Math 3B – Calculus II
Berkeley City College

Section 7.7 – Numerical Integration

We know that if a function $f(x)$ is continuous on an interval $[a, b]$, then $\int_a^b f(x)dx$ exists; however, there are many functions with no known antiderivatives. This makes it nearly impossible to apply various integration techniques we have studied this semester.

Some examples of antiderivatives that cannot be expressed algebraically as an elementary function (non-elementary functions) include

$$\int e^{-x^2}dx,\ \int \sin(x^2)dx,\ \int \frac{\sin x}{x}dx,\ \int \frac{e^x}{x}dx,\ \int \frac{1}{\ln x}dx,\ \int \sqrt{1+x^3}\,dx,\ \&\ \int \sqrt{1-k\sin^2 x}\,dx \text{ where } 0<k<1.$$

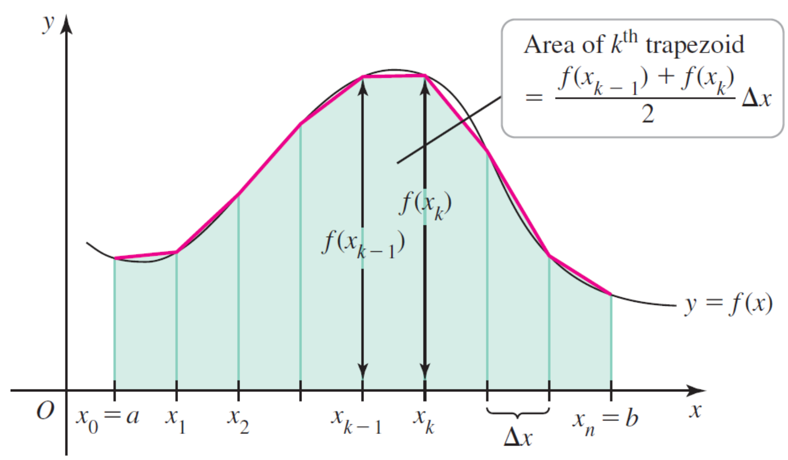
Scientists rely on various numerical techniques programmed into technology to compute definite integrals that involve the integrands of such functions. We have had experience using Right, Left, and Midpoint Riemann Sums to approximate definite integrals; however, those methods are rather inefficient and take too long before giving accurate answers. Today we will take our knowledge to the next level and talk about the use of the Trapezoid & Simpson's Rule.

I. Trapezoid Rule

Let $f(x)$ be continuous on $[a, b]$.

Suppose we wanted compute $\int_a^b f(x)dx$ by using trapezoids instead of rectangles.

We first break up $[a, b]$ into n-subintervals of size Δx and build trapezoids on each subinterval, then approximate the total area under the curve by adding up the areas of all trapezoids.



$$\begin{aligned}
\int_a^b f(x)dx &\approx \frac{1}{2}\left(f(x_0)+f(x_1)\right)\Delta x+\frac{1}{2}\left(f(x_1)+f(x_2)\right)\Delta x+\frac{1}{2}\left(f(x_2)+f(x_3)\right)\Delta x+\dots+\frac{1}{2}\left(f(x_{n-1})+f(x_n)\right)\Delta x\\
&=\frac{\Delta x}{2}\left[\left(f(x_0)+f(x_1)\right)+\left(f(x_1)+f(x_2)\right)+\dots+\left(f(x_{n-2})+f(x_{n-1})\right)+\left(f(x_{n-1})+f(x_n)\right)\right]\\
&=\frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+\dots+2f(x_{n-1})+f(x_n)\right]\\
&=\frac{\Delta x}{2}\left[f(x_0)+2\sum_{i=1}^{n-1}f(x_i)+f(x_n)\right]
\end{aligned}$$

Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = (b-a)/n$ and $x_i = a + i\,\Delta x$.

Example 1: We can easily show that $\int_0^1 \frac{4}{1+x^2}dx = \pi$. Approximate this integral using the Trapezoid Rule for the specified value of *n.* Round your final answer to 6 decimal places.

A. $n = 2$

$$\Delta x = \frac{1-0}{2} = \frac{1}{2}$$

$$\Rightarrow \int_0^1 \frac{4}{1+x^2}dx \approx \frac{\Delta x}{2}\left[f\left(x_0\right)+2f\left(x_1\right)+f\left(x_2\right)\right]$$

$$= \frac{1/2}{2}\left[f\left(0\right)+2f\left(\frac{1}{2}\right)+f\left(1\right)\right]$$

$$= \frac{1}{4}\left[\left(\frac{4}{1+0^2}\right)+2\left(\frac{4}{1+(1/2)^2}\right)+\left(\frac{4}{1+1^2}\right)\right]$$

$$= 3.1$$

B. $n = 4$

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

$$\Rightarrow \int_0^1 \frac{4}{1+x^2}dx \approx \frac{\Delta x}{2}\left[f\left(x_0\right)+2f\left(x_1\right)+2f\left(x_2\right)+2f\left(x_3\right)+f\left(x_4\right)\right]$$

$$= \frac{1/4}{2}\left[f\left(0\right)+2f\left(\frac{1}{4}\right)+2f\left(\frac{2}{4}\right)+2f\left(\frac{3}{4}\right)+f\left(1\right)\right]$$

$$= \frac{1}{8}\left[\left(\frac{4}{1+0^2}\right)+2\left(\frac{4}{1+(1/4)^2}\right)+2\left(\frac{4}{1+(2/4)^2}\right)+2\left(\frac{4}{1+(3/4)^2}\right)+\left(\frac{4}{1+1^2}\right)\right]$$

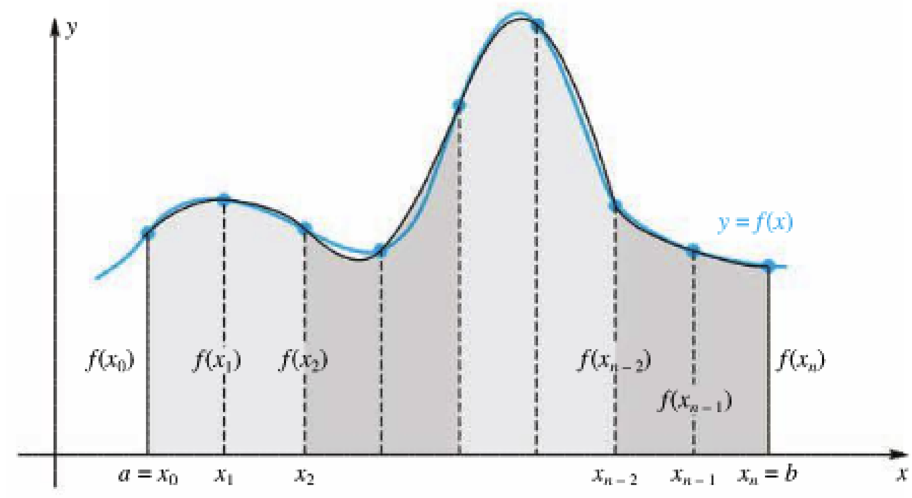
$$\approx 3.131176$$

C. $n = 6$

II. Simpsons Rule

Simpson's Rule estimates $\int_a^b f(x)dx$ by approximating $f(x)$ on a partition using a series of parabolas going through consecutive sets of 3 points where the next starts at the endpoint from the previous 3.

Using these three points and a technique called interpolation, a quadratic function has the same values as $f(x)$ at the midpoint and 2 endpoints of the interval.



Adding up the area under these parabolas approximates the total area under the curve.

Note: Simpson's Rule can only be applied when the area is divided into an even number of strips due to the way the points are chosen for the interpolation.

Simpson's Rule

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where n is even and $\Delta x = (b - a)/n$.

Note the pattern of coefficients: 1, 4, 2, 4, 2, 4, 2, . . . , 4, 2, 4, 1.

Example 2: We can easily show that $\int_0^1 \frac{4}{1+x^2}dx = \pi$. Approximate this integral using Simpson's Rule for the specified value of n. Round your final answer to 6 decimal places.

A. $n = 2$

$$\Delta x = \frac{1-0}{2} = \frac{1}{2}$$

$$\Rightarrow \int_0^1 \frac{4}{1+x^2}dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$= \frac{1/2}{3}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$$

$$= \frac{1}{6}\left[\left(\frac{4}{1+0^2}\right) + 4\left(\frac{4}{1+(1/2)^2}\right) + \left(\frac{4}{1+1^2}\right)\right]$$

$$\approx 3.133333$$

B. $n = 4$

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

$$\Rightarrow \int_0^1 \frac{4}{1+x^2}\,dx \approx \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)\right]$$

$$= \frac{1/4}{3}\left[f(0)+4f\left(\frac{1}{4}\right)+2f\left(\frac{2}{4}\right)+4f\left(\frac{3}{4}\right)+f(1)\right]$$

$$= \frac{1}{12}\left[\left(\frac{4}{1+0^2}\right)+4\left(\frac{4}{1+(1/4)^2}\right)+2\left(\frac{4}{1+(2/4)^2}\right)+4\left(\frac{4}{1+(3/4)^2}\right)+\left(\frac{4}{1+1^2}\right)\right]$$

$$\approx 3.141569$$

C. $n = 6$.

III. Error Analysis for the Trapezoid & Simpson's Rule

A. Absolute & Relative Error

Knowing the true value of $\int_a^b f(x)dx$, we can approximate the absolute and relative error attained for each method.

DEFINITIONS **Absolute and Relative Error**

Suppose c is a computed numerical solution to a problem having an exact solution x. There are two common measures of the error in c as an approximation to x:

$$\textbf{absolute error} = |c - x|$$

and

$$\textbf{relative error} = \frac{|c - x|}{|x|} \quad (\text{if } x \neq 0)$$

Example 3: In this example we will compare the errors attained for each method. Round all final answers to 6 decimal places.

A. Fill in the chart below for the trapezoid method.

n	Absolute Error	Relative Error
2	$\|3.1-\pi\| \approx 0.041593$	$\left\|\frac{3.1-\pi}{\pi}\right\| \approx 0.013239$ (1.3239%)
4	$\|3.131176-\pi\| \approx 0.010417$	$\left\|\frac{3.131176-\pi}{\pi}\right\| \approx 0.003316$ (0.3316%)
6		

B. Fill in the chart below for Simpson's Rule.

n	Absolute Error	Relative Error
2	$\lvert 3.133333-\pi \rvert \approx 0.008260$	$\left\lvert \frac{3.133333-\pi}{\pi} \right\rvert \approx 0.002629$ (0.2629%)
4	$\lvert 3.141569-\pi \rvert \approx 0.000024$	$\left\lvert \frac{3.141569-\pi}{\pi} \right\rvert \approx 0.000008$ (0.0008%)
6		

C. Comparing both methods, which gives better accuracy in a faster amount of time?

B. Estimating Absolute Error

In the previous example, we saw that we could compute the absolute error as we knew the exact value of the definite integral. This may not always be the case. We are able to give a "worst case scenario" approximation of the error attained by using these methods.

Error Estimate for the Trapezoid Rule

Assume f'' is continuous on $[a,b]$ and k is a real number such that $|f''(x)| \le k$ for all x in $[a,b]$. Then the absolute error in approximating $\int_a^b f(x)\,dx$ by the Trapezoid Rule using n-subintervals satisfy the inequality

$$\left|E_T\right| \le \frac{k(b-a)^3}{12n^2}.$$

We need Taylor's Theorem to prove why these error estimates are valid, but for now let's know how to apply them.

<u>**Example 4**</u>: It can be shown that $\left|\frac{d^2}{dx^2}\left(e^{x^2}\right)\right| \le 6e$ on $[0,1]$. Approximate $\int_0^1 e^{x^2}\,dx$ using T_4, then estimate the error obtained by approximating the integral with T_4. Round your answers to 6 decimal places.

Error Estimate for Simpson's Rule

Assume $f^{(4)}$ is continuous on $[a,b]$ and K is a real number such that $|f^{(4)}(x)| \le K$ for all x in $[a,b]$. Then the absolute error in approximating $\int_a^b f(x)\,dx$ by the Simpson's Rule using n-subintervals satisfy the inequality

$$\left|E_S\right| \le \frac{K(b-a)^5}{180n^4}.$$

<u>**Example 5**</u>: It can be shown that $\left|\frac{d^4}{dx^4}\left(e^{x^2}\right)\right| \le 76e$ on [0,1]. Find the smallest value of n need to approximate $\int_0^1 e^{x^2}\,dx$ with S_n up to an order of error no more than 10^{-3}, then find S_n up to this order of error.

III. Applications of Numerical Integration – Simpson's Rule

Example 6: The speedometer reading (*v*) on a car was observed at 1-minute intervals and recorded in the chart. Use Simpson's Rule to estimate the distance traveled by the car.

t (min)	v (mi/h)	t (min)	v (mi/h)
0	40	6	56
1	42	7	57
2	45	8	57
3	49	9	55
4	52	10	56
5	54		

Example 7: To meet the demand of parking, a city has allocated the area shown below to build a new parking lot. Excluding labor, it costs \$0.10 per square foot to clear the land and \$2.00 per square foot to pave the land. Use Simpson's Rule to help determine the total cost to build the parking lot.