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Math 3B – Calculus II
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Section 7.4 – Integration of Rational Functions by Partial Fractions

When adding (or subtracting) 2 rational expressions with distinct linear factors, we find the LCD and use this to combine the fractions.

$$\frac{2}{x+2}+\frac{1}{x+3}=\frac{2(x+3)}{(x+2)(x+3)}+\frac{1(x+2)}{(x+3)(x+2)}=\frac{(2x+6)+(x+2)}{(x+2)(x+3)}=\frac{3x+8}{(x+2)(x+3)}.$$

Now suppose we wanted to evaluate $\int \frac{3x+8}{x^2+5x+6}dx$. Notice there is no obvious substitution; however,

$$\int \frac{3x+8}{x^2+5x+6}dx=\int \frac{3x+8}{(x+2)(x+3)}dx=\int\left(\frac{2}{x+2}+\frac{1}{x+3}\right)dx=2\ln|x+2|+\ln|x+3|+C.$$

Notice the denominator factored to 2 distinct linear factors, and decomposed the rational function into a sum of rational functions that are the ratio of a constant to a linear term.

To see how the method of partial fractions works in general, let's consider a rational function

$$f(x)=\frac{P(x)}{Q(x)}$$

where P and Q are polynomials. It's possible to express f as a sum of simpler fractions provided that the degree of P is less than the degree of Q. Such a rational function is called *proper*.

If f is *improper*, that is, $\deg(P) \geq \deg(Q)$, then we must take the preliminary step of dividing Q into P (by long division) until a remainder $R(x)$ is obtained such that $\deg(R) < \deg(Q)$.

The division statement is

$$\boxed{1} \quad f(x)=\frac{P(x)}{Q(x)}=S(x)+\frac{R(x)}{Q(x)}$$

where S and R are also polynomials.

As the next example illustrates, sometimes this preliminary step is all that is required.

Example 1: Evaluate $\int \frac{x^3-1}{2x+1}dx$.

The next step is to factor the denominator $Q(x)$ as far as possible.

It can be shown that any polynomial Q can be factored as a product of linear factors (of the form $ax + b$) and irreducible quadratic factors (of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$).

For instance, if $Q(x) = x^4 - 16$, we could factor it as

$$Q(x) = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$$

The third step is to express the proper rational function $R(x)/Q(x)$ (from Equation 1) as a sum of ***partial fractions*** of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

A theorem from algebra guarantees that it is always possible to do this. We explain the details for the four cases that occur.

I. $Q(x)$ is a Product of Distinct Linear Factors

This means that we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another).

In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\boxed{2} \quad \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$$

These constants can be determined as in the next example.

<u>**Example 2**</u>**:** Evaluate $\int \frac{1}{4-x^2}\,dx$.

II. $Q(x)$ is a Product of Distinct Linear Factors with Some Repeated

Suppose the first linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equation 2, we would use

$$\boxed{7} \quad \frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

Example 3: Evaluate $\int \frac{5t^2 + 20t + 6}{t^3 + 2t^2 + t}\,dt$.

III. $Q(x)$ contains Irreducible Quadratics

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations 2 and 7, the expression for $R(x)/Q(x)$ will have a term of the form

$$\boxed{9} \quad \frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined.

For instance, the function given by $f(x) = x/[(x - 2)(x^2 + 1)(x^2 + 4)]$ has a partial fraction decomposition of the form

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{x^2 + 4}$$

The term given in Equation 9 can be integrated with the help of completing the square (if necessary) and using the formula

$$\boxed{10} \quad \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Example 4: Evaluate $\int \frac{x^2 + 2x + 9}{x^2 + 2x + 10}\,dx$.

Example 5: Evaluate $\int \frac{x^2 - 4x + 1}{x^3 + 4x}\,dx$.

IV. $Q(x)$ Contains Repeated Irreducible Quadratics

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of the single partial fraction, the sum

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r} \tag{11}$$

occurs in the partial fraction decomposition of $R(x)/Q(x)$. Each of the terms can be integrated by using a substitution or by first completing the square if necessary.

Example 6: Evaluate $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx$.

The form of the partial fraction decomposition is

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying both sides by $x(x^2 + 1)^2$, we have

$$\begin{aligned} -x^3 + 2x^2 - x + 1 &= A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x \\ &= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\ &= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A \end{aligned}$$

If we equate coefficients, we get the system

$$A + B = 0 \qquad C = -1 \qquad 2A + B + D = 2 \qquad C + E = -1 \qquad A = 1$$

which has the solution $A = 1$, $B = -1$, $C = -1$, $D = 1$ and $E = 0$. Thus,

$$\begin{aligned} \int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx &= \int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\right)dx \\ &= \int \frac{dx}{x} - \int \frac{x}{x^2+1}dx - \int \frac{dx}{x^2+1} + \int \frac{x\,dx}{(x^2+1)^2} \\ &= \ln|x| - \tfrac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + K \end{aligned}$$