Math 3B – Calculus II
Berkeley City College

Section 7.8 – Improper Integrals

In today's lecture, we will discuss how to evaluate integrals where the limits of integration are infinite or where we integrate over an infinite discontinuity as these integrals are encountered in different areas of math and physics.

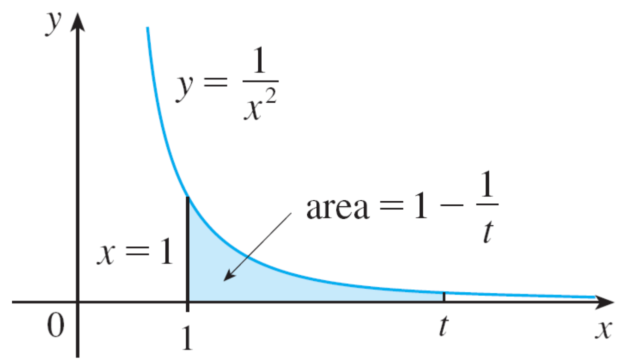
I. Type 1: Infinite Limits of Integration

Consider the infinite region S that lies under the curve $y = 1/x^2$, above the x-axis, and to the right of the line $x = 1$.

You might think that, since S is infinite in extent, its area must be infinite, but let's take a closer look.

The area of the part of S that lies to the left of the line $x = t$ is

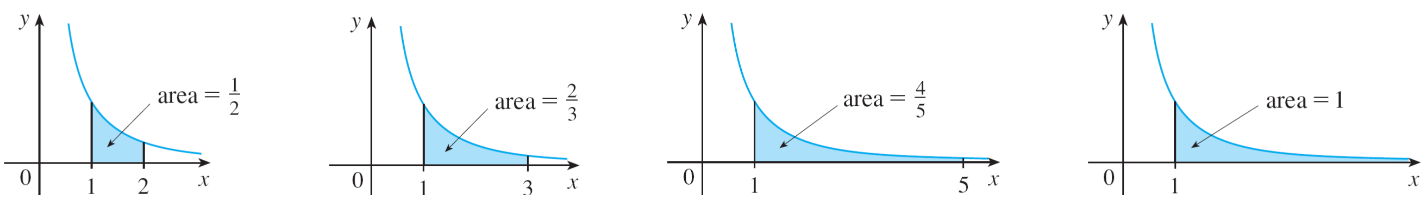
$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_1^t = -\frac{1}{t} + 1$$



Notice that $A(t) < 1$ no matter how large t is chosen. We also observe that

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(-\frac{1}{t} + 1\right) = 1$$

The area of the shaded region approaches 1 as $t \to \infty$ as illustrated below



We say that the area of the infinite region S is equal to 1 and write $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = 1$

Using this example as a guide, we define the integral of f (not necessarily a positive function) over an infinite interval as the limit of integrals over finite intervals.

1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used.

Example 1: Determine if the following improper integrals converge or diverge. If the integral is convergent, find the value it converges to.

A. $\int_1^\infty (x-1)e^{-x}\,dx$

B. $\int_5^\infty \frac{1}{\sqrt{x-4}}\,dx$

C. $\int_{-\infty}^{-3} \frac{1}{x^2 - 4}\,dx$

D. $\int_{-\infty}^{\infty} \frac{1}{x^2 + 4}\,dx$

Example 2: Determine the values of p for which $\int_1^\infty \frac{1}{x^p}\,dx$ converges.

From the above example we deduce the following

2 $\quad \int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

Example 3: Use the result of **Example 2** to determine if the following improper integrals converge or diverge. If the integral is convergent, state the value it converges to.

A. $\int_1^\infty \frac{1}{\sqrt{\pi x}}\,dx$	**B.** $\int_4^\infty \frac{1}{x\sqrt{x}}\,dx$

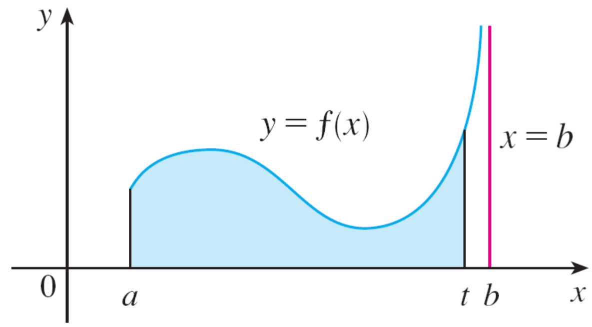
II. Type 2: Infinite Integrands

Suppose that f is a positive continuous function defined on a finite interval $[a, b)$ but has a vertical asymptote at b.

Let S be the unbounded region under the graph of f and above the x-axis between a and b.

The area of the part of S between a and t is

$$A(t) = \int_a^t f(x)\,dx$$



If it happens that $A(t)$ approaches a definite number A as $t \to b^-$, then we say that the area of the region S is A and we write

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_1^t f(x)\,dx$$

We use this equation to define an improper integral of Type 2 even when f is not a positive function, no matter what type of discontinuity f has at b.

3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example 4: Determine if the following improper integrals converge or diverge. If the integral is convergent, find the value it converges to.

A. $\int_0^2 \frac{1}{\sqrt{4 - x^2}}\,dx$

B. $\int_{20}^{4} \frac{1}{\sqrt{x-4}}\,dx$

C. $\int_{-1}^{4} \frac{1}{x^2}\,dx$