# 9.8/Taylor and Maclaurin Series

4 Find the terms through $x^5$ in the Maclaurin series for $f(x)=e^{-x}\cos x$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

MULTIPLY

$$(e^{-x})(\cos x) = \quad 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$

$$\times \quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

(vertical multiplication of polynomials.
* include first terms through $x^5$ term in each series in the multiplication.
* don't need to include terms above $x^5$.

$$\frac{x^4}{4!} \quad - \frac{x^5}{4!}$$ unnecessary terms. above $x^5$. . .

$$-\frac{x^2}{2!} \quad + \frac{x^3}{2!} \quad - \frac{x^4}{2!2!} \quad + \frac{x^5}{3!2!}$$

$$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!}$$

$$1 - x + \left(\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{1}{24} - \frac{1}{4} + \frac{1}{24}\right)x^4 + \left(\frac{-1}{24} + \frac{1}{12} - \frac{1}{120}\right)x^5$$

$$\boxed{1 - x + \frac{x^3}{3} - \frac{x^4}{6} + \frac{x^5}{30} - \cdots}$$

9.8 / Taylor and Maclaurin series.

11 Find the terms through $x^5$ in the Maclaurin series for $f(x) = \frac{1}{1+x+x^2}$. You can use operations on known Maclaurin series.

Finding multiple derivatives of $f(x)$ is cumbersome. The only 2 algebraic known Maclaurin series -- are those for $\frac{1}{1-x}$ and $(1+x)^p$

$$(1+x+x^2)(1-x) = 1-x^3$$

so
$$\frac{1}{1+x+x^2} = \frac{1}{(1+x+x^2)} \cdot \frac{(1-x)}{(1-x)} = (1-x) \cdot \frac{1}{1-x^3}$$

$$\frac{1}{1-x^3} = 1 + (x^3) + (x^3)^2 + (x^3)^3 + \cdots$$
$$= 1 + x^3 + x^6 + x^9 + \cdots \quad \text{for } |x| < 1$$

$$(1-x) \cdot \frac{1}{1-x^3} = (1-x)(1+x^3+x^6+x^9+\cdots)$$

vertical multiplication →

$$\begin{array}{rrrrr} 1 & +x^3 & +x^6 & + & \cdots \\ 1 & -x & & & \\ \hline -x & -x^4 & & + & \cdots \\ + \;\; 1 & +x^3 & +x^6 & + & \cdots \\ \hline \end{array}$$

$$= \boxed{1 - x + x^3 - x^4 + x^6 + \cdots} \quad |x| < 1$$

9.8/ Taylor and Maclaurin series

(20) Find the Taylor series in x-a through the term $(x-a)^3$

$\sin x$ , $a = \frac{\pi}{6}$

$$\sin x = C_0 + C_1\left(x - \frac{\pi}{6}\right) + C_2\left(x - \frac{\pi}{6}\right)^2 + C_3\left(x - \frac{\pi}{6}\right)^3 + \cdots$$

where for $n = 0, 1, 2, 3$

$$C_n = \frac{f^n\left(\frac{\pi}{6}\right)}{n!}$$

| | | |
|---|---|---|
| $f(x) = \sin(x)$ | $\sin(\pi/6) = 1/2$ | $C_0 = 1/2 / 0! = \frac{1}{2}$ |
| $f'(x) = \cos(x)$ | $\cos(\pi/6) = \sqrt{3}/2$ | $C_1 = \sqrt{3}/2 / 1! = \sqrt{3}/2$ |
| $f''(x) = -\sin(x)$ | $-\sin(\pi/6) = -1/2$ | $C_2 = -1/2 / 2! = -1/4$ |
| $f'''(x) = -\cos(x)$ | $-\cos(\pi/6) = -\sqrt{3}/2$ | $C_3 = -\sqrt{3}/2 / 3! = -\frac{\sqrt{3}}{12}$ |

$$\therefore \sin x = C_0 + C_1(x - \pi/6) + C_2(x - \pi/6)^2 + C_3(x - \pi/6)^3 + \cdots$$

$$\boxed{\simeq \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}(x - \pi/6)^2 - \frac{\sqrt{3}}{12}(x - \pi/6)^3 + \cdots}$$

9.8 / Taylor and Maclaurin Series

(28) Given that $\sinh^{-1} x = \int_0^x \frac{1}{\sqrt{1+t^2}}\, dt$

find the first four nonzero terms in the maclaurin series for $\sinh^{-1} x$.

$\frac{1}{\sqrt{1+t}} = (1+t)^{-1/2}$ $\quad f(0) = 1 \quad C_0 = 1$

$f'(t) = -\frac{1}{2}(1+t)^{-3/2}$ $\quad f'(0) = -1/2 \quad C_1 = {}^{-1/2}/_{1!} = -1/2$

$f''(t) = 3/4\,(1+t)^{-5/2}$ $\quad f''(0) = 3/4 \quad C_2 = {}^{3/4}/_{2!} = 3/8$

$f'''(t) = \frac{-15}{8}(1+t)^{-7/2}$ $\quad f'''(0) = -15/8 \quad C_3 = {}^{-15/8}/_{3!} = -5/16$

$\therefore \frac{1}{\sqrt{1+t}} = C_0 + C_1 t + C_2{}^{2} + C_3{}^{3} + \cdots$

$= 1 - 1/2\, t + 3/8\, t^2 - 5/16\, t^3 + \cdots$

$\therefore \frac{1}{\sqrt{1+t^2}} = 1 - \frac{1}{2}(t^2) + \frac{3}{8}(t^2)^2 - \frac{5}{16}(t^2)^3 + \cdots$

$= 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \frac{5}{16}t^6 + \cdots$

$\int_0^x \frac{1}{\sqrt{1+t^2}} = \int_0^x \left(1 - \frac{t^2}{2} + \frac{3t^4}{8} - \frac{5}{16}t^6 + \cdots\right) dt$

$$\boxed{\sinh^{-1} x = 1 - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{5x^7}{112} + \cdots}$$