Math 3B – Calculus II
Berkeley City College

Section 7.3 – Trigonometric Substitution

Consider the integral $\int x\sqrt{a^2 - x^2}\, dx$.

In this case, it is clear that we apply a substitution of $u = a^2 - x^2$ to integrate

Now consider $\int \sqrt{a^2 - x^2}\, dx$ where $a > 0$.

These is no clear method of integration; however, notice that the inside of the square root vaguely resembles $1 - \sin^2\theta = \cos^2\theta$.

Let us try to eliminate the radical with a substitution of $x = a\sin\theta$.

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = \left|a\cos\theta\right| = a\left|\cos\theta\right|$$

Notice the difference between the substitution $u = a^2 - x^2$ (in which the new variable is a function of the old one) and the substitution $x = a\sin\theta$ (the old variable is a function of the new one).

In general, we can make a substitution of the form $x = g(t)$ by using the Substitution Rule in reverse.

To make our calculations simpler, we assume that g has an inverse function; that is, g is one-to-one.

In this case, if $x = g(t)$ and $dx = g'(t)dt$, the Substitution Rule tells us

$$\int f(x)\, dx = \int f(g(t))g'(t)\, dt$$

This kind of substitution is called *inverse substitution*.

We can make the inverse substitution $x = a\sin\theta$ when it is a one-to-one function. Namely, we restrict $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ so our function is invertible.

Hence, $\sqrt{a^2 - x^2} = a\left|\cos\theta\right| = a\cos\theta$ since $\cos\theta \ge 0$ on $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

In general, how do we integrate functions involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$?

Applying a similar technique, we use the following substitution on the restricted interval.

Radical	Substitution	Restiction on θ	Rationalized Radical
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$\sqrt{a^2 - x^2} = a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\sqrt{a^2 + x^2} = a\sec\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$0 < \theta < \frac{\pi}{2}$ or $\frac{3\pi}{2} < \theta < \pi$	$\sqrt{x^2 - a^2} = a\tan\theta$

When applying a trigonometric substitution, we will encounter various trigonometric integrals. Below is a list of helpful trigonometric identities and integrals

Trigonometric Identities	Trigonometric Integrals
$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$	$\int \sec\theta\, d\theta = \ln\lvert\sec\theta + \tan\theta\rvert + C$
$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$	$\int \csc\theta\, d\theta = \ln\lvert\csc\theta - \cot\theta\rvert + C$
$\tan^2\theta = \sec^2\theta - 1$	$\int \tan d\theta = \ln\lvert\sec\theta\rvert + C$
$\cot^2\theta = \csc^2\theta - 1$	$\int \cot\theta\, d\theta = \ln\lvert\sin\theta\rvert + C$
$\sin 2\theta = 2\sin\theta\cos\theta$	$\int \sec^2\theta\, d\theta = \tan\theta + C$
	$\int \csc^2\theta\, d\theta = -\cot\theta + C$

Example 1: Evaluate $\int \frac{\sqrt{16 - x^2}}{x} dx$.

Example 2: Show that the area of an ellipse is πab.

Example 3: Evaluate $\int \frac{x^3}{\sqrt{1+x^2}}dx$

Example 4: Evaluate $\int \frac{x}{\sqrt{x^2-16}}\,dx$

Example 5: Evaluate $\int_3^6 \frac{\sqrt{x^2-9}}{x}\,dx$.

Example 6: Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$.