Section 7.1 – Integration by Parts

I. Integration by Parts

We know that $\int xe^{x^2}\,dx = e^x + C$ using the substitution of $u = x^2$, but what about $\int xe^x\,dx$?

$\frac{d}{dx}\left(xe^x\right) = e^x + xe^x$ by the product rule $\Rightarrow \int \left(e^x + xe^x\right)dx = xe^x$

$$\Rightarrow \int e^x\,dx + \int xe^x\,dx = xe^x$$
$$\Rightarrow \int xe^x\,dx = xe^x - \int e^x\,dx$$
$$\Rightarrow \int xe^x\,dx = xe^x - e^x + C$$

It can be difficult integrate products of functions, where there is no obvious substitution, simply by guessing.

Let $f(x)$ and $g(x)$ by differentiable on some interval.

$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$ by the product rule $\Rightarrow \int \left(f'(x)g(x) + f(x)g'(x)\right)dx = f(x)g(x)$

$$\Rightarrow \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx = f(x)g(x)$$

1 $$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

To make this easier, we let $u = g(x)$ and $v = g(x)$, then $du = g(x)dx$ and $dv = g(x)dx$.

2 $$\int u\,dv = uv - \int v\,du$$

In applying integration by party we need to choose a term in the integral to be u and dv.

How do we use this for $\int xe^x\,dx$?

We note for definite integrals, our formula becomes

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$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx$$

Example 1: Find the volume of the solid generated by rotating bounded by $y = x\sin x$ and the x-axis where $0 \le x \le \pi$.

Example 2: Evaluate $\int \left(t^3 + 1\right)\ln t\,dt$.

Example 3: Find the average value of $f(x) = \arctan(4x)$ on $\left[0, \frac{1}{4}\right]$.

Example 4: Evaluate the following integrals.

A. $\int 5x^9 e^{x^5} dx$

B. $\int \cos\sqrt{x}\, dx$

Example 5: Evaluate $\int e^{-t}\cos(2t)\,dt$.

III. Options for Integration

Example 6: Evaluate the following integrals.

A. $\int \sin^2 x\,dx$

B. $\int x\sqrt{x+1}\,dx$

II. Reduction Formulas

A reduction formula is an integration tool used to integrate large powers of functions.

Example 7: Prove the reduction formula

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx \text{ where } n \text{ is a positive integer.}$$

After, use your result to integrate $\int \sin^2 x\,dx$.