7.5 Integration by Partial Fractions

21) Use partial fraction decomposition to perform the required integration

$$\int \frac{x+1}{(x-3)^2}\,dx$$

$$\frac{x+1}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$

$$x+1 = A(x-3) + B$$

when $x=3$

$$3+1 = A(3-3) + B$$

$$\boxed{4 = B}$$

when $x=0$

$$0+1 = A(0-3)+4$$

$$1 = -3A+4$$

$$-3 = -3A$$

$$\boxed{1 = A}$$

$$\frac{x+1}{(x-3)^2} = \frac{1}{(x-3)} + \frac{4}{(x-3)^2}$$

$$\int \frac{x+1}{(x-3)^2}\,dx = \int \frac{dx}{(x-3)} + 4\int \frac{dx}{(x-3)^2}$$

let $u = x-3$ let $v = x-3$

$du = dx$ $dv = dx$

$$= \int \frac{du}{u} + 4\int v^{-2}\,dv$$

$$= \ln|u| - \frac{4}{v} + C$$

$$= \boxed{\ln|x-3| - \frac{4}{x-3} + C}$$

7.5 Partial Fraction

(27) $\int \frac{2x^2+x-8}{x^3+4x}$ $\qquad x^3+4x = x(x^2+4)$

$$\frac{2x^2+x-8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$2x^2+x-8 = A(x^2+4) + (Bx+C)x$$
$$= Ax^2+4A + Bx^2 + Cx$$
$$2x^2+x-8 = (A+B)x^2 + Cx + 4A$$

$$-8 = 4A$$
$$\boxed{-2 = A}$$

$$2x^2+x-8 = (-2+B)x^2 + Cx - 8$$

$$\boxed{1 = C}$$

and

$$2 = -2 + B$$
$$\boxed{4 = B}$$

So,
$$\frac{2x^2+x-8}{x(x^2+4)} = \frac{-2}{x} + \frac{4x+1}{x^2+4}$$

$$\int \frac{2x^2+x-8}{x(x^2+4} = -2\int \frac{dx}{x} + 4\int \frac{x\,dx}{x^2+4} + \int \frac{1\,dx}{x^2+4}$$

let $u = x^2+4$ $\quad$ use
$du = 2x\,dx$ $\quad \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C$

$$= -2\ln|x| + 2\int \frac{du}{u} + \int \frac{dx}{x^2+2^2}$$
$$= -2\ln|x| + 2\ln|u| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$$

$$= \boxed{-2\ln|x| + 2\ln|x^2+4| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C}$$

7.5 Partial Fractions

36 $\displaystyle\int \frac{(\sin t)(4\cos^2 t - 1)}{(\cos t)(1+2\cos^2 t+\cos^4 t)}\,dt$

let $x = \cos t$
$dx = -\sin t\,dt$

$$= -\int \frac{4x^2-1}{x(1+2x^2+x^4)}\,dx$$

$1+2x^2+x^4 = (1+x^2)(1+x^2)$

$$\frac{4x^2-1}{x(1+2x^2+x^4)} = \frac{A}{x} + \frac{Bx+C}{(1+x^2)} + \frac{Dx+E}{(1+x^2)^2}$$

$4x^2-1 = A(1+x^2)^2 + (Bx+C)x\cdot(1+x^2) + (Dx+E)x$

when $x=0$

$-1 = A + 0 + 0 \quad \Rightarrow \boxed{A=-1}$

So, $A=-1 \Rightarrow$

- expand polynomials
- collect like terms of x^n
- equate coefficients of powers x^n

$4x^2-1 = -1(1+x^2)^2 + (Bx+C)(x+x^3) + (Dx+E)x$

$4x^2-1 = -1(1+2x^2+x^4) + Bx^2 + Bx^4 + Cx + Cx^3 + Dx^2 + Ex$

$= -1 - 2x^2 - x^4 + Bx^2 + Bx^4 + Cx + Cx^3 + Dx^2 + Ex$

$4x^2-1 = (-1+B)x^4 + Cx^3 + (-2+B+D)x^2 + (C+E)x \; -1$

$4 = -2+B+D$ and $-1=-1$

$5 = D$

the rest of the coefficients for $x^4, x^3, x = 0$.

$\therefore\; -1+B=0 \quad \boxed{C=0} \quad C+E=0$

$\boxed{B=1} \quad 0+E=0 \; \boxed{E=0}$

$\boxed{D=5}$ from above and $B=1$

$$\frac{4x^2-1}{x(1+x^2)^2} = \frac{-1}{x} + \frac{1\cdot x+0}{(1+x^2)} + \frac{5x+0}{(1+x^2)^2}$$

$$= \frac{-1}{x} + \frac{x}{1+x^2} + \frac{5x}{(1+x^2)^2}$$

next page

$$\frac{4x^2-1}{x(1+x^2)^2} = \frac{-1}{x} + \frac{x}{1+x^2} + \frac{5x}{(1+x^2)^2}$$

$$\int \frac{4x^2-1}{x(1+2x^2+x^4)} = -\int \frac{dx}{x} + \int \frac{x}{1+x^2}\,dx + 5\int \frac{x\,dx}{(1+x^2)^2}$$

let $u = 1+x^2$ let $v = 1+x^2$

$du = 2x\,dx$ $dv = 2x\,dx$

$$= -\int \frac{dx}{x} + \frac{1}{2}\int \frac{du}{u} + \frac{5}{2}\int v^{-2}\,dv$$

$$= -\ln|x| + \frac{1}{2}\ln|u| - \frac{5}{2v} + C$$

$$= -\ln|x| + \frac{1}{2}\ln|1+x^2| - \frac{5}{2(1+x^2)} + C$$

Recall $x = \cos t$ and we need to evaluate

$$\int \frac{\sin t\,(4\cos^2 t-1)}{\cos t\,(1+2\cos^2 t+\cos^4 t)}\,dt = -\int \frac{4x^2-1}{x(1+2x+x^4)}\,dx$$

$\therefore$ Answer $= +\ln|x| - \frac{1}{2}\ln|1+x^2| + \frac{5}{2(1+x^2)} + C$

and $x = \cos t$

$$= \boxed{\ln|\cos t| - \frac{1}{2}\ln|1+\cos^2 t| + \frac{5}{2(1+\cos^2 t)} + C}$$

7.5

49 Suppose that the Earth will not support a population of more than 16 billion and that there were 2 billion people in 1925 and 4 billion in 1975. Then, if y is the population t years after 1925 an appropriate model is the logistic differential equation $\frac{dy}{dt} = ky(16-y)$

(a) Solve the differential equation

(b) Predict the population in 2015

(c) When will the population be 9 billion?

(a) Solve $\frac{dy}{dt} = ky(16-y)$

$$\frac{dy}{y(16-y)} = k\,dt$$

$$\frac{1}{y(16-y)} = \frac{A}{y} + \frac{B}{16-y}$$

$$1 = A(16-y) + By$$

when $y=0$ $\quad 1 = A(16) + B\cdot 0 \Rightarrow 1 = 16A \Rightarrow \boxed{A = \frac{1}{16}}$

when $y=16$ $\quad 1 = A(0) + B\cdot 16 \Rightarrow 1 = B\cdot 16 = \boxed{B = \frac{1}{16}}$

$$\frac{1}{y(16-y)} = \frac{\frac{1}{16}}{y} + \frac{\frac{1}{16}}{16-y}$$

$$= \frac{1}{16y} + \frac{1}{16(16-y)}$$

$$\therefore \frac{dy}{y(16-y)} = \frac{dy}{16y} + \frac{dy}{16(16-y)}$$

7.5

49 Continued... page 2

$$\frac{dy}{y(16-y)} = k\,dt$$

$$\int \frac{dy}{16y} + \frac{dy}{16(16-y)} = \int k\,dt$$

$$\frac{1}{16}\ln|y| - \frac{1}{16}\ln|16-y| = kt + C$$

$$\frac{1}{16}\left(\ln|y| - \ln|16-y|\right) = kt + C$$

$$\frac{1}{16}\ln\left|\frac{y}{16-y}\right| = kt + C$$

$$\ln\left|\frac{y}{16-y}\right| = 16kt + C$$

$$\frac{y}{16-y} = e^{16kt+C}$$

$$\frac{y}{16-y} = Ce^{16kt}$$

1925 $y(0) = 2$ billion

$y = 2$ when $t = 0$

$$\frac{2}{14} = Ce^{16k(0)}$$

$$\boxed{\frac{1}{7} = C}$$

Now Find k.

$y(50) = 4$ billion

$y = 4$ when $t = 50$

$$\frac{y}{16-y} = \frac{1}{7}e^{16kt}$$

$$\frac{4}{16-4} = \frac{1}{7}e^{16\cdot k\cdot 50}$$

$$\frac{1}{3} = \frac{1}{7}e^{800k}$$

7.5

49 continued.... page 3

$$\frac{1}{3} = \frac{1}{7} e^{800K}$$

$$\frac{7}{3} = e^{800K}$$

$$\ln\left(\frac{7}{3}\right) = 800K$$

$$\frac{\ln\left(\frac{7}{3}\right)}{800} = K$$

$$.00106 \approx K$$

$$\therefore \frac{y}{16-y} = \frac{1}{7} e^{16Kt}$$

$$\frac{y}{16-y} = \frac{1}{7} e^{.0169t}$$

$$7y = (16-y) e^{.0169t}$$

$$7y = 16e^{.0169t} - ye^{.0169t}$$

$$7y + ye^{.0169t} = 16e^{.0169t}$$

$$y(7 + e^{.0169t}) = 16e^{.0169t}$$

$$\boxed{y = \frac{16e^{.0169t}}{7 + e^{.0169t}}}$$

solution to differential equation!

7.5

49 continued... page 4.

(b) Predict the population in 2015

$$y = \frac{16e^{.0169t}}{7+e^{.0169t}}$$

1925 $t=0$
2015 $t=90$

$$y = \frac{16e^{.0169(90)}}{7+e^{.0169(90)}}$$

$$= \frac{16e^{1.525}}{7+e^{1.525}}$$

$$= \frac{16(4.596)}{7+4.596}$$

$$= \frac{73.532}{11.596}$$

$$\boxed{y = 6.34 \text{ billion}}$$

7.5

(49) continued ... page 5

(c) When will the population be 9 billion?

$$y = \frac{16e^{.0169t}}{7+e^{.0169t}}$$

$$9 = \frac{16e^{.0169t}}{7+e^{.0169t}}$$

$$9(7+e^{.0169t}) = 16e^{.0169t}$$

$$63 + 9e^{.0169t} = 16e^{.0169t}$$

$$63 = 16e^{.0169t} - 9e^{.0169t}$$

$$63 = 7e^{.0169t}$$

$$9 = e^{.0169t}$$

$$\ln 9 = .0169t$$

$$\frac{\ln 9}{.0169} = t$$

$$\frac{2.197}{.0169} = t$$

$$130.01 = t$$

$t=0$ in 1925 130 years later

$\boxed{\text{In 2055 there will be 9 billion people}}$