Math 3B – Calculus II
Berkeley City College

Section 7.5: Integration Strategies

We will outline some strategies to approach integrating various functions. Before doing so, we will assume the following integrals as fact.

Table of Integration Formulas Constants of integration have been omitted.

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$	**2.** $\int \frac{1}{x}\,dx = \ln\|x\|$
3. $\int e^x\,dx = e^x$	**4.** $\int a^x\,dx = \frac{a^x}{\ln a}$
5. $\int \sin x\,dx = -\cos x$	**6.** $\int \cos x\,dx = \sin x$
7. $\int \sec^2 x\,dx = \tan x$	**8.** $\int \csc^2 x\,dx = -\cot x$
9. $\int \sec x \tan x\,dx = \sec x$	**10.** $\int \csc x \cot x\,dx = -\csc x$
11. $\int \sec x\,dx = \ln\|\sec x + \tan x\|$	**12.** $\int \csc x\,dx = \ln\|\csc x - \cot x\|$
13. $\int \tan x\,dx = \ln\|\sec x\|$	**14.** $\int \cot x\,dx = \ln\|\sin x\|$
15. $\int \sinh x\,dx = \cosh x$	**16.** $\int \cosh x\,dx = \sinh x$
17. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$	**18.** $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \quad a > 0$

Let us follow these steps when presented with any integral.

1. Simplify the Integrand if Possible

Sometimes the use of algebraic manipulation or trigonometric identities will simplify the integrand and make the method of integration obvious. Here are some examples:

$$\int \sqrt{x}(1+x)\,dx = \int (x^{1/2} + x^{3/2})\,dx \text{ or } \int \frac{\tan\theta}{\sec^2\theta}\,d\theta = \int \frac{\sin\theta}{\cos\theta}\cdot\cos^2\theta\,d\theta = \int \sin\theta\cos\theta\,d\theta.$$

2. Look for an Obvious Substitution

Try to find some function $u = g(x)$ in the integrand whose differential $du = g'(x)dx$ also occurs, apart from a constant factor. For instance, in the integral

$$\int \frac{x}{1-x^2}\,dx \text{ we notice that if } u = x^2 - 1, \text{ then } du = 2x\,dx.$$

Therefore we use the substitution $u = x^2 - 1$ instead of the method of partial fractions.

3. Classify the Integrand According to Its Form

If Steps 1 and 2 have not led to the solution, then we take a look at the form of the integrand $f(x)$.

a. ***Trigonometric functions***: If $f(x)$ is a product of powers of sin x and cos x, of tan x and sec x, or of cot x and csc x, then we use the substitutions. Here are some examples:

$$\int \sin^3\theta\,d\theta \qquad \int \cos^5 2x \sin 2x\,d\theta \qquad \int \cos^4 x\,dx \qquad \int \sec^4\theta \tan^3\theta\,d\theta.$$

b. ***Rational functions***: If f is a rational function, we use the procedure involving partial fractions. Here are some examples:

$$\int \frac{x-1}{x^3+x^2}dx \qquad \int \frac{t^3-t^2+1}{t(t^2+1)}dt.$$

c. ***Integration by Parts:*** If $f(x)$ is a product of a power of x (or a polynomial) and a transcendental function (such as a trigonometric, exponential, or logarithmic function), then we try integration by parts, choosing u and dv. Here are some examples:

$$\int x^2 \sin 2x\,dx \qquad \int t \ln t\,dt \qquad \int e^{-x}\cos x\,dx.$$

d. ***Trigonometric Substituiton:*** If any radical of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, or $\sqrt{x^2-a^2}$ occurs, we use a trigonometric substitution.

4.Try Again

If the first three steps have not produced the answer, remember that there are basically only two methods of integration: substitution and parts.

a. ***Try substitution:*** Even if no substitution is obvious (Step 2), some inspiration or ingenuity may suggest an appropriate substitution. Here are some examples:

$$\int 5x^9 e^{x^5}\,dx = \int 5x^4 x^5 e^{x^5}\,dx = \int ue^u\,du \text{ where } u = x^5 \ \&\ du = 5x^4 dx$$

$$\int \sin\sqrt{x}\,dx = \int 2u \sin u\,du \text{ where } u = \sqrt{x} \ \&\ du = \frac{1}{2\sqrt{x}}dx = \frac{1}{2u}dx$$

$$\int x^2\sqrt{x-1}\,dx = \int (u+1)^2\sqrt{u}\,du \text{ where } u = x-1,\ u+1 = x,\ \&\ du = dx$$

b. ***Try Parts:*** Although integration by parts is used most of the time on products of the form described in Step 3(c), it is sometimes effective on single functions. Here are some examples:

$$\int \arctan 2x\,dx \text{ where } u = \arctan 2x \ \&\ dv = dx.$$

c. ***Manipulate the integrand***: Algebraic manipulations may be useful in transforming the integral into an easier form. These manipulations may be more substantial than in Step 1 and may involve some ingenuity. Here ere some examples:

$$\int \sec^3 x\,dx = \int \sec^2 x \sec x\,dx \text{ where } u = \sec x \ \&\ dv = \sec^2 x dx$$

$$\int \frac{x}{\sqrt{3-2x-x^2}}dx = \int \frac{x}{\sqrt{4-(x+1)^2}}dx = \int \frac{u-1}{\sqrt{4-u^2}}du \text{ where } u = x+1 \text{ \& } du = dx$$

e. ***Use several methods***: Sometimes two or three methods are required to evaluate an integral. The evaluation could involve several successive substitutions of different types, or it might combine integration by parts with one or more substitutions. We can see this in many of the examples presented.

On a separate sheet of paper, evaluate the following integrals. Make note of the strategy you are using and why.

1. $\int xe^{-x^2}\,dx$

2. $\int e^{\sqrt[3]{x}}\,dx$

3. $\int \frac{x^2}{x^2-4}dx$

4. $\int \frac{1}{z^2(z^2+1)}dz$

5. $\int \frac{\tan^3\theta}{\cos^3\theta}dz$

6. $\int_0^{1/4} \arcsin 2x\,dx$

7. $\int_1^e \frac{\ln t}{t}dt$

8. $\int \left(t^2+1\right)\ln t\,dt$

9. $\int \sqrt{4-x^2}dx$

10. $\int_0^{\pi/2} \frac{\cos x}{(1+\sin^2 x)^{3/2}}dx$

11. $\int \cos^5 x\,dx$

12. $\int \frac{x}{\sqrt{x^2-4}}dx$

13. $\int \frac{1}{x^2+3x+3}dx$

14. $\int e^t \cos 2t\,dt$

Key

1. $-\frac{1}{2}e^{-x^2}+C$ - Substituiton

2. $e^{\sqrt[3]{x}}\left(3\sqrt[3]{x^2}-6\sqrt[3]{x}+6\right)+C$ - Substitution & By Parts

3. $x+\ln\left|\frac{x-2}{x+2}\right|+C$ - Partial Fractions

4. $-\frac{1}{x}-\arctan x+C$ - Partial Fractions

5. $\int\frac{\tan^3\theta}{\cos^3\theta}d\theta=\int\tan^3\theta\sec^3\theta\,d\theta$ or $\int\sin^3\theta\cos^{-6}\theta\,d\theta$ - Manipulation & Trig Integral Methods

6. $-\frac{1}{2}+\frac{\sqrt{3}}{4}+\frac{\pi}{24}$ - By Parts & Substitution

7. $\frac{1}{2}$ - Subtitution

8. $\left(\frac{t^3}{3}+t\right)\ln t-t-\frac{t^3}{9}+C$ - By Parts

9. $\frac{1}{2}x\sqrt{4-x^2}+2\arcsin\left(\frac{x}{2}\right)+C$ - Trig Substitution & Trig Identity

10. $\frac{1}{\sqrt{2}}$ - Substitution & Trig Substitution

11. $-\cos x+\frac{2\cos^3 x}{3}-\frac{\cos^5 x}{5}+C$ - Trig Integral Methods

12. $\sqrt{x^2-4}+C$ - Substitution or Trig Substitution

13. $\frac{2}{\sqrt{3}}\arctan\left(\frac{2x+3}{\sqrt{3}}\right)+C$ - Manipulation by Completing Square

14. $\frac{1}{5}e^t\left(\cos(2t)+2\sin(2t)\right)+C$ - By Parts