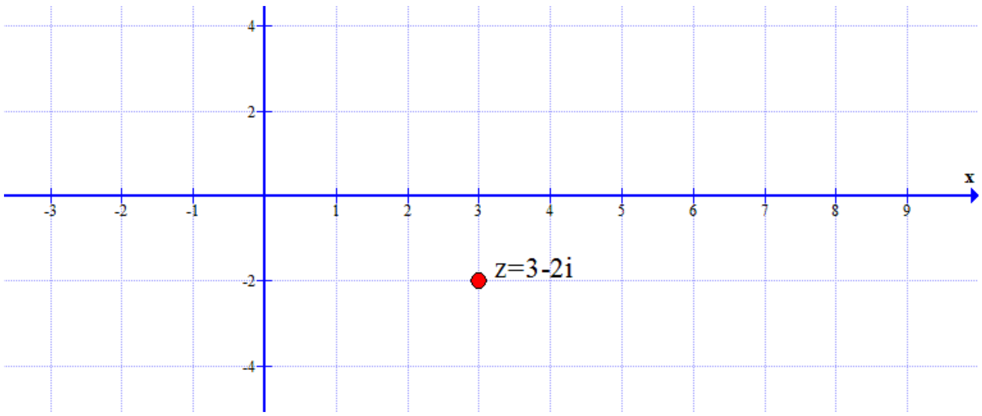
We can think of a complex number $a + bi$ as the point (a, b) in the xy plane. In this representation, a is the x coordinate and b is the y coordinate.The $x-$axis is called the **real axis** and the $y-$axis the **imaginary axis**, and we refer to this plane as the **complex plane**.

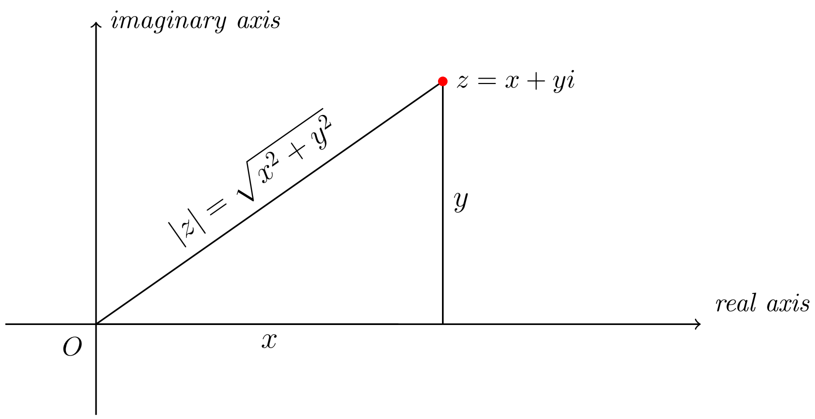
Example: The complex number $3 - 2i$ can be viewed as the point $(3, -2)$ in the complex plane.



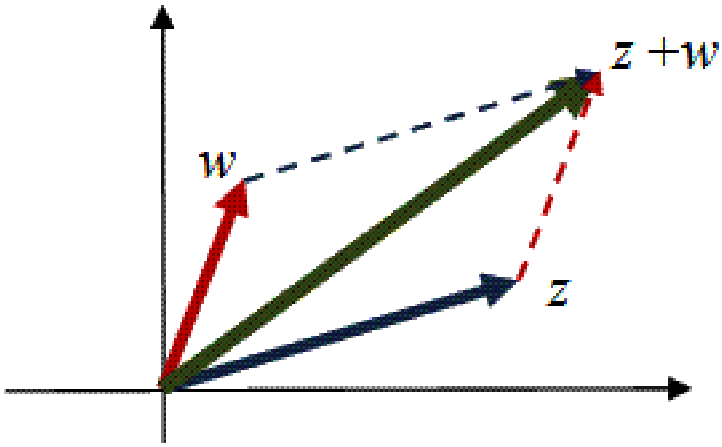
Given a complex number $z = x + yi$, we define the **magnitude** of z, written $|z|$, as:

$|z| = \sqrt{x^2 + y^2}$.

Graphically, the magniture of z is the distance between z (viewed as a point on the xy plane) and the origin.

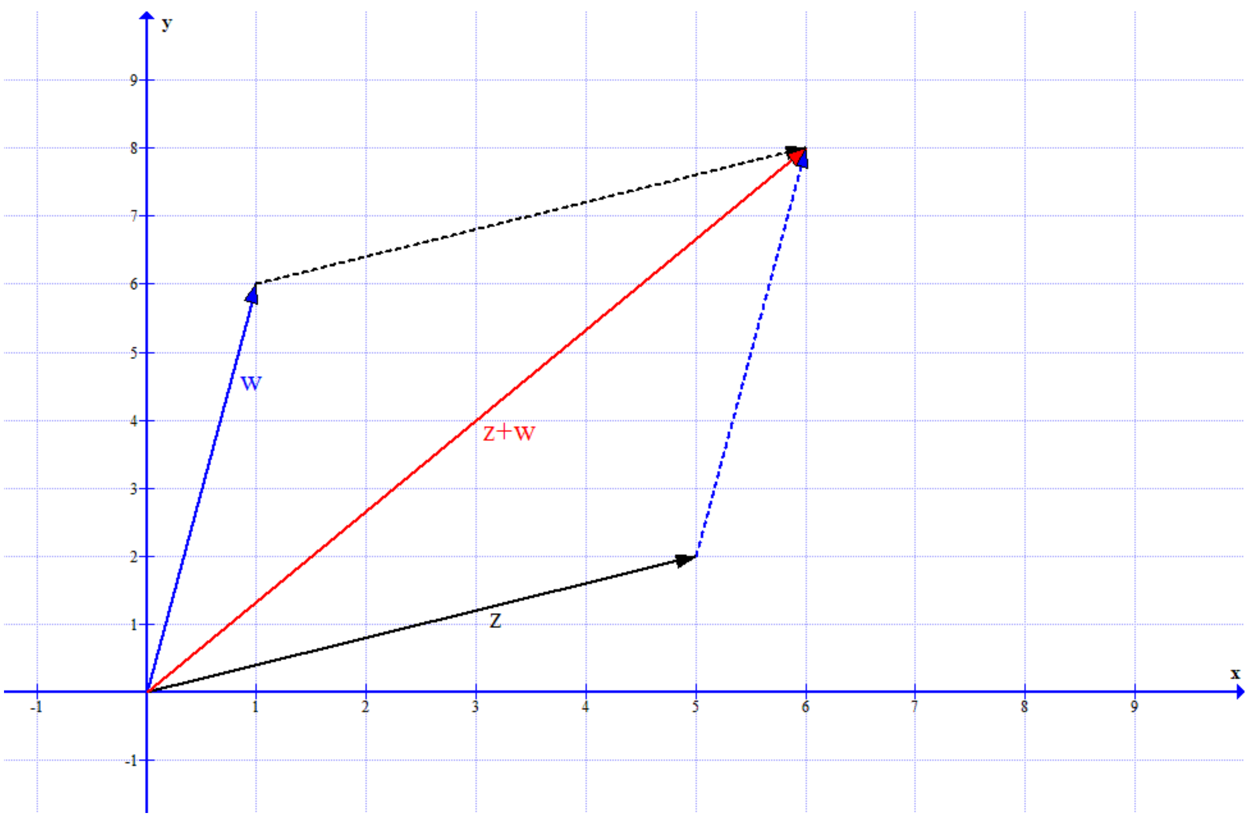


Graphically, if we **add** two complex number, we are adding the two numbers by treating them as **vectors** and adding like **vector addition**.

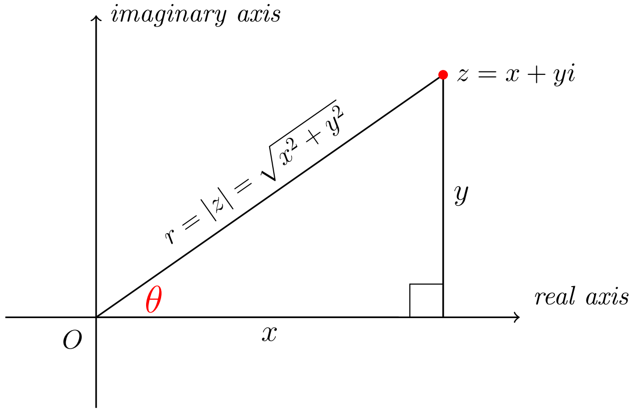


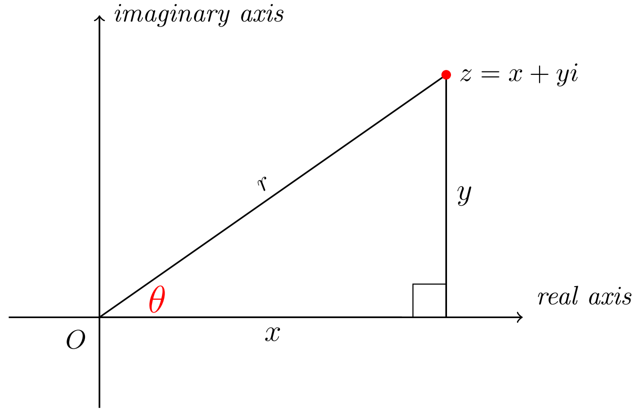
For example, Let $z = 5 + 2i$, $w = 1 + 6i$, then

$z + w = (5 + 2i) + (1 + 6i) = (5 + 1) + (2 + 6)i = 6 + 8i$



In order to interpret multiplication of two complex numbers, let's look again at the complex number represented as a point on the complex plane. This time, we let $r = \sqrt{x^2 + y^2}$ be the magnitude of z. Let $0 \le \theta < 2\pi$ be the angle in standard position with z being its terminal point. We call θ the **argument** of the complex number z:





By definition of sine and cosine, we have

$\cos(\theta) = \dfrac{x}{r} \Rightarrow x = r\cos(\theta)$

$\sin(\theta) = \dfrac{y}{r} \Rightarrow y = r\sin(\theta)$

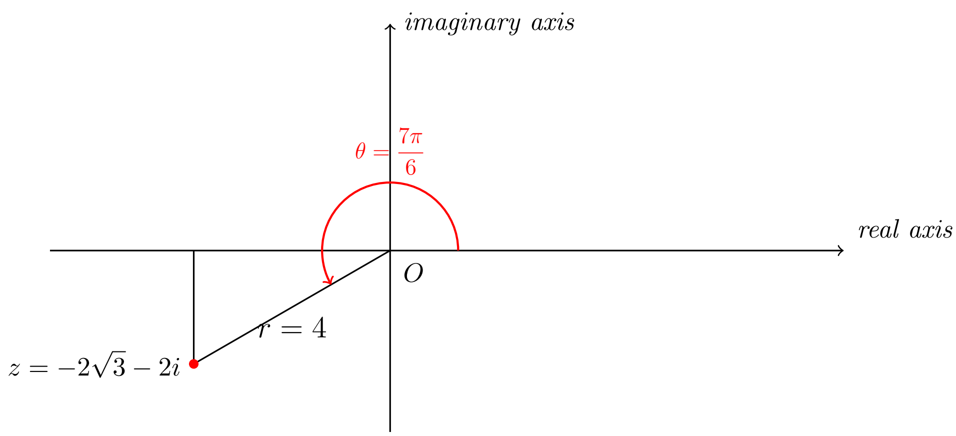
We have obtained the **polar representation** of a complex number:

Suppose $z = x + yi$ is a complex number with (x, y) in rectangular coordinate. Let $r = |z| = \sqrt{x^2 + y^2}$, let $0 \leq \theta < 2\pi$ be an angle in standard position whose terminal point is (x, y) (in other words, (r, θ) is the polar coordinate of (x, y)), then

$z = x + yi = r\cos(\theta) + r(\sin(\theta))i = r\left[\cos(\theta) + i\sin(\theta)\right]$

Example: Write $-2\sqrt{3} - 2i$ in polar form:

Ans: $r = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{(4 \cdot 3) + 4} = \sqrt{12 + 4} = 4$



The sides of the triangle formed has length 2, $2\sqrt{3}$, and 4. This is the ratio of a $30° - 60° - 90°$ triangle, meaning that $\theta = \dfrac{7\pi}{6}$

Therefore,

$z = -2\sqrt{3} - 2i = 4\left[\cos\left(\dfrac{7\pi}{6}\right) + i\sin\left(\dfrac{7\pi}{6}\right)\right]$

Example: Express the (polar form) complex number $z = 3\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$ as a complex number in rectangular form.

Ans: $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$, and, $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$, we have:

$$z = 3\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right] = 3\left[-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right] = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

Let $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$, let $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$, then,

$z_1 \cdot z_2 = [r_1(\cos(\theta_1) + i\sin(\theta_1))][r_2(\cos(\theta_2) + i\sin(\theta_2))]$

$= r_1r_2[(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2))]$

$= r_1r_2\left[\cos(\theta_1)\cos(\theta_2) + i\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2) + i^2\sin(\theta_1)\sin(\theta_2)\right]$

$= r_1r_2[\cos(\theta_1)\cos(\theta_2) + i\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)]$

$= r_1r_2[(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2))]$

$= r_1r_2[(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2))]$

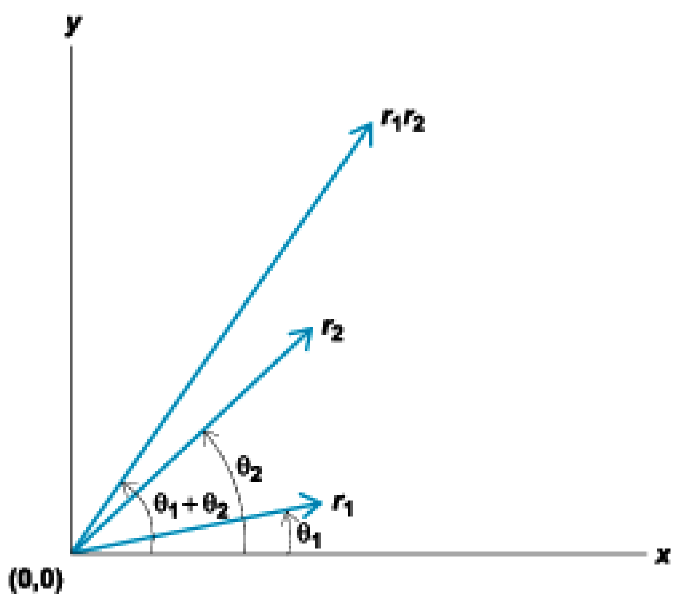
$= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$ (double angle identity)

We have just proved the following formula:

Let $z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)]$, $z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)]$, be complex numbers in polar form, then

$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

This formula tells us that, goemetrically, to multiply two complex numbers is to multiply their magniture and add their argument. In other words, to multiply $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$ and $z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$, we multiply their magniture, r_2 and r_2, and add their argument, θ_1 and θ_2. The result (product) is a complex number whose magnitude is r_1r_2 and whose argument is $\theta_1 + \theta_2$.



We can use a similar method to prove the following formula for dividing two complex numbers in polar form:

Let $z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)]$, let $z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)]$, $z_2 \neq 0$, be complex numbers in polar form, then

$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

Example: Let $z_1 = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$, $z_2 = 5\left[\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right]$

then by the formula,

$$z_1 z_2 = (2)(5)\left[\cos\left(\frac{\pi}{4} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{4\pi}{3}\right)\right] = 10\left[\cos\left(\frac{19\pi}{12}\right) + i\sin\left(\frac{19\pi}{12}\right)\right]$$

Example: Let $z = r\left(\cos(\theta) + i\sin(\theta)\right)$, find z^2 and z^3

Ans: Using the formula, $z^2 = z \cdot z = \left[r\left(\cos(\theta) + i\sin(\theta)\right)\right]\left[r\left(\cos(\theta) + i\sin(\theta)\right)\right]$

$= r \cdot r\left[\cos(\theta + \theta) + i\sin(\theta + \theta)\right] = r^2\left[\cos(2\theta) + i\sin(2\theta)\right]$

Similarly, to find z^3, we note that $z^3 = z^2 \cdot z$

$= \left[r^2\left(\cos(2\theta) + i\sin(2\theta)\right)\right]\left[r\left(\cos(\theta) + i\sin(\theta)\right)\right]$

$= r^2 \cdot r\left[\cos(2\theta + \theta) + i\sin(2\theta + \theta)\right] = r^3\left[\cos(3\theta) + i\sin(3\theta)\right]$

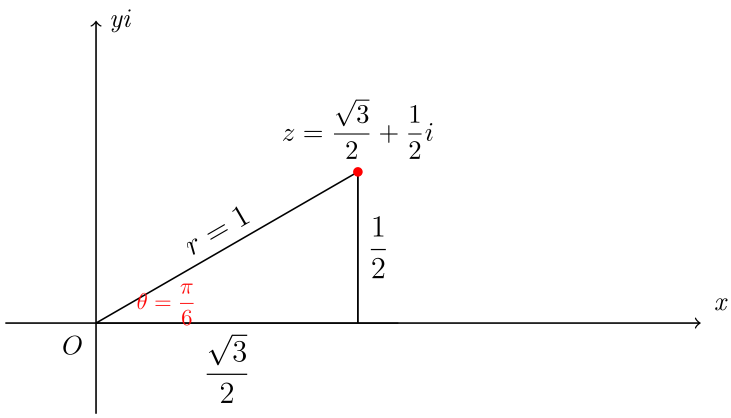
This pattern tells us that, each time we raise a complex number z to an integer power, n, we raise its magniture, r, to the $n-$th power, and multiply its argument, θ, by n. This is the:

De Moivre's Theorem:

Suppose $z = r\left[\cos(\theta) + i\sin(\theta)\right]$ is a complex number is polar form, and $n \geq 1$ is a positive integer, then:

$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$

Example: Let $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ be a complex number in rectangular form. Find z^{17}



Ans: We first write z in polar form. We note that θ, the argument of z, is the angle of the special $30^\circ - 60^\circ - 90^\circ$ triangle, so $\theta = \frac{\pi}{6}$

$$r = |z| = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{2}} = \sqrt{1} = 1$$

Therefore, $z = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i = 1\left[\cos\left(\dfrac{\pi}{6}\right) + i\sin\left(\dfrac{\pi}{6}\right)\right]$

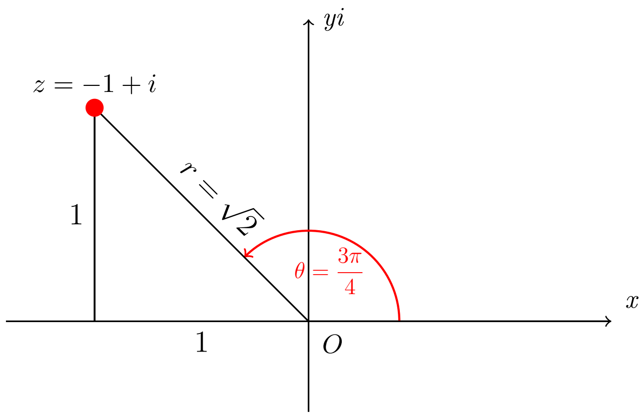
Using De Moivre's Theorem, we have:

$$z^{17} = 1^{17}\left[\cos\left(17\cdot\frac{\pi}{6}\right) + i\sin\left(17\cdot\frac{\pi}{6}\right)\right] = 1\left[\cos\left(\frac{17\pi}{6}\right) + i\sin\left(\frac{17\pi}{6}\right)\right]$$

$$= \left[\cos\left(2\pi + \frac{5\pi}{6}\right) + i\sin\left(2\pi + \frac{5\pi}{6}\right)\right] = \left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right] = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Example: Let $z = -1 + i$, find z^{14}

Ans: We again first write z in polar form:



We find $r = |z| = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

The $x-$ and $y-$ coordinate of z forms an isoceles right triangle with the $x-$axis, so the reference angle of z is $\dfrac{\pi}{4}$, meaning that the argument of z, θ, is $\theta = \dfrac{3\pi}{4}$.

We have:

$$z = -1 + i = \sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$$

Using De Moivre's Theorem, we have:

$$z^{14} = \left(\sqrt{2}\right)^{14}\left[\cos\left(14\cdot\frac{3\pi}{4}\right) + i\sin\left(14\cdot\frac{3\pi}{4}\right)\right] = (2)^7\left[\cos\left(\frac{21\pi}{2}\right) + i\sin\left(\frac{21\pi}{2}\right)\right]$$

$$= 128\left[\cos\left(10\pi + \frac{\pi}{2}\right) + i\sin\left(10\pi + \frac{\pi}{2}\right)\right] = 128\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$$

$$= 128\left[0 + i(1)\right] = 128i$$

We know the meaning of the n-th root of real numbers like $\sqrt{5}$ or $\sqrt[3]{13}$. With the introduction of complex numbers, even numbers like $\sqrt{-3}$ can now be expressed as complex numbers. But what about numbers like the square root or cube roots of complex numbers? For example, what is the meaning of $\sqrt{i}$ or $\sqrt[3]{1+i}$, and does it exist?

By definition of the $n-$th root, we know that if $b = \sqrt[n]{a}$, this means that $b^n = a$. So if w is a complex number, if we say $z = \sqrt[n]{w}$, we are looking for a number z with the property that $z^n = w$. If we write w as a complex number in polar form, then De Movire's Theorem allows us to find all the $n-$th roots of w.

The n-th roots of a complex number:

Let $n \geq 1$ be a positive integer, let $w = r\left[\cos(\theta) + i\sin(\theta)\right]$ be a complex number in polar form with $r > 0$ be the magniture of w, then the equation $z^n = w$ has n **distinct** (complex numbers) solutions. Each solution z_k is of the form:

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right],$$

where $k = 0, 1, 2, 3, \ldots, n-2, n-1$

The above theorem tells us that each complex number w always has n distinct (complex) $n-$th roots.

Example: Find all the cube roots of i. In other words, find all the (complex) solutions of the equation: $z^3 = i$. Write your answer in rectangular form if possible.

Ans: According to the theorem just mentioned, there will be three distinct complex numbers z that satisfies the equation. In order to apply the theorem, we need to represent i in polar form. Notice that $|i| = 1$, and the terminal side of i is the $y-$ axis, with an argument of $\theta = \frac{\pi}{2}$, therefore, we have:

$$i = 1\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$$

According to theorem, we have:

$$z_0 = \sqrt[3]{1}\left[\cos\left(\frac{\pi/2}{3} + \frac{2(0)\pi}{3}\right) + i\sin\left(\frac{\pi/2}{3} + \frac{2(0)\pi}{3}\right)\right]$$

$$= \left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right] = \frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)$$

$$z_1 = \sqrt[3]{1}\left[\cos\left(\frac{\pi/2}{3} + \frac{2(1)\pi}{3}\right) + i\sin\left(\frac{\pi/2}{3} + \frac{2(1)\pi}{3}\right)\right]$$

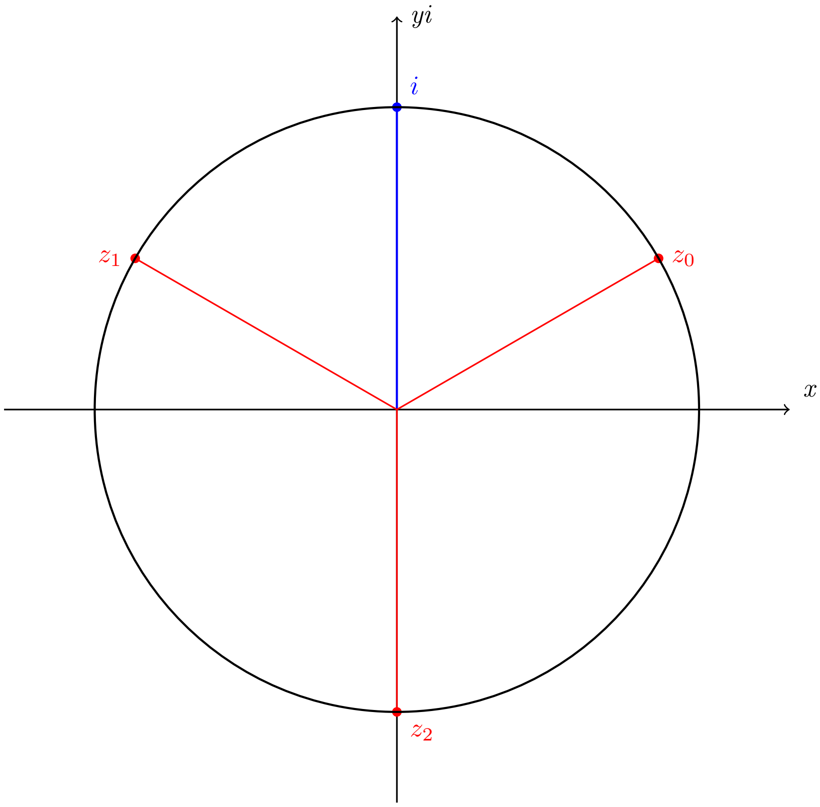
$$= \left[\cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)\right]$$

$$= \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)$$

$$z_2 = \sqrt[3]{1}\left[\cos\left(\frac{\pi/2}{3} + \frac{2(2)\pi}{3}\right) + i\sin\left(\frac{\pi/2}{3} + \frac{2(2)\pi}{3}\right)\right]$$

$$= \left[\cos\left(\frac{\pi}{6}+\frac{4\pi}{3}\right)+i\sin\left(\frac{\pi}{6}+\frac{4\pi}{3}\right)\right]$$
$$= \cos\left(\frac{3\pi}{2}\right)+i\sin\left(\frac{3\pi}{2}\right) = 0+i(-1) = -i$$

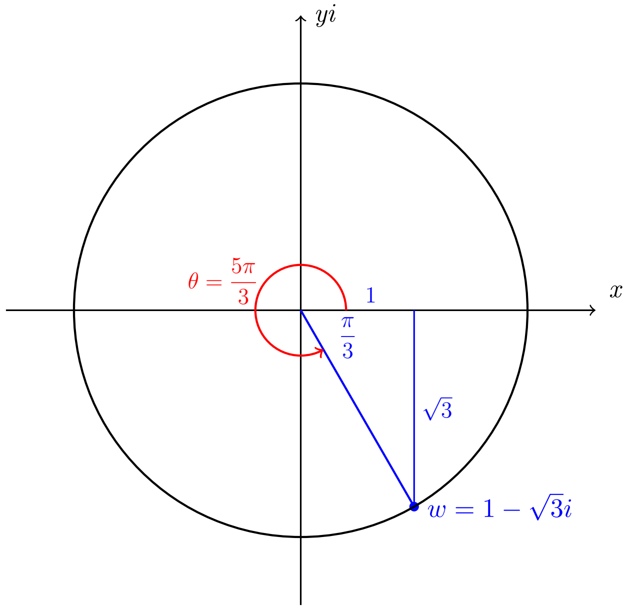
Graphically, these three roots all lie on the same circle (on the complex plane) with radius 1 (magniture of each z_k for this example), and they are equally spaced from each other.



Example: Find all the 5th roots of the complex number $w = 1-\sqrt{3}i$. Write your solution in rectangular form, if possible.

Ans: We are looking for all five of the (complex) solutions to the equation $z^5 = w$. We first express w in polar form:

$$|w| = \sqrt{(1)^2+\left(-\sqrt{3}\right)^2} = \sqrt{1+3} = 2$$



The sides of the triangle formed by the terminal side of w and the x and y is in the ratio of a $30° - 60° - 90°$ triangle. So the reference angle of w is $\frac{\pi}{3}$, so the argument of w is $\theta = \frac{5\pi}{3}$.

Therefore, $w = 2\left[\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right]$

Using the formula, we know there are five unique solutions to the equation:

$z^5 = w = 2\left[\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right]$, these solutions are:

$$z_0 = \sqrt[5]{2}\left[\cos\left(\frac{5\pi/3}{5} + \frac{2(0)\pi}{5}\right) + i\sin\left(\frac{5\pi/3}{5} + \frac{2(0)\pi}{5}\right)\right]$$

$$= \sqrt[5]{2}\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right] = \sqrt[5]{2}\left[\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = \frac{\sqrt[5]{2}}{2} + i\left(\frac{\sqrt[5]{2}\cdot\sqrt{3}}{2}\right)$$

$$z_1 = \sqrt[5]{2}\left[\cos\left(\frac{5\pi/3}{5} + \frac{2(1)\pi}{5}\right) + i\sin\left(\frac{5\pi/3}{5} + \frac{2(1)\pi}{5}\right)\right]$$

$$= \sqrt[5]{2}\left[\cos\left(\frac{\pi}{3} + \frac{2\pi}{5}\right) + i\sin\left(\frac{\pi}{3} + \frac{2\pi}{5}\right)\right] = \sqrt[5]{2}\left[\cos\left(\frac{11\pi}{15}\right) + i\sin\left(\frac{11\pi}{15}\right)\right]$$

$$z_2 = \sqrt[5]{2}\left[\cos\left(\frac{5\pi/3}{5} + \frac{2(2)\pi}{5}\right) + i\sin\left(\frac{5\pi/3}{5} + \frac{2(2)\pi}{5}\right)\right]$$

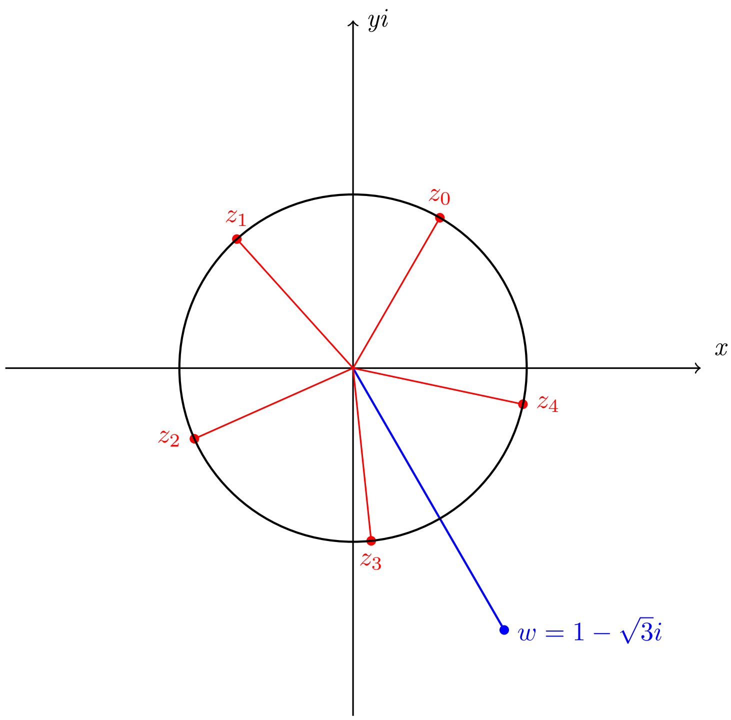
$$= \sqrt[5]{2}\left[\cos\left(\frac{\pi}{3} + \frac{4\pi}{5}\right) + i\sin\left(\frac{\pi}{3} + \frac{4\pi}{5}\right)\right] = \sqrt[5]{2}\left[\cos\left(\frac{17\pi}{15}\right) + i\sin\left(\frac{17\pi}{15}\right)\right]$$

$$z_3 = \sqrt[5]{2}\left[\cos\left(\frac{5\pi/3}{5} + \frac{2(3)\pi}{5}\right) + i\sin\left(\frac{5\pi/3}{5} + \frac{2(3)\pi}{5}\right)\right]$$
$$= \sqrt[5]{2}\left[\cos\left(\frac{\pi}{3} + \frac{6\pi}{5}\right) + i\sin\left(\frac{\pi}{3} + \frac{6\pi}{5}\right)\right] = \sqrt[5]{2}\left[\cos\left(\frac{23\pi}{15}\right) + i\sin\left(\frac{23\pi}{15}\right)\right]$$

$$z_4 = \sqrt[5]{2}\left[\cos\left(\frac{5\pi/3}{5} + \frac{2(4)\pi}{5}\right) + i\sin\left(\frac{5\pi/3}{5} + \frac{2(4)\pi}{5}\right)\right]$$
$$= \sqrt[5]{2}\left[\cos\left(\frac{\pi}{3} + \frac{8\pi}{5}\right) + i\sin\left(\frac{\pi}{3} + \frac{8\pi}{5}\right)\right] = \sqrt[5]{2}\left[\cos\left(\frac{29\pi}{15}\right) + i\sin\left(\frac{29\pi}{15}\right)\right]$$

While we can use special triangle ratio to turn z_0 into rectangular form, it will not be easy to express the other z_1, z_2, z_3, or z_4 into rectangular form, so we leave the answer in polar form.

Each of the five five roots lie on the circle (on the complex plane) with radius equal to $\sqrt[5]{2}$, and equally spaced from each other.



Let $n \geq 1$ be a positive integer, we say that an **n-th root of unity** is a (complex) number z that solves the equation $z^n = 1$. From the theorem we just discussed, for each n, there are n many $n-$th roots of unity.

Example: Find all the $6th$ roots of unity. Express your answers in rectangular form if possible.

Ans: We are looking for all the (complex) solutions of the equation $z^6 = 1$. Writing 1 in polar form, we have:

$1 = 1 + 0i = 1\left[\cos(0) + i\sin(0)\right]$

The six roots of unity are given by:

$$z_0 = 1\left[\cos\left(\frac{0}{6} + \frac{2(0)\pi}{6}\right) + i\sin\left(\frac{0}{6} + \frac{2(0)\pi}{6}\right)\right] = \cos(0) + i\sin(0) = 1$$

$$z_1 = 1\left[\cos\left(\frac{0}{6} + \frac{2(1)\pi}{6}\right) + i\sin\left(\frac{0}{6} + \frac{2(1)\pi}{6}\right)\right]$$

$$= \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)$$

$$z_2 = 1\left[\cos\left(\frac{0}{6} + \frac{2(2)\pi}{6}\right) + i\sin\left(\frac{0}{6} + \frac{2(2)\pi}{6}\right)\right]$$

$$= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)$$

$$z_3 = 1\left[\cos\left(\frac{0}{6} + \frac{2(3)\pi}{6}\right) + i\sin\left(\frac{0}{6} + \frac{2(3)\pi}{6}\right)\right]$$

$$= \cos(\pi) + i\sin(\pi) = -1 + 0i = -1$$

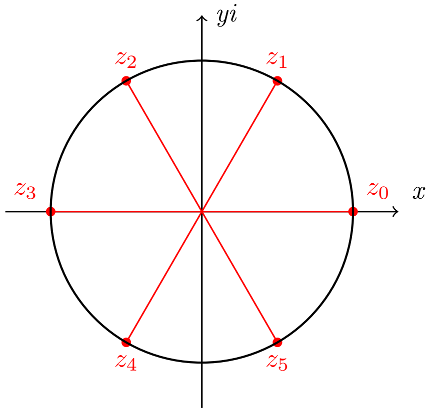
$$z_4 = 1\left[\cos\left(\frac{0}{6} + \frac{2(4)\pi}{6}\right) + i\sin\left(\frac{0}{6} + \frac{2(4)\pi}{6}\right)\right]$$

$$= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right) = -\frac{1}{2} - i\left(\frac{\sqrt{3}}{2}\right)$$

$$z_5 = 1\left[\cos\left(\frac{0}{6} + \frac{2(5)\pi}{6}\right) + i\sin\left(\frac{0}{6} + \frac{2(5)\pi}{6}\right)\right]$$

$$= \cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{2} - i\left(\frac{\sqrt{3}}{2}\right)$$

The six solutions are equally spaced on the unit circle.



yi
x
z0
z1
z2
z3
z4
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