Section 7.2 – Trigonometric Integrals

I. Integrals of the form $\int \sin^m(\theta)\cos^n(\theta)\,d\theta$

Example 1: Evaluate the following integrals.
A. $\int_0^{\pi} \sin^3(x)\,dx$
B. $\int \sin^2(4x)\cos^5(4x)\,dx$
C. $\int \cos^2 x\,dx$

D. $\int_0^{\pi} 4\sin^2 x\cos^2 x\,dx$

We want to use the method of substitution when attempting to deal with the product above. Below outlines a strategy for these integrals.

Power of Sine *m* is *odd*

1. Split off a $\sin(\theta)$.
   - ie, $\sin^5(3x) = \sin^4(3x)\sin(3x)$
2. Rewrite the remaining even power of $\sin(\theta)$ in terms of $\cos(\theta)$ using $\sin^2\theta + \cos^2\theta = 1$.
   - ie, $\sin^5(3x) = \sin^4(3x)\sin(3x) = \left(\sin^2(3x)\right)^2\sin(3x) = \left(1-\cos^2(3x)\right)^2\sin(3x)$
3. Apply a *u*-substitution by letting $u = \cos(\theta)$ and integrate.

Power of Cosine *n* is odd

1. Split off a $\cos(\theta)$.
2. Rewrite the remaining even power of $\cos(\theta)$ in terms of $\sin(\theta)$ using $\sin^2\theta + \cos^2\theta = 1$.
3. Apply a *u*-substitution by letting $u = \sin(\theta)$ and integrate.

Both *m* & *n* are even

1. Rewrite the even powers of $\cos(\theta)$ & $\sin(\theta)$ in terms of there half angle identities $\sin^2(\theta) = \dfrac{1-\cos(2\theta)}{2}$ & $\cos^2(\theta) = \dfrac{1+\cos(2\theta)}{2}$ and simplify.
   - ie, $\sin^4(2x)\cos^2(2x) = \left(\sin^2(2x)\right)^2\cos^2(2x) = \left(\dfrac{1-\cos(4x)}{2}\right)^2\left(\dfrac{1+\cos(4x)}{2}\right)$
2. Integrate the remaining powers of cosine by further applying the half angle formula *and/or* by using the cosine reduction formula.

Note: If both *m* & *n* are odd, you can use either of the first 2 methods. Both will look different, but yield trigonometrically identical answers.

II. Antiderivatives of the 6 Trigonometric Functions

We have yet to establish the antiderivatives for the trigonometric functions sec*x*, csc*x*, tanx, and cot*x.*

In Section 5.5, we saw to find antiderivatives of tan*x* (and cot*x*) by breaking them down into sines and cosines and applying a simple *u*-substitution. The antiderivatives for sec*x* and csc*x* can also be found using a substitution, but these results are not at all trivial.

The table below show the remaining antiderivatives. After today, you can take these as fact and add them to our library of antiderivatives.

$$\int \tan x\,dx = -\ln|\cos x| + C = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C \qquad \int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

__Proof__ The method for finding antiderivatives for tan*x* and cot*x* are outlined in Section 5.5. We now derive those for sec*x* and csc*x*. Consider $\int \sec x\,dx$.

Let us multiply the numerator and denominator of the integrand by $\sec x + \tan x$.

$$\int \sec x\,dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx$$

$$= \int \frac{1}{u}du$$

$$= \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

← Notice that the derivative of the bottom appears on the top. Therefore, we let $u = \sec x + \tan x$ & $du = (\sec x \tan x + \sec^2 x)dx$.

I leave it as a challenge for you to configure a similar method to compute $\int \csc x\,dx$. **Q.E.D.**

III. Integrating Products of secant and tangent like $\int \tan^m(\theta)\sec^n(\theta)\,d\theta$

__Example 2:__ Evaluate the following integrals.

A. $\int \tan^3(x)\sec^4(x)\,dx$

B. $\int_0^{\pi/4} \tan^3(x)\,dx$

B. $\int \sec^3(x)\,dx$

We want to use the method of substitution when attempting to deal with the product above. Below outlines a strategy for these integrals.

<u>Power of Secant *n* is even</u>

1. Split off a $\sec^2(\theta)$.
   - ie, $\tan^3 x \sec^4 x = \tan^3 x \sec^2 x \sec^2 x$
2. Rewrite the remaining even power of $\sec(\theta)$ in terms of $\tan(\theta)$ using $1+\tan^2\theta = \sec^2\theta$.
   - ie, $\tan^3 x \sec^4 x = \tan^3 x \sec^2 x \sec^2 x = \tan^3 x\left(1+\tan^2 x\right)\sec^2 x$
3. Apply a *u*-substitution by letting $u = \tan(\theta)$ and integrate.

Power of Tangent *m* is odd

1. Split off a $\sec(\theta)\tan(\theta)$.
   - $\tan^3 x \sec^4 x = \tan^2 x \sec^3 x \sec x \tan x$
2. Rewrite the remaining even power of $\tan(\theta)$ in terms of $\sec(\theta)$ using $1+\tan^2\theta = \sec^2\theta$.
   - $\tan^3 x \sec^4 x = \tan^2 x \sec^3 x = (\sec^2 x - 1)\sec^3 x \sec x \tan x$
3. Apply a *u*-substitution by letting $u = \sec(\theta)$ and integrate.

Note

- If both *n* is even & *m* is odd, you can use either of the first 2 methods. Both will look different, but yield trigonometrically identical answers.
- Methods for integrals of the form $\int \cot^m(\theta)\csc^n(\theta)\,d\theta$ are similar to the above.

IV. Integrals of the form $\int \sin(mx)\cos(nx)\,dx$, $\int \sin(mx)\sin(nx)\,dx$, $\int \cos(mx)\cos(nx)\,dx$

2 To evaluate the integrals (a) $\int \sin mx \cos nx\,dx$, (b) $\int \sin mx \sin nx\,dx$, or (c) $\int \cos mx \cos nx\,dx$, use the corresponding identity:

(a) $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

(b) $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

(c) $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Just remember when simplifying these expression that cosine is an even function and sine is an odd function, ie something like $\cos(-8x) = \cos(8x)$ & $\sin(-3x) = -\sin(3x)$.

Example 3: Evaluate $\int \sin(3x)\cos(7x)\,dx$.