Berkeley City College
Math 3B – Calculus II

Section 5.5 – The Substitution Rule

I. Prelude to the Method of Substitution

Recall the chain rule: $\frac{d}{dx}(F(g(x)) = F'(g(x))g'(x)$. When faced with a composition of functions, we differentiate the outside and multiply the by the derivative of the inside. Thinking backwards, it makes sense that $\int f(g(x))g'(x)\,dx = F(g(x)) + C$ where $F' = f$. Let's explore this idea.

Example 1: Evaluate the indefinite Integral.	
A. $\int 2x\cos(x^2)\,dx$	**B.** $\int x\cos(x^2)\,dx$
C. $\int \cos(2x)\,dx$	**D.** $\int \frac{1}{2t-1}\,dt$

Note: If the inside function, $g(x)$, is linear, then $\int f(ax+b)\,dx = \frac{1}{a}\int af(ax+b)\,dx = \frac{1}{a}F(ax+b)+C$ where $F' = f$.

II. The Method of Substitution

In the previous example, we saw that integrands contained a derivative relationship between an inside and outside function, hinting that this came from a chain rule.

Rather than guess or force in the relationship, we need a strategy that will work when we see this kind of relationship.

4 The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Proof

The method of substitution allows us to transform an integral when we notice a derivative relationship between the inside and outside function in the integrand.

Let us follow these steps when applying the method of substitution:

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that the a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)dx$ (or multiple of) in the integral
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using $u = g(x)$.

Example 2: Evaluate the definite Integral.	
A. $\int (5x-1)^2\,dx$	
B. $\int \tan(3x)\,dx$	**C.** $\int \frac{1}{\sqrt{t}\left(1+\sqrt{t}\right)^2}\,dt$

D. $\int \frac{w}{4+w^4}\,dw$

What about definite integrals? The same concept applies *except* we change the limits of integration when applying the substitution.

6 The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Proof

Geometrically, the substitution rule for definite integrals tells us that we transform from one coordinate system to another.

Example 3: Evaluate the definite Integral.

A. $\int_0^1 (2e^t + 2)\sqrt{e^t + t}\, dt$

B. $\int_1^e \frac{5\ln x}{x} dx$

Example 4: An oil storage tank ruptures at time $t = 0$ and oil leaks from the tank at a rate of $r(t) = 100e^{-0.01t}$ liters per minute. How much oil leaks out after the first hour? Round your answer to the nearest tenth of a liter.

Example 5: A particle moves back and forth along a straight line where its velocity at time t seconds is given by $v(t) = \sin(\pi t)$ m/s. Find the displacement and distance travelled by the particle when $0 \le t \le 3/2$.

III. Back Substitutions

Sometimes, a clever substitution is needed to simplify complex looking integrals.

Example 6: Evaluate the definite Integral.

A. $\int x\sqrt{x+1}\,dx$

B. $\int_0^1 \frac{x^2}{x+1}\,dx$

IV. Integrals of Symmetric Functions

Recall that

i) If $f(-x) = f(x)$ for all x in the domain of f, then $f(x)$ is even.
- the graph of f is symmetric about the y-axis.

ii) If $f(-x) = -f(x)$ for all x in the domain of f, then $f(x)$ is odd.
- the graph of f is symmetric about the origin.

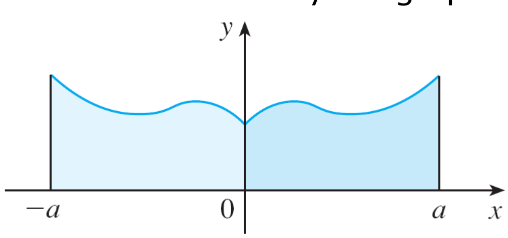
When integrating these functions over a symmetric interval $[-a, a]$, we attain useful integral properties.

7 Integrals of Symmetric Functions Suppose f is continuous on $[-a, a]$.

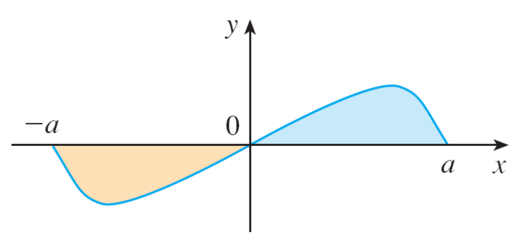
(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

This theorem is illustrated by the graphs below.



(a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$



(b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

__**Proof**__ Consider the case when f is odd. Let us break up the integrals at 0.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

Consider $\int_{-a}^{0} f(x)\,dx$. By substitution,

Let $\boxed{u = -x}$ & $du = -dx$

$u(0) = -0 = \boxed{0}$

$\boxed{-du = dx}$

$u(-a) = -(-a) = \boxed{a}$

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-u)(-du) = -\int_{a}^{0} f(-u)\,du = \int_0^a f(-u)\,du = \int_0^a -f(u)\,du$$

$$= -\int_0^a f(u)\,du$$

Therefore, $\int_{-a}^{0} f(x)dx = -\int_{0}^{a} f(x)dx$, and

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 0 .$$

Similarly, we can prove the case when *f* is even. **Q.E.D**

Strategy for Integration: We will follow the steps below when we see an integral with limits of integration from –*a* to *a*.

1. Determine if the function is odd or even.
2. If there are multiple terms, break up any sum in the integrand by its even and odd parts.

Example 7: Evaluate the following definite integrals.	
A. $\int_{-\pi}^{\pi} \frac{\sin t}{t^4 + t^2 + 1} dt$	**B.** $\int_{-3}^{3} \left(xe^{5x^2} + x^2 - x + 1 \right) dt$