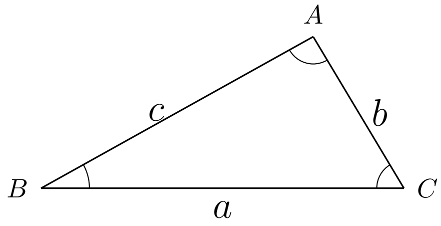
Law of Sine:



If ΔABC is any triangle whose angles are A, B, C and the sides opposite these angles are, correspondingly, a, b, c, then:

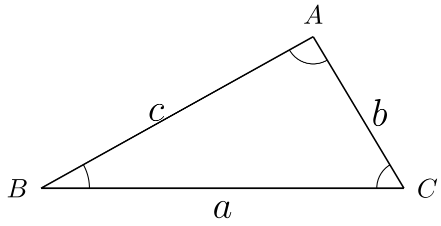
$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

Depending on what you are interested in, the law of sine can also be stated in the other form:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

The formula tells us that, as long as we know the value of two angles in a triangle and a side of the triangle, we can find the value of the other missing angles and sides. In other words, if two angles and a side of a triangle is fixed, the triangle is fixed. This corresponds to the **(ASA)** (angle-side-angle) and **(AAS)** (angle-angle-side) triangle congruency in geometry.

Law of Cosine:



If ΔABC is any triangle whose angles are A, B, C and the sides opposite these angles are, correspondingly, a, b, c, then:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$
$$b^2 = a^2 + c^2 - 2ac\cos(B)$$
$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

Note that if $C = \dfrac{\pi}{2}$, then $\cos(C) = \cos\left(\dfrac{\pi}{2}\right) = 0$ and the first formula becomes: $c^2 = a^2 + b^2$, which is just the Pythegorean Theorem.

You may think of the law of cosine as an *adjustment* to the Pythegorean Theorem for an arbitrary triangle.

The law of cosine tells us that, if two sides of a triangle and the angle between the two sides is known **(SAS)** or if all three sides of a triangle is known **(SSS)**, the other parts of the triangle can be found.

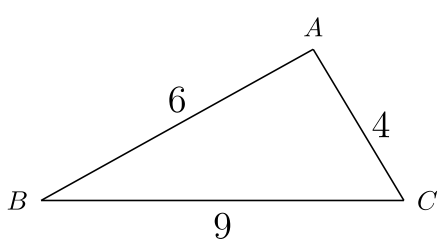
Using the law of cosine, we may prove the following fact:

In any triangle, if the sum of the squares of the two shorter sides is **greater than** the square of the longest side, the triangle is acute.

In any triangle, if the sum of the squares of the two shorter sides is **less than** the square of the longest side, the triangle is obtuse.

In any triangle, if the sum of the squares of the two shorter sides is **equal to** the square of the longest side, the triangle is right.

Example:

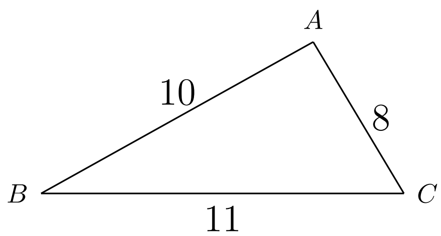


For the above triangle, the sum of the square of the two shorter sides is $4^2 + 6^2 = 16 + 36 = 52$.

The square of the longest side is $9^2 = 81$.

Since $52 < 81$, the above triangle is obtuse. In other words, angle A is an obtuse angle (greater than $90°$).

Example:

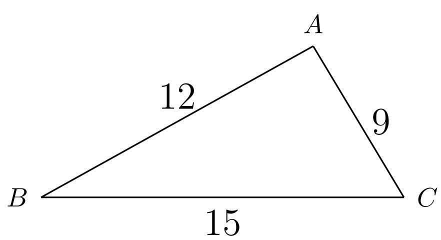


For the above triangle, the sum of the square of the two shorter sides is $8^2 + 10^2 = 64 + 100 = 164$.

The square of the longest side is $11^2 = 121$.

Since $164 > 121$, the above triangle is acute. In other words, angle A is an acute angle (less than $90°$).

Example:



For the above triangle, the sum of the square of the two shorter sides is $9^2 + 12^2 = 81 + 144 = 225$.

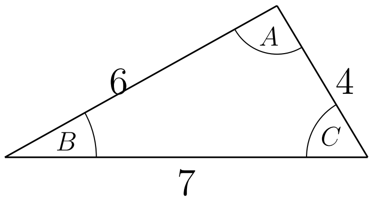
The square of the longest side is $15^2 = 225$.

Since the sum of the square of the two shorter sides is **equal** to the square of the longest side, the above triangle is right. In other words, angle A is a right ($90°$) angle.

Triangle Inequality:

In any triangle, the sum of the length of any two sides is always greater than the third side.

Example: Solve the given triangle:



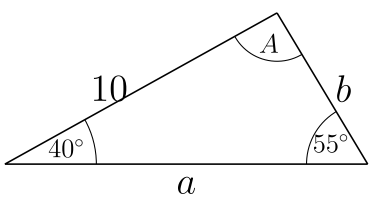
Ans: This is a SSS case, we need to use the law of cosine.

$$7^2 = 4^2 + 6^2 - 2(4)(6)\cos(A)$$
$$49 = 16 + 36 - 48\cos(A)$$
$$49 = 52 - 48\cos(A)$$
$$-3 = -48\cos(A)$$
$$\cos(A) = \frac{-3}{-48} = \frac{1}{16}$$
$$A = \cos^{-1}\left(\frac{1}{16}\right) \approx 86.4^\circ$$

We may similarly use the law of cosine to find the value of B and C:

$$4^2 = 6^2 + 7^2 - 2(6)(7)\cos(B)$$
$$6^2 = 4^2 + 7^2 - 2(4)(7)\cos(C)$$

Example:

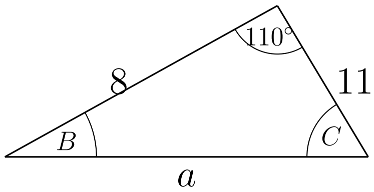


Ans: This is an AAS case, we use law of sine:

$$A = 85^\circ$$

$$\frac{b}{\sin(40^\circ)} = \frac{10}{\sin(55^\circ)} \Rightarrow b = \frac{10\sin(40^\circ)}{\sin(55^\circ)} \approx 7.85$$
$$\frac{a}{\sin(85^\circ)} = \frac{10}{\sin(55^\circ)} \Rightarrow a = \frac{10\sin(85^\circ)}{\sin(55^\circ)} \approx 12.2$$

Example: Solve the triangle:



Ans: This is an SAS case. We use law of cosine:

$a^2 = 8^2 + 11^2 - 2(8)(11)\cos(110°)$

$a^2 = 64 + 121 - 176\cos(110°)$

$a^2 = 185 - 176\cos(110°)$

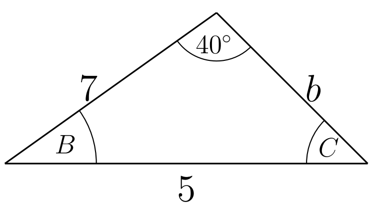
$a = \sqrt{185 - 176\cos(110°)} \approx 15.7$

Once we found the value of a, we can use the law of cosine again to find the value of B and C.

$11^2 = 8^2 + a^2 - 2(8)(a)\cos(B)$

$8^2 = 11^2 + a^2 - 2(11)(a)\cos(C)$

Example: Solve the triangle:



Ans: This is an ambiguous case (SSA). If we try to use the law of sine to find angle C:

$\frac{\sin(C)}{7} = \frac{\sin(40°)}{5} \Rightarrow \sin(C) = \frac{7\sin(40°)}{5} \approx 0.9$

There are two values of C, between 0° to 180°, that would solve the above equation, namely, $C \approx 64.2°$ or C $\approx 115.8°$. There are two possible triangles that can be formed, one with an obtuse C, and another one with an acute C.

What if we had used law of cosine instead? In order to use the law of cosine, we must solve for b:

$5^2 = b^2 + 7^2 - 2(7)(b)\cos(40°)$

$25 = b^2 + 49 - 14(b)\cos(40°)$

$b^2 - 14(b)\cos(40°) + 24 = 0$

$b^2 - 10.72b + 24 = 0$

This is a quadratic equation in b, which has two solutions:

$b = \dfrac{10.72 \pm \sqrt{(-10.72)^2 - 4(1)(24)}}{2} \Rightarrow b \approx \dfrac{10.72 \pm 4.35}{2} \Rightarrow b \approx 7.54$ or b $\approx$ 3.19

Whether we used law of sine or law of cosine, we ended up with two possible triangles.