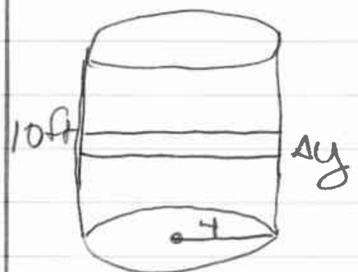
6.5

(13) Find the work done in pumping all the oil (density $\delta = 50$ pounds per cubic foot.) over the edge of a cylindrical tank that stands on one of its bases. Assume that the radius of the base is 4 feet, the height is 10 ft, and the tank is full of oil.



Work = Force · Distance

Find Force $F(x)$ first.

Force needed to lift oil = weight

= Volume · δ

Δy slice is a circle

Volume of slice $= \pi r^2 \cdot \Delta y$

$= \pi (4)^2 \Delta y$

$= 16\pi \Delta y$ ft^3

$\delta = 50 \frac{lbs}{ft^3}$

$F(x) = Vol \cdot \delta$

The Δy slice is at height $= y$ and must be pumped just over top, so it must travel distance $10 - y$.

ΔWork = Force · Distance

$= 16\pi \Delta y \cdot \delta \cdot (10 - y)$ where $0 \le y \le 10$

$$W = \int_0^{10} 16\pi \delta (10 - y)\,dy = 16\pi\delta \int_0^{10} 10 - y\,dy$$

$$= 16\pi\delta \left(10y - \frac{y^2}{2}\right)\Big|_0^{10}$$

$$= 16\pi\delta\,(100 - 50)$$

$$= 16\pi(50)(50) = 125{,}664$$

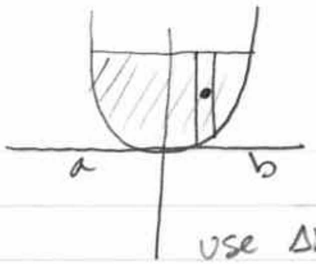
$\boxed{W = 125{,}664 \text{ ft-lbs.}}$

Finding Centroids $(\bar{x}, \bar{y})$

* Find slice Δx, or Δy
* remember m_y is moment wrt y-axis
* distance to y-axis $= x$.
* m_x is moment wrt x-axis

arbitrary distance to x-axis $= y$.

* midpoint between $f(x)$ and $g(x)$ $= \dfrac{f(x)+g(x)}{2}$



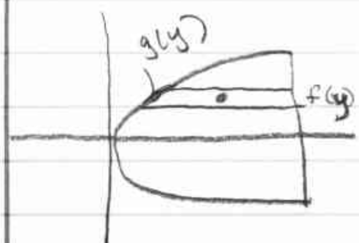
use Δx.

y-axis symmetry

$$\bar{x} = 0$$

$$\bar{y} = \frac{\frac{1}{2}\int_a^b \left[(f(x))^2 - (g(x))^2\right]dx}{\int_a^b [f(x) - g(x)]\,dx}$$

* center of mass of rectangle slice = geom center



use Δy if you have x-axis symmetry because then

$$\bar{y} = 0 = \frac{m_x}{m} \quad \frac{\Sigma \text{mass} \cdot \text{dist } y}{\Sigma \text{mass}}$$

and $$\bar{x} = \frac{M_y}{m} = \frac{\frac{1}{2}\int_c^d (f(y))^2 - (g(y))^2\,dy}{\int_c^d (f(y) - g(y))\,dy}$$

mass = Density $\cdot$ area or mass = Density $\cdot$ volume depending on density units.

Centroids = center of mass of plane region it doesn't depend on density or mass really because δ factors and ultimately cancels out of $\frac{M_y}{m}$ and $\frac{M_x}{m}$

$\bar{x}$ and $\bar{y}$.

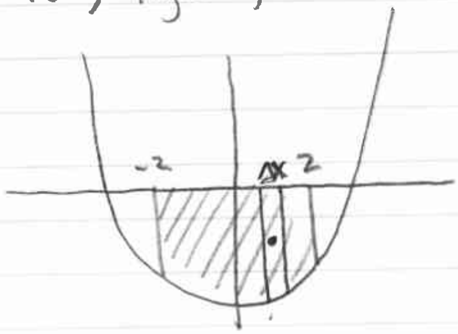
$$\bar{x} = \frac{M_y}{m} = \frac{\text{Total Moment wrt y axis}}{\text{Total mass}}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\text{Total Moment WRT x-axis}}{\text{Total mass}}$$

$\boxed{\text{ANSWER: } (\bar{x}, \bar{y}) = \left(0, \frac{-287}{130}\right)}$

(12) Find the centroid of the region bounded by the given curves. Make a sketch and use symmetry where possible.

$y = \frac{1}{2}(x^2 - 10)$, $y = 0$, and between $x = -2$ and $x = 2$



y-axis symmetry ∴ choose Δx and $\bar{x} = 0$.

$$\bar{y} = \frac{m_x}{m} = \frac{\frac{1}{2}\int_{-2}^{2} 0^2 - \left(\frac{1}{2}(x^2-10)\right)^2 dx}{-\frac{1}{2}\int_{-2}^{2}(x^2-10)\,dx}$$

$-\left(\frac{1}{2}\right)^2 = -\frac{1}{4}$ and $-\frac{1}{4}\cdot\frac{1}{2} = -\frac{1}{8}$

$$= \frac{-\frac{1}{8}\int_{-2}^{2}(x^2-10)^2\,dx}{-\frac{1}{2}\int_{-2}^{2}(x^2-10)\,dx}$$

$$= \frac{\frac{1}{4}\int_{-2}^{2} x^4 - 20x + 100\,dx \ ①}{\int_{-2}^{2}(x^2-10)\,dx \ ②} = \frac{\frac{1148}{15}}{-\frac{104}{3}} = \frac{1148}{15}\cdot\frac{-3}{104}$$

$\bar{y} = -287/130$

① $\frac{1}{4}\int_{-2}^{2} x^4 - 20x^2 + 100\,dx = \frac{1}{4}\left[\frac{x^5}{5} - \frac{20x^3}{3} + 100x\right]\Big|_{-2}^{2} = \frac{1}{4}\left(\frac{32}{5} - \frac{160}{3} + 200 + \frac{32}{5} - \frac{160}{3} + 200\right)$

$= \frac{1}{4}\left(400 + \frac{64}{5} - \frac{320}{3}\right)$

$= \frac{1}{4}\left(\frac{6000}{15} + \frac{192}{15} - \frac{1600}{15}\right)$

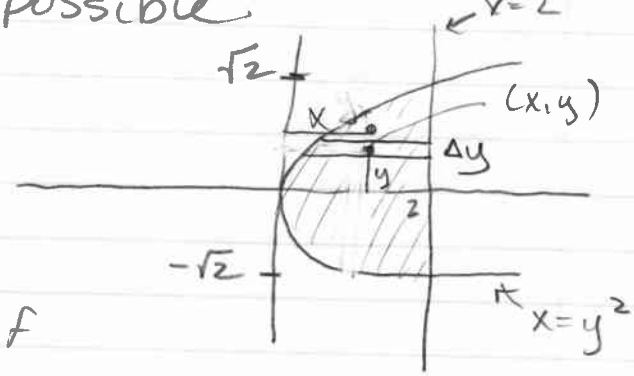
② $\int_{-2}^{2} x^2 - 10\,dx = 2\int_{0}^{2} x^2 - 10\,dx = 2\left(\frac{x^3}{3} - 10x\right)\Big|_{0}^{2} = 2\cdot\left(\frac{8}{3} - 20\right) = 2\left(\frac{8}{3} - \frac{60}{3}\right)$

$= -\frac{104}{3}$

6.6 — center of mass of plane region

(15) Find the centroid of the region bounded by the given curves. Make a sketch and use symmetry where possible.

$x = y^2$, $x = 2$
$g(y)$ $\quad f(y)$



To find the points of intersection solve

$y^2 = 2$ (set boundary functions equal)

$y = \pm\sqrt{2}$

center of mass $\bar{x} = \dfrac{M_y}{m} = \dfrac{\text{Total Moment}}{\text{Total mass}}$

$\Delta M_y = \overline{\Delta x} \cdot \Delta\text{mass}$ $\qquad$ $\Delta\text{mass} = \delta\,[f(y) - g(y)]\,\Delta y$

$\overline{\Delta x}$ = geometric center of Δy rectangle slice.

$$\Delta M_y = \left(\frac{f(y)+g(y)}{2}\right) \cdot \delta\,[f(y)-g(y)]\,\Delta y$$

$$= \frac{\delta}{2}\left[f(y)^2 - g(y)^2\right]\Delta y \qquad = \left(\frac{f(y)+g(y)}{2}\right)$$

$$\bar{x} = \frac{M_y}{m} = \frac{\int \Delta M_y}{\int \Delta m} = \frac{\frac{\delta}{2}\int_{\sqrt{2}}^{\sqrt{2}} 2^2 - y^4\,dy}{\delta \int_{\sqrt{2}}^{\sqrt{2}} 2 - y^2\,dy}$$

the δ's cancel!

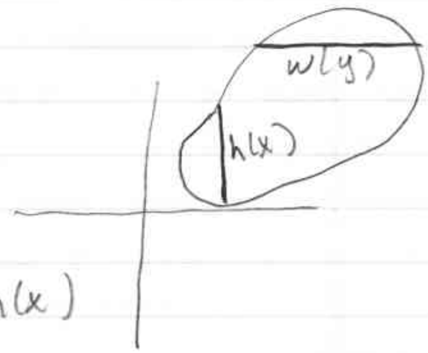
$\bar{y} = 0$ by symmetry of plane region

centroid $= (\bar{x}, \bar{y})$

6.6

28 Prove Pappus's Theorem by assuming that the region of area A in Figure 20 is to be revolved about the y-axis. Hint: $V = 2\pi \int_a^b x\,h(x)\,dx$

and $\bar{x} = \dfrac{\int_a^b (x\,h(x)\,dx)}{A}$



ΔV can be found using shell method where $r = x$ and $h = h(x)$

$\Delta V = 2\pi r h \Delta x$
$= 2\pi x\, h(x) \Delta x$ $\quad\therefore$ Volume $= 2\pi \int_a^b x\,h(x)\,dx.$

$\Delta m \approx h(x)\Delta x$ $\quad\therefore m = \int_a^b h(x)\,dx = A.$

$\Delta M_y \approx \overset{\text{dist} \cdot \text{mass}}{x \cdot h(x)\Delta x}$ $\quad\therefore M_y = \int_a^b x\,h(x)\,dx$

centroid
$\bar{x} = \dfrac{M_y}{m} = \dfrac{\int_a^b x\,h(x)\,dx}{A}$

The distance travelled by centroid around y-axis is $2\pi\bar{x}$.

and from above, multiplying both sides by $2\pi A$ we get

$2\pi A \bar{x} = 2\pi \int_a^b x\,h(x)\,dx$

Pappus's Theorem! $\longrightarrow$ $= \underbrace{(2\pi\bar{x})}_{\substack{\text{distance travelled} \\ \text{by centroid}}} \underbrace{A}_{\text{Area}} = \underbrace{\text{Volume of solid.}}_{\text{around y-axis}}$