9.3 / Positive Series: The Integral Test

(8) Use the Integral Test to determine convergence or divergence of the series

$$\sum_{k=1}^{\infty} \frac{k^2}{e^k}$$

$\frac{x^2}{e^x}$ is positive, continuous

$f(x) = \frac{x^2}{e^x}$

$\boxed{\text{converges}}$

$f'(x) = \frac{2xe^x - x^2 \cdot e^x}{e^{2x}} \leq 0$

when $2xe^x - x^2 \cdot e^x \leq 0$

$e^x \cdot x \cdot (2-x) \leq 0$

$\cancel{e^x \leq 0} \quad x \geq 0 \quad 2 - x \leq 0$

$2 \leq x$

$\therefore f(x)$ nonincreasing on $[2, \infty)$

Integral test applies

$$\int_2^{\infty} x^2 \cdot e^{-x}\,dx$$

By PARTS let $u = x^2$ $\quad v = -e^{-x}$

$du = 2x\,dx \quad dv = e^{-x}\,dx$

$$= u \cdot v - \int v \cdot du$$

$$= \frac{-x^2}{e^x} + \int e^{-x} \cdot 2x\,dx$$

By parts again let $u = 2x$ $\quad v = -e^{-x}$

$du = 2\,dx \quad dv = e^{-x}\,dx$

$$= \frac{-x^2}{e^x} - \frac{2x}{e^x} + 2\int e^{-x}\,dx$$

use L'Hopital's Rule

$$= \left.\frac{-x^2}{e^x} - \frac{2x}{e^x} - \frac{2}{e^x}\right|_2^{\infty} = \lim_{b \to \infty}\left(\frac{-b^2}{e^b} - \frac{2b}{e^b} - \frac{2}{e^b}\right) - \left(\frac{-4}{e^2} - \frac{4}{e^2} - \frac{2}{e^2}\right)$$

$$= 0 + \frac{10}{e^2} \quad \text{converges}$$

9.3/ Positive Series : The Integral Test

(10) $\sum_{k=1}^{\infty} \frac{1000k^2}{1+k^3}$

$\frac{1000x^2}{1+x^3}$ is positive
continuous
and non-increasing
on $[2, \infty)$
Integral Test applies.

$\int_2^{\infty} \frac{1000x^2}{1+x^3}dx$

$= 1000 \int_2^{\infty} \frac{x^2}{1+x^3} dx$

let $u = 1+x^3$
$du = 3x^2 dx$

$= \frac{1000}{3} \int_2^{\infty} \frac{3x^2 dx}{1+x^3}$

$= \frac{1000}{3} \int \frac{du}{u}$

$= \frac{1000}{3} \ln|1+x^3| \Big|_2^{\infty}$

$= \frac{1000}{3}(\infty - \ln 9) = \infty$ diverges

$\therefore$ the series $\sum_{k=1}^{\infty} \frac{1000k^2}{1+k^3}$ also diverges.

9.3 / Positive Series: The Integral Test

(21) $\displaystyle\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{1+k^2}$

$\tan^{-1} x > 0 \quad \forall x \in [1,\infty)$

(ranges from $\pi/4$ to $\pi/2$)

$\frac{1}{1+x^2} > 0 \quad \forall x \in [1,\infty)$

$\therefore \frac{\tan^{-1} k}{1+x^2} > 0 \quad \forall k \in [1,\infty)$

positive

Continuous (product of 2 continuous functions.)

$$f(x) = \frac{\tan^{-1} x}{1+x^2}$$

$$f'(x) = \frac{\frac{1}{1+x^2} \cdot 1+x^2 - \tan^{-1} x \cdot (2x)}{(1+x^2)^2}$$

$$= \frac{1 - 2x \cdot \tan^{-1} x}{(1+x^2)^2}$$

$$< 0$$

when $1 - 2x \tan^{-1}(x) < 0$

$1 < 2x \tan^{-1}(x)$

true $\forall x \in [1,\infty)$

$\therefore f'(x) < 0 \quad \forall x \in [1,\infty)$

$\Rightarrow f(x)$ is nonincreasing on $[1,\infty)$.

Integral Test applies.

$$\int_1^{\infty} \frac{\tan^{-1} x}{1+x^2}\,dx$$

let $u = \tan^{-1} x$

$du = \frac{1}{1+x^2}\,dx$

$$\int u\,du = \frac{u^2}{2} = \frac{(\tan^{-1} x)^2}{2}\Bigg|_1^{\infty} = \lim_{b\to\infty} \frac{(\tan^{-1} b)^2}{2} - \frac{1}{2}\left(\frac{\pi}{4}\right)^2$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\left(\frac{\pi}{4}\right)^2$$

$< \infty$ converges

$\therefore$ The series $\displaystyle\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{1+k^2}$ converges

9.3/ Positive Series: The Integral Test

(24) Estimate the error that is made by approximating the sum of the given series by the sum of the first five terms.

$$\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k}}$$

$$f(x) = \frac{1}{x\sqrt{x}} = \frac{1}{x^1 \cdot x^{1/2}} = \frac{1}{x^{3/2}}$$

continuous
positive
nonincreasing
$\forall x \in [5, \infty)$

$$E = \sum_{k=6}^{\infty} \frac{1}{k\sqrt{k}} \leq \int_5^{\infty} \frac{1}{x^{3/2}}\,dx$$

$$= \int_5^{\infty} x^{-3/2}\,dx$$

$$= \frac{x^{-1/2}}{-1/2} = \left.\frac{-2}{\sqrt{x}}\right|_5^{\infty}$$

$$= \lim_{b \to \infty} \frac{-2}{\sqrt{b}} + \frac{2}{\sqrt{5}}$$

$$= 0 + \frac{2}{\sqrt{5}}$$

$$\approx \boxed{0.8944}$$

9.3/ Positive Series : The Integral Test

(32) Determine how large n must be so that using the nth partial sum to approximate the series gives an error of no more than 0.0002

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

Use Partial Fractions!

$$E_n = \sum_{k=n+1}^{\infty} \frac{1}{k(k+1)} < \int_n^{\infty} \frac{1}{x(x+1)}\,dx = \int_n \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= \ln|x| - \ln|x+1|$$

$$= \ln\left|\frac{x}{x+1}\right| \Bigg|_n^{\infty}$$

$$= \lim_{b\to\infty} \ln\left(\frac{b}{b+1}\right) - \ln\left(\frac{n}{n+1}\right)$$

$$= 0 - \ln\left(\frac{n}{n+1}\right) = \ln\left(\frac{n}{n+1}\right)^{-1}$$

$$= \ln\left(\frac{n+1}{n}\right)$$

$$E_n = \ln\left(\frac{n+1}{n}\right) \le 0.0002$$

$$\frac{n+1}{n} \le e^{.0002} \approx 1.0002$$

$$1 + \frac{1}{n} \le 1.0002$$

$$\frac{1}{n} \le .0002$$

$$\frac{1}{.0002} \le n$$

$$\boxed{5000 \le n}$$