9.9 / The Taylor Approximation to a Function

(4) Find the Maclaurin polynomial of order 4 for $f(x) = \tan x$ and use it to approximate $f(0.12)$

order 4

$$f(x) \approx P_4(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$$

degree 4

$$\text{where} \quad c_n = \frac{f^{(n)}(0)}{n!}$$

$f(x) = \tan x \qquad \tan(0) = 0 \qquad \boxed{c_0 = \frac{f(0)}{0!} = 0}$

$f'(x) = \sec^2 x \qquad \sec^2(0) = \frac{1}{\cos^2(0)} = 1 \qquad \boxed{c_1 = \frac{1}{1!} = 1}$

$f''(x) = 2\sec x \cdot \sec x \tan x$
$= 2\sec^2 x \tan x \qquad 2\sec^2(0)\tan(0) = 0 \qquad \boxed{c_2 = 0}$

$f^3(x) = [4\sec x \cdot \sec x \tan x] \cdot \tan x + [2\sec^2 x][\sec^2 x]$
$= 4\sec^2 x \tan^2 x + 2\sec^4 x$
$f^3(0) = 4(1)(0) + 2(1) = 0 + 2 = 2$

$$\boxed{c_3 = \frac{2}{3!} = \frac{1}{3}}$$

$f^4(x) = D_x\{4\sec^2 x \tan^2 x + 2\sec^4 x\}$

$= [(8\sec x \cdot \sec x \tan x) \cdot \tan^2 x + (4\sec^2 x)(2\tan x \cdot \sec^2 x)] + 8\sec^3 x \cdot \sec x \tan x$

$= 8\sec^2 x \tan^3 x + 8\sec^4 x \tan x + 8\sec^4 x \tan x$

$f^4(0) = 8(1)(0) + 8(1)(0) + 8(1)(0) = 0 + 0 + 0 = 0.$

$$\boxed{c_4 = 0}$$

9.9/ Taylor Approximation to a Function

4 for $f(x) = \tan x$

$$f(x) \approx P_4(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$$

$$= 0 + 1x + 0x^2 + \frac{1}{3}x^3 + 0x^4$$

$$\boxed{P_4(x) = x + \frac{x^3}{3}}$$

$$f(0.12) \approx P_4(0.12) = (0.12) + \frac{(0.12)^3}{3}$$

$$\boxed{f(0.12) \approx 0.1206}$$

9.9 / Taylor Approximation to a Function

(10) Find the Taylor polynomial of order 3 based at a for the given function.

$f(x) = \sin x \qquad a = \pi/4$

$f(x) \approx P_3(x) = C_0 + C_1(x-\pi/4) + C_2(x-\pi/4)^2 + C_3(x-\pi/4)^3$

where $C_n = \dfrac{f^n(\pi/4)}{n!}$

$f(x) = \sin x \qquad \sin(\pi/4) = \frac{\sqrt{2}}{2} \qquad \boxed{C_0 = \frac{f(\pi/4)}{0!} = \frac{\sqrt{2}}{2}}$

$f'(x) = \cos x \qquad \cos(\pi/4) = \frac{\sqrt{2}}{2} \qquad \boxed{C_1 = \frac{f'(\pi/4)}{1!} = \frac{\sqrt{2}}{2}}$

$f''(x) = -\sin x \qquad -\sin(\pi/4) = -\frac{\sqrt{2}}{2} \qquad \boxed{C_2 = f''(\pi/4)/2! = \frac{-\sqrt{2}}{4}}$

$f^3(x) = -\cos x \qquad -\cos(\pi/4) = -\frac{\sqrt{2}}{2} \qquad \boxed{C_3 = \frac{f^3(\pi/4)}{3!} = \frac{-\sqrt{2}}{2 \cdot 6} = \frac{-\sqrt{2}}{12}}$

$P_3(x) = C_0 + C_1(x-\frac{\pi}{4}) + C_2(x-\frac{\pi}{4})^2 + C_3(x-\frac{\pi}{4})^3$

$$\boxed{P_3(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x-\frac{\pi}{4}) - \frac{\sqrt{2}}{4}(x-\frac{\pi}{4})^2 - \frac{\sqrt{2}}{12}(x-\frac{\pi}{4})^3}$$

9.9 / Taylor Approximation to a Function

16 Find the Taylor polynomial of order 4 based at 2 for $f(x) = x^4$ and show that it represents $f(x)$ exactly.

$\leftarrow$ Taylor polynomial of order 4 based at 2

$$f(x) \approx P_4(x) = c_0 + c_1(x-2) + c_2(x-2)^2 + c_3(x-2)^3 + c_4(x-2)^4$$

where $c_n = \dfrac{f^n(2)}{n!}$

$f(x) = x^4 \quad f(2) = 2^4 = 16 \quad \boxed{c_0 = \dfrac{f(2)}{0!} = 16}$

$f'(x) = 4x^3 \quad f'(2) = 4(2)^3 = 32 \quad \boxed{c_1 = \dfrac{f'(2)}{1!} = 32}$

$f''(x) = 12x^2 \quad f''(2) = 12(2)^2 = 48 \quad \boxed{c_2 = f''(2)/2! = 48/2 = 24}$

$f^3(x) = 24x \quad f^3(2) = 24 \cdot 2 = 48 \quad \boxed{c_3 = \dfrac{f^3(2)}{3!} = \dfrac{48}{6} = 8}$

$f^4(x) = 24 \quad f^4(2) = 24 \quad \boxed{c_4 = \dfrac{f^4(2)}{4!} = \dfrac{24}{24} = 1}$

$$\boxed{\begin{aligned} f(x) \approx P_4(x) &= c_0 + c_1(x-2) + c_2(x-2)^2 + c_3(x-2)^3 + c_4(x-2)^4 \\ &= 16 + 32(x-2) + 24(x-2)^2 + 8(x-2)^3 + (x-2)^4 \end{aligned}}$$

over for proof that $P_4(x) = f(x)$

9.9 16 continued

show $P_4(x)$ represents $f(x)$ exactly

Prove $P_4(x) = f(x)$ Prove ?

$$16+32(x-2)+24(x-2)^2+8(x-2)^3+(x-2)^4 = x^4$$

$$P_4(x) = \left[16+32(x-2)+24(x-2)^2\right] + \left[8(x-2)^3+(x-2)^4\right]$$

$$= \left[16+32x-64+24x^2-96x+96\right] + \left[(x-2)^2\left[8(x-2)+(x-2)^2\right]\right]$$

$$= \left[24x^2-64x+48\right] + \left[(x^2-4x+4)\left[8x-16+x^2-4x+16\right]\right]$$

$$= \left[24x^2-64x+48\right] + \left[(x^2-4x+4)(x^2+4x-12)\right]$$

$$= \left[24x^2-64x+48\right] + \left[x^4+4x^3-12x^2-4x^3-16x^2+48x+4x^2+16x-48\right]$$

$$= x^4+(4-4)x^3+(24-12-16+4)x^2+(-64+48+16)x+(48-48)$$

$$= x^4+0x^3+0x^2+0x+0$$

$$= x^4$$

$$= f(x)$$