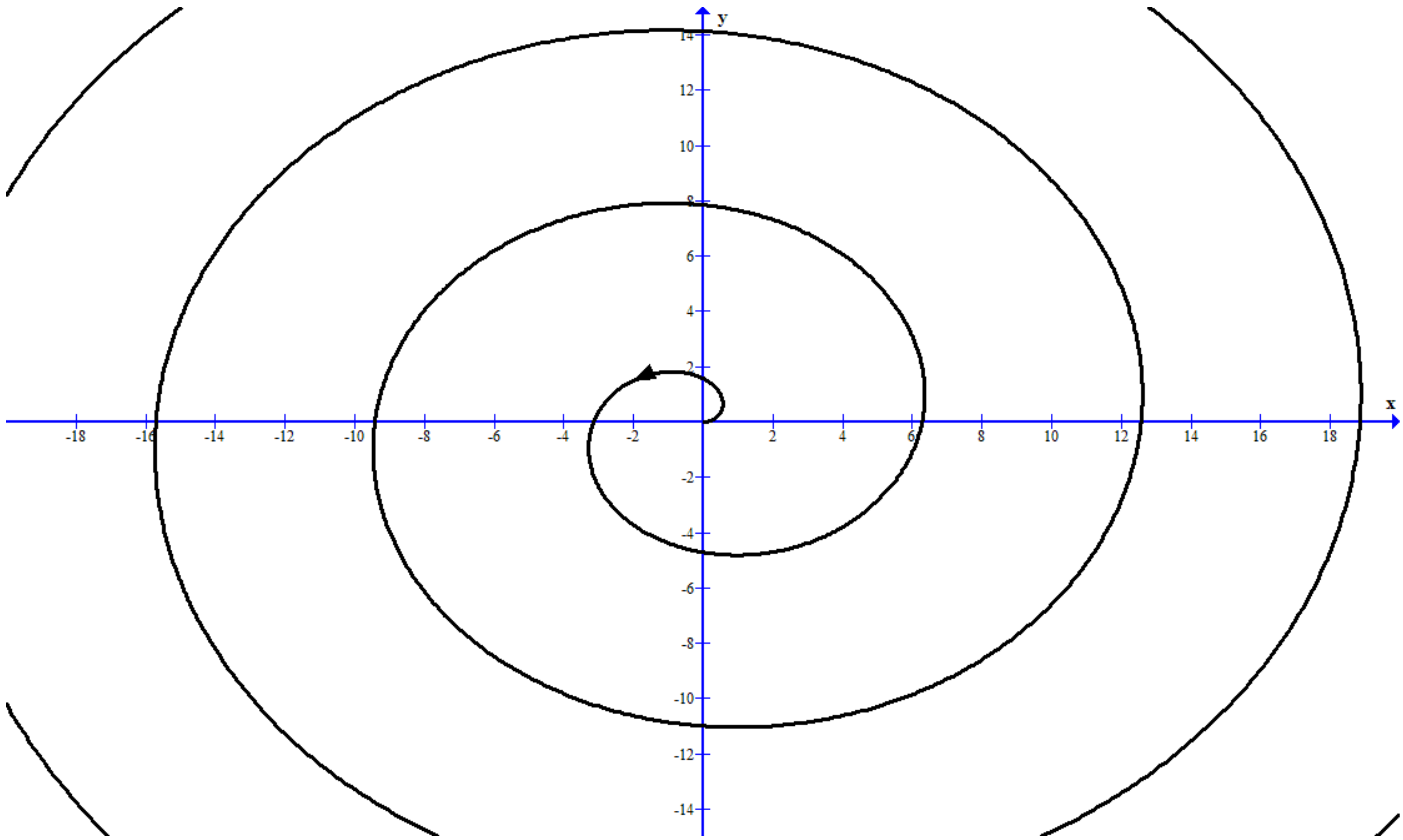
In general, the polar equation $r = c$, ($c > 0$ a constant), is the equation of a circle with radius c centered at the origin.

The polar equation $\theta = c$, (c a constant), is the equation of a straight line through the origin with slope $m = \tan(c)$

Example: Describe the graph of the polar equation:

$r = \theta$, $0 \leq \theta < \infty$

As θ increases, so does r, so we have a curve that keeps on expanding from the origin in a counter-clockwise, circular manner.



Example: Describe the graph of the polar equation:

$r = 6\cos\theta$

It may be a little easier if we change this back to rectangular equation:

$r = 6\left(\dfrac{x}{r}\right)$

$r^2 = 6x$

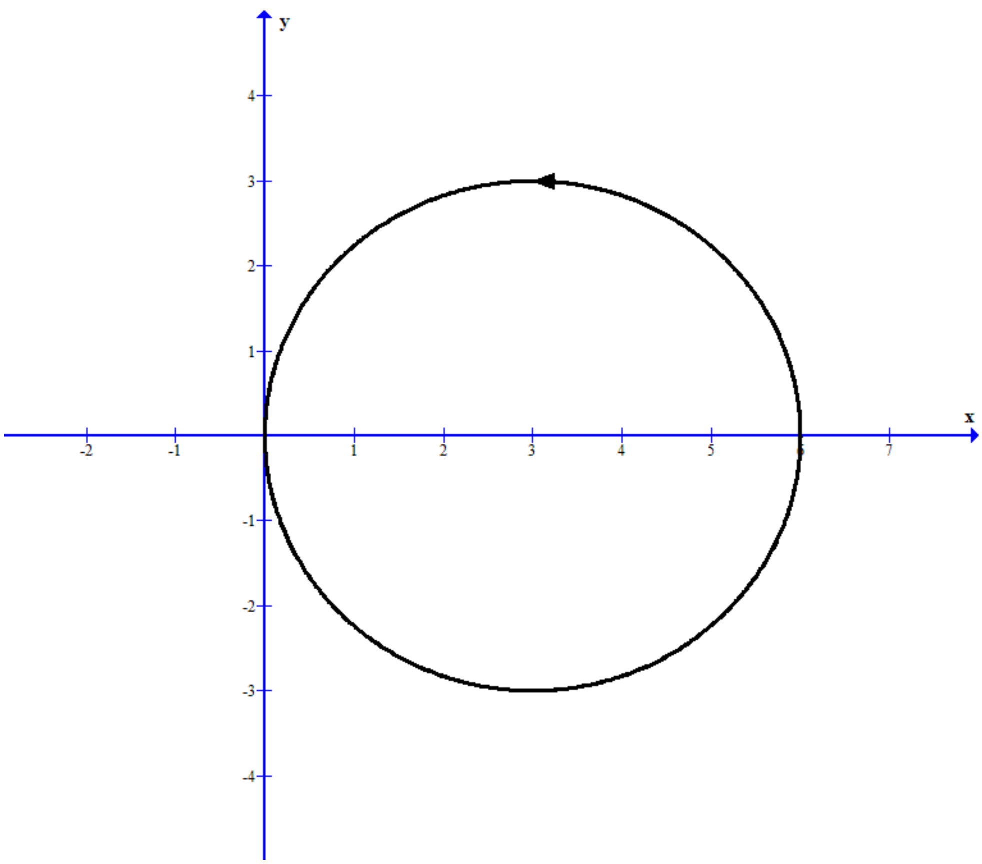
$x^2 + y^2 = 6x$

$x^2 - 6x + y^2 = 0$

$x^2 - 6x + 9 + y^2 = 9$

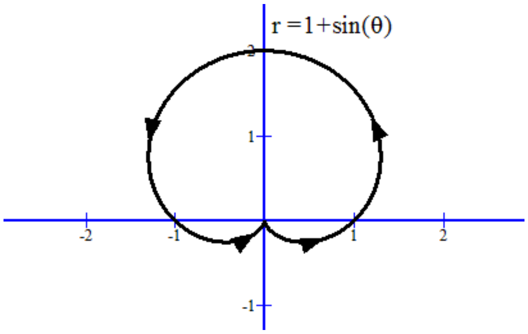
$(x-3)^2 + y^2 = 9$

This is a circle with radius 3 centered at (rectangular point) $(3, 0)$

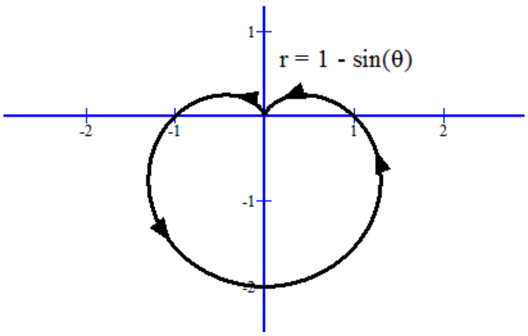


Below are some polar equations that cannot be easily written in retangular form, and their corresponding graphs.

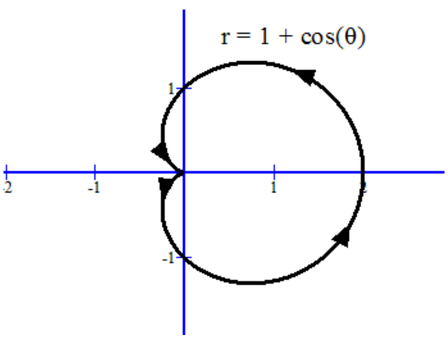
$r = 1 + \sin\theta$



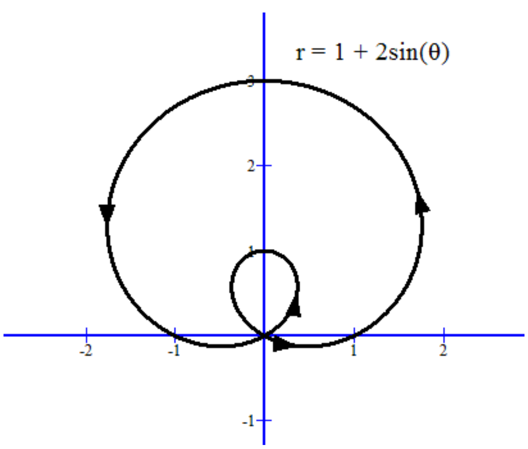
$r = 1 - \sin\theta$



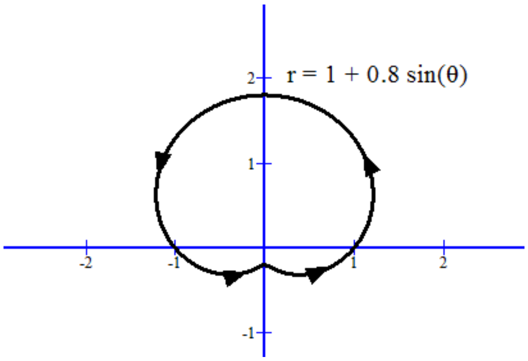
$r = 1 + \cos\theta$



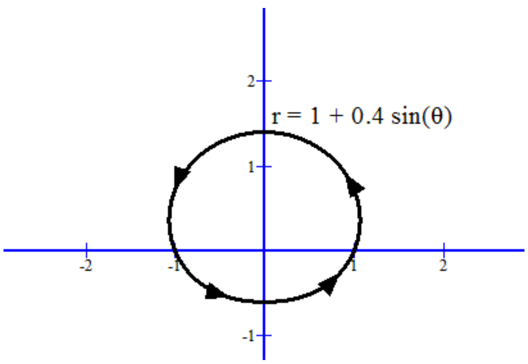
$r = 1 + 2\sin\theta$



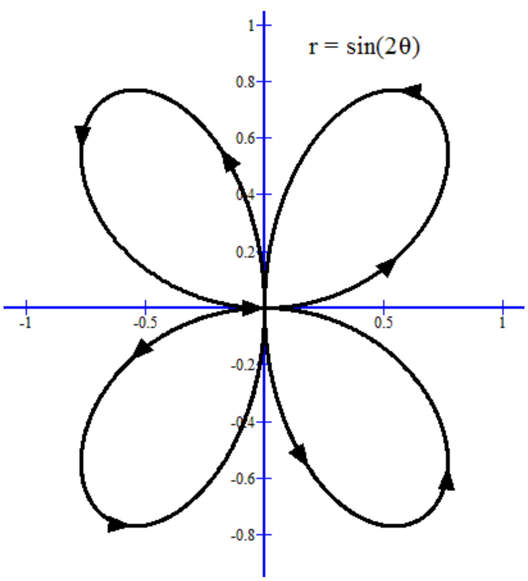
$$r = 1 + \frac{4}{5}(\sin\theta)$$



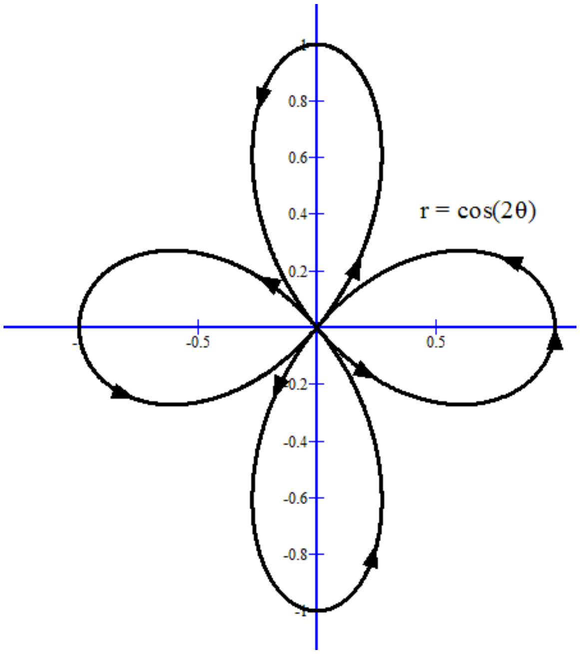
$$r = 1 + \frac{2}{5}(\sin\theta)$$



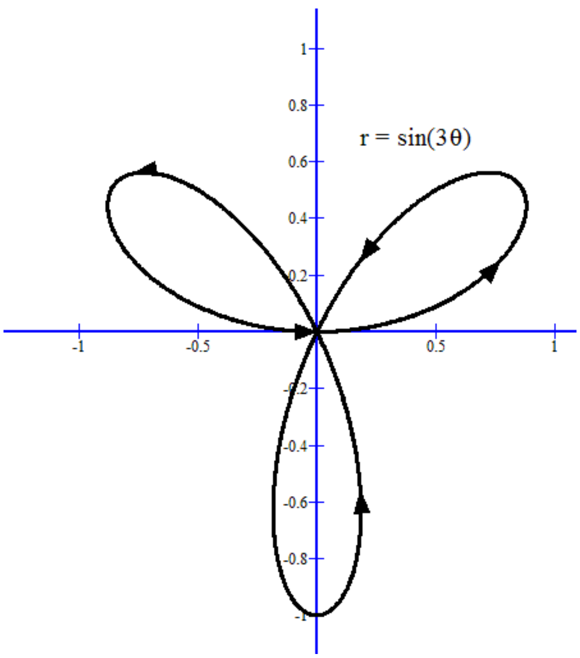
$$r = \sin 2\theta$$



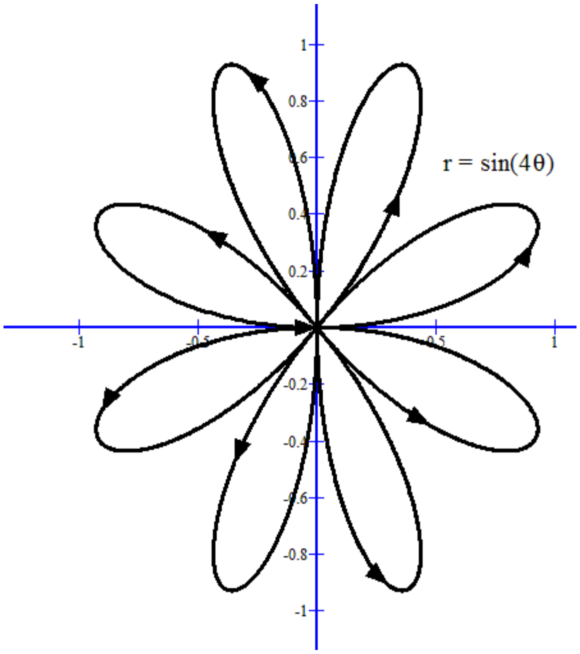
$r = \cos 2\theta$



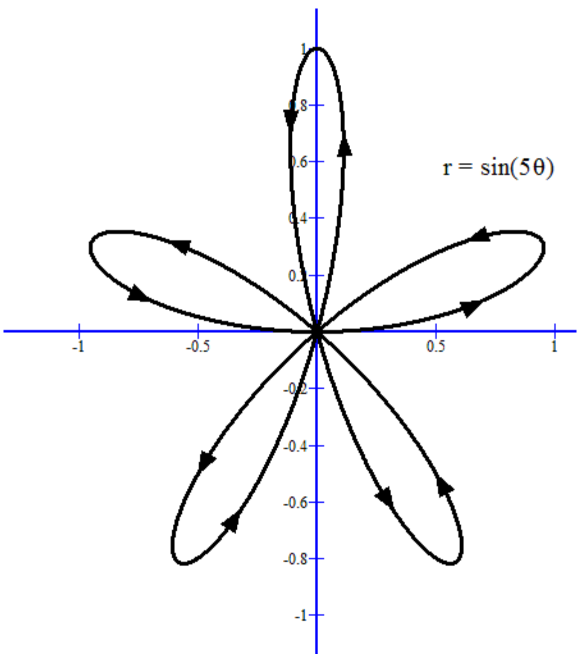
$r = \sin 3\theta$



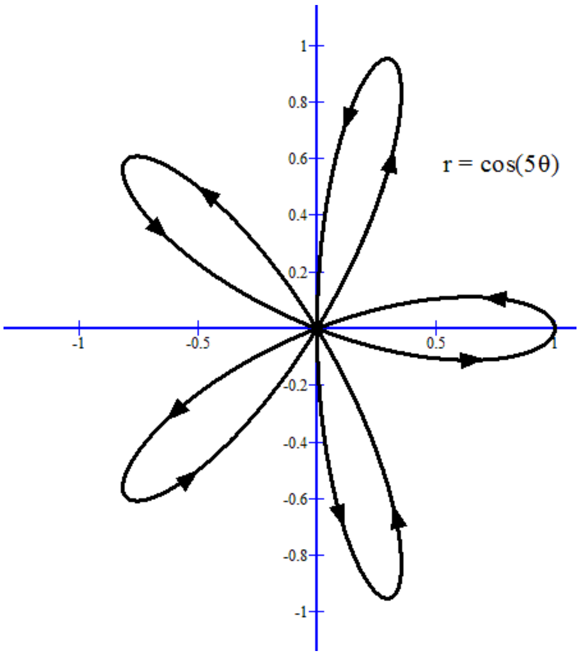
$r = \sin 4\theta$



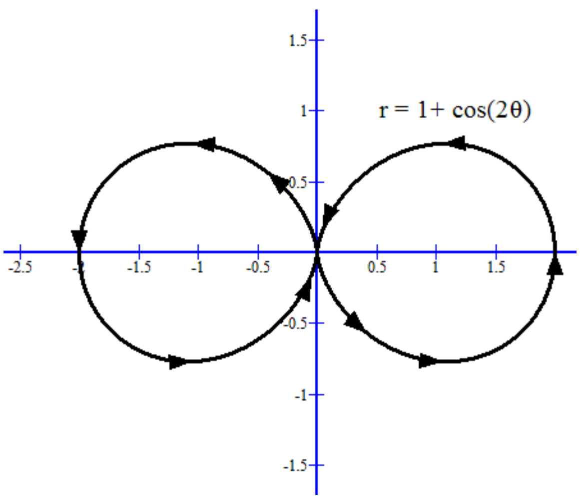
$r = \sin 5\theta$



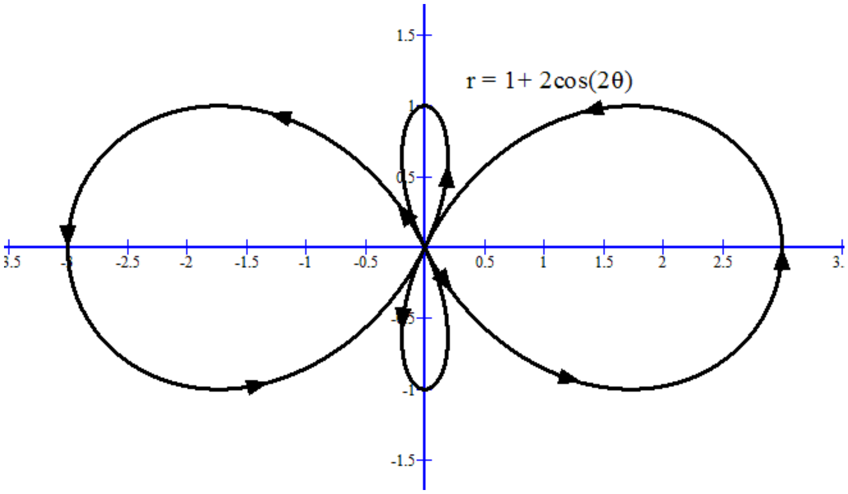
$r = \cos 5\theta$



$r = 1 + \cos 2\theta$



$r = 1 + 2\cos 2\theta$



$r = 1 + \frac{4}{5}\cos 2\theta$