7.4 Rationalizing Substitutions

2 $\int x\sqrt[3]{x+\pi}\,dx$

let $\boxed{u=\sqrt[3]{x+\pi}}$

$u^3 = x+\pi$

$\boxed{3u^2du = dx}$ and $u^3 = x+\pi$

$\boxed{u^3-\pi = x}$

$$\int x\cdot\sqrt[3]{x+\pi}\cdot dx$$

$$= \int (u^3-\pi)\cdot u\cdot 3u^2\,du$$

$$= \int (3u^6 - 3u^3\pi)\,du$$

$$= \frac{3u^7}{7} - \frac{3\pi u^4}{4} + C$$

$$= \boxed{\frac{3(x+\pi)^{7/3}}{7} - \frac{3\pi(x+\pi)^{4/3}}{4} + C}$$

7.4

(6) $\int_0^1 \frac{\sqrt{t}}{t+1}\,dt$

let $\boxed{u = \sqrt{t}}$

$u^2 = t$

$\boxed{2u\,du = dt}$

$u^2 = t$

$\Rightarrow \boxed{u^2+1 = t+1}$

change limits of integration so no back substitution $\begin{cases} t=0 & u=\sqrt{t} = \sqrt{0} = 0 \\ t=1 & u=\sqrt{t}=\sqrt{1}=1 \end{cases}$

$$\int_0^1 \frac{\sqrt{t}}{t+1}\,dt = \int_0^1 \frac{u}{u^2+1}\cdot 2u\,du$$

$$= \int_0^1 \frac{2u^2}{u^2+1}\,du$$

$$= 2\int_0^1 \frac{u^2}{u^2+1}\,du$$

$$= 2\int_0^1 \frac{(u^2+1)-1}{u^2+1}\,du$$

$$= 2\int_0^1 \frac{u^2+1}{u^2+1}\,du - 2\int_0^1 \frac{1}{u^2+1}\,du$$

$$= 2\int_0^1 du - 2\int_0^1 \frac{du}{1+u^2}$$

$$= 2u - 2\tan^{-1}(u)\Big|_0^1 = 2 - 2\cdot\frac{\pi}{4} - (0-0)$$

$$= \boxed{2 - \frac{\pi}{2}}$$

7.4

(10) $\int \frac{x^2\,dx}{\sqrt{16-x^2}}$

$\sqrt{16-x^2}$ is of the form $\sqrt{a^2-x^2}$

so let $x = a\sin t \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$x = 4\sin t$

then

$\sqrt{16-x^2} = \sqrt{16-16\sin^2 t} = \sqrt{16(1-\sin^2 t)}$

$= \sqrt{16}\sqrt{\cos^2 t}$

$= |4\cos t|$

$\boxed{\sqrt{16-x^2} = 4\cos t.}$

$x = 4\sin t$

$\boxed{\begin{aligned} dx &= 4\cos t\,dt \\ x^2 &= 16\sin^2 t \end{aligned}}$

$$\int \frac{x^2\,dx}{\sqrt{16-x^2}} = \int \frac{(16\sin^2 t)(4\cos t\,dt)}{4\cos t}$$

$$= 16\int \sin^2 t\,dt$$

identity: $\sin^2 t = \frac{1-\cos 2t}{2}$

$$= 16\int \frac{1-\cos 2t}{2}\,dt$$

$$= 16\int \tfrac{1}{2}\,dt - 16\int \frac{\cos 2t}{2}\,dt$$

$$= \int 8\,dt - 8\int \cos 2t\,dt$$

let $u = 2t$

$du = 2\,dt$

$$= \int 8\,dt \quad -4\int 2\cdot\cos 2t\,dt$$

$$= \int 8\,dt \quad -4\int \cos u\,du$$

$$= 8t \quad -4\sin 2t + C$$

7.4 (10) continued.

$8t - \underline{4\sin 2t + C}$

Identity

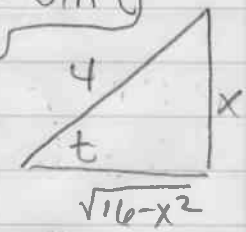
$\sin 2t = 2\sin t \cos t$

$= \underline{8t - 8\sin t \cos t + C}$

ORIGINAL SUBSTITUTION

$x = 4\sin t$

$\boxed{\frac{x}{4} = \sin t}$



By Pythagorean Thm

$\Rightarrow \boxed{\cos t = \frac{\sqrt{16-x^2}}{4}}$

$\Rightarrow \frac{x}{4} = \sin t$

$\boxed{\sin^{-1}\left(\frac{x}{4}\right) = t}$

therefore

$$\int \frac{x^2\,dx}{\sqrt{16-x^2}} = 8\sin^{-1}\left(\frac{x}{4}\right) - 8\cdot\frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4} + C$$

$$\boxed{= 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C}$$

Final answer must be in terms of x not t.

7.4

15. $\displaystyle\int \frac{2z-3}{\sqrt{1-z^2}}\,dz$ $\qquad \sqrt{1-z^2}$ is of the form $\sqrt{a^2-x^2}$

So let

$z = a\sin t$

$z = 1\cdot\sin t = \sin t$

$\boxed{dz = \cos t\,dt}$

and

$\sqrt{1-z^2} = \sqrt{1-\sin^2 t}$

$= \sqrt{\cos^2 t} = \cos t$

$\therefore$

$$\int \frac{2z-3}{\sqrt{1-z^2}}\,dz = \int \frac{2\cdot\sin t - 3}{\cos t}\cdot \cos t\,dt$$

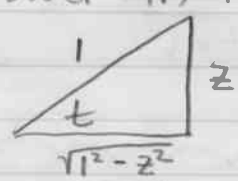
$$= \int (2\sin t - 3)\,dt$$

$$= -2\cos t - 3t + C$$

write answer in terms of z.

$z = \sin t$

$\sin^{-1}(z) = t$



$= \sqrt{1-z^2}$

$\sqrt{1-z^2} = \frac{\sqrt{1-z^2}}{1} = \cos t$

$\therefore$ $\displaystyle\int \frac{2z-3}{\sqrt{1-z^2}}\,dz = -2\cos t - 3t + C$

$$\boxed{= -2\sqrt{1-z^2} - 3\sin^{-1}(z) + C}$$

7.4

(18) $$\int \frac{dx}{\sqrt{x^2+4x+5}} = \int \frac{dx}{\sqrt{(x^2+4x+4)+1}}$$

WAY YOU GO

complete the square

$$= \int \frac{dx}{\sqrt{(x+2)^2+1}}$$

u subsitution

let $u = x+2 \quad du = dx$

$$= \int \frac{du}{\sqrt{u^2+1}}$$

trig substitution

let $u = 1 \cdot \tan t \qquad -\pi/2 \le t \le \pi/2$

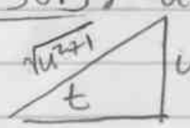
$du = \sec^2 t \, dt$

$$u = \tan t$$
$$\sqrt{u^2+1} = \sqrt{\tan^2 t + 1}$$
$$= \sqrt{\sec^2 t}$$
$$\sqrt{u^2+1} = \sec t$$

$$\int \frac{du}{\sqrt{u^2+1}} = \int \frac{\sec^2 t \, dt}{\sec t} = \int \sec t \, dt$$

$$= \ln|\sec t + \tan t| + C$$

TRIG SUB: $u = \tan t$



$\sec t = \frac{\sqrt{u^2+1}}{1} = \sqrt{u^2+1}$

$\tan t = u$

$$= \ln\left|\sqrt{u^2+1} + u\right| + C$$

now u-substution

$u = x+2$

$$= \ln\left|\sqrt{(x+2)^2+1} + x+2\right| + C$$

$$= \boxed{\ln\left|\sqrt{x^2+4x+5} + x+2\right| + C}$$

(To back sub, go back the way you came trig first, then finally u-substitution.)

Show

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C$$

Identity $\sec x = \frac{\sin x}{\cos x} + \frac{\cos x}{1+\sin x}$

$$\therefore \int \sec x\, dx = \int \left(\frac{\sin x}{\cos x} + \frac{\cos x}{1+\sin x} \right) dx$$

$$= \int \frac{\sin x}{\cos x}\, dx + \int \frac{\cos x}{1+\sin x}\, dx$$

let $u = \cos x$ let $v = 1+\sin x$

$du = -\sin x$ $dv = \cos x\, dx$

$$= -\int \frac{du}{u} + \int \frac{dv}{v}$$

$$= -\ln|u| + \ln|v| + C$$

$$= -\ln|\cos x| + \ln|1+\sin x| + C$$

$$= \ln|1+\sin x| - \ln|\cos x| + C$$

$$= \ln\left|\frac{1+\sin x}{\cos x}\right| + C$$

$$= \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + C$$

$$= \ln|\sec x + \tan x| + C$$

(used $\int \sec x\, dx = \ln|\sec x + \tan x| + C$ in 7.4 #18.)

7.4

(26) $\displaystyle\int \frac{2x-1}{x^2-6x+18}\,dx$

Complete the square

$x^2-6x+18 = x^2-6x+9+9$

$= (x-3)^2+9$

let $u = x-3 \Rightarrow \boxed{x = u+3}$

$\boxed{du = dx}$

and $(x-3)^2+9 = u^2+9$.

$$\int \frac{2x-1}{x^2-6x+18}\,dx = \int \frac{2(u+3)-1}{u^2+9}\,du$$

$$= \int \frac{2u+6-1}{u^2+9}\,du$$

$$= \int \frac{2u+5}{u^2+9}\,du$$

$$= \int \frac{2u}{u^2+9}\,du + 5\int \frac{du}{u^2+9}$$

let $v = u^2+9$ use $\displaystyle\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$

$dv = 2u\,du$

$$= \int \frac{dv}{v} + \frac{5}{3}\tan^{-1}\left(\frac{u}{3}\right) + C$$

$$= \ln|v| + \frac{5}{3}\tan^{-1}\left(\frac{u}{3}\right) + C$$

$$= \ln|u^2+9| + \frac{5}{3}\tan^{-1}\left(\frac{u}{3}\right) + C$$

BACK substitute for x.

$$= \boxed{\ln|x^2-6x+18| + \frac{5}{3}\tan^{-1}\left(\frac{x-3}{3}\right) + C}$$