Ch8

Arc Length

7) Find the exact length of the curve

$$y = 1 + 6x^{3/2} \qquad 0 \le x \le 1$$

$$\frac{dy}{dx} = 9x^{1/2} = 9\sqrt{x}$$

$$\left(\frac{dy}{dx}\right)^2 = 81x$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + 81x$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + 81x}$$

$$\text{Arc Length} = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 81x}\, dx$$

$$\text{let } u = 1 + 81x \qquad x = 0 \quad u = 1$$

$$du = 81\,dx \qquad x = 1 \quad u = 82$$

$$= \frac{1}{81}\int_1^{82} u^{1/2}\, du \qquad = \frac{1}{81} \cdot \frac{2}{3} u^{3/2} \Big|_1^{82}$$

$$= \frac{2}{243}(82)^{3/2} - \frac{2}{243}(1)^{3/2}$$

$$= \boxed{\frac{2}{243}\left(82\sqrt{82} - 1\right)}$$

(10) Find the exact length of the curve.

$$x = \frac{y^4}{8} + \frac{1}{4y^2} \qquad 1 \le y \le 2$$

$$x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$$

$$\frac{dx}{dy} = \frac{1}{2}y^3 - \frac{1}{2}y^{-3} = \frac{1}{2}(y^3 - y^{-3})$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{1}{4}(y^6 - 2 + y^{-6})$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4}y^6 - \frac{1}{2} + \frac{1}{4}y^{-6} = \frac{1}{4}y^6 + \frac{1}{2} + \frac{1}{4}y^{-6}$$

$$= \frac{1}{4}(y^6 + 2 + y^{-6})$$

$$= \frac{1}{4}(y^3 + y^{-3})^2$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \frac{1}{2}(y^3 + y^{-3})$$

$$\text{Arc Length} = \int_1^2 \frac{1}{2}(y^3 + y^{-3})\,dy = \frac{1}{2}\left(\frac{y^4}{4} - \frac{1}{2y^2}\right)\Bigg|_1^2$$

$$= \frac{1}{2}\left(\frac{16}{4} - \frac{1}{8}\right) - \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= \frac{1}{2}\left(\frac{32 - 1}{8} - \frac{2}{8} + \frac{4}{8}\right)$$

$$= \frac{1}{2}\left(\frac{31 - 2 + 4}{8}\right) = \boxed{\frac{33}{16}}$$

Arc Length

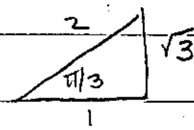
(12) $y = \ln(\cos x) \qquad 0 \le x \le \frac{\pi}{3}$

$$\frac{dy}{dx} = \frac{1}{\cos x} \cdot -\sin x = -\tan x.$$

$$\left(\frac{dy}{dx}\right)^2 = \tan^2 x$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\tan^2 x} = \sqrt{\sec^2 x} = \sec x$$

$$\text{Arc Length} = \int_0^{\pi/3} \sec x \, dx = \ln|\sec x + \tan x| \Big|_0^{\pi/3}$$



$\sec\frac{\pi}{3} = 2 \qquad \tan\frac{\pi}{3} = \sqrt{3}$

$\sec 0 = 1 \qquad \tan 0 = 0$

$$= F(\pi/3) - F(0)$$

$$= \ln|2+\sqrt{3}| - \ln|1+0|$$

$$= \boxed{\ln|2+\sqrt{3}|}$$

Find the exact length of the curve.

16 $y = \sqrt{x - x^2} + \sin^{-1}(\sqrt{x})$ domain $= [0, 1]$

endpts $(0, 0)$ $0 \le x \le 1$

$(1, \pi/2)$

$y = (x - x^2)^{1/2} + \sin^{-1}(x^{1/2})$

$\frac{dy}{dx} = \frac{1}{2}(x - x^2)^{-1/2}(1 - 2x) + \frac{1}{\sqrt{1 - (\sqrt{x})^2}} \cdot \frac{1}{2}x^{-1/2}$

$= \frac{1 - 2x}{2\sqrt{x - x^2}} + \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1-2x}{2\sqrt{x}\sqrt{1-x}} + \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{2-2x}{2\sqrt{x}\sqrt{1-x}}$

$= \frac{\cancel{2}(\cancel{1-x})^{\sqrt{1-x}}}{\cancel{2}\sqrt{x}\cancel{\sqrt{1-x}}} = \frac{\sqrt{1-x}}{\sqrt{x}}$

$= \sqrt{\frac{1-x}{x}}$

$\left(\frac{dy}{dx}\right)^2 = \left(\sqrt{\frac{1-x}{x}}\right)^2 = \frac{1-x}{x}$

$1 + \left(\frac{dy}{dx}\right)^2 = \frac{x}{x} + \frac{1-x}{x} = \frac{1}{x}$

$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{1}{x}} = x^{-\frac{1}{2}}$

Arc Length $= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 x^{-\frac{1}{2}}\, dx = 2\sqrt{x}\Big|_0^1$

$= 2\sqrt{1} - 2\sqrt{0}$

$= \boxed{2}$