9.4 / Positive Series: Other Tests

(2) Use the Limit Comparison Test to determine convergence or divergence.

$$\sum_{n=1}^{\infty} \frac{3n+1}{n^3-4}$$

$$a_n = \frac{3n+1}{n^3-4} \qquad b_n = \frac{1}{n^2} \quad \left(\text{coefficients don't matter}\right)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3n+1}{n^3-4}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{n^2(3n+1)}{n^3-4}$$

$$= \lim_{n\to\infty} \frac{3n^3+n^2}{n^3-4} = \lim_{n\to\infty} \frac{\cancel{n^3}\left(3+\frac{1}{n}\right)}{\cancel{n^3}\left(1-4/n^3\right)}$$

$$= \frac{3+0}{1-0} = 3$$

Since $0 < 3 < \infty$
(Limit a_n/b_n)

and $\sum b_n = \sum \frac{1}{n^2}$ converges by p-series test ($p = 2 > 1$.)

then by the Limit Comparison Test

$$\boxed{\sum a_n = \sum_{n=1}^{\infty} \frac{3n+1}{n^3-4} \text{ also converges.}}$$

9.4/ Positive Series: Other Tests.

(10) Use the Ratio Test to determine convergence or divergence.

$$\sum_{k=1}^{\infty} \frac{3^k + k}{k!}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{3^{n+1}+n+1}{(n+1)!}\right)}{\left(\frac{3^n+n}{n!}\right)}$$

$$= \lim_{n\to\infty} \frac{n!\,(3^{n+1}+n+1)}{(n+1)!\,(3^n+n)}$$

$$= \lim_{n\to\infty} \frac{\cancel{n!}\,(3^{n+1}+n+1)}{\cancel{n!}\,(n+1)(3^n+n)}$$

$$= \lim_{n\to\infty} \frac{3^{n+1}+n+1}{(n+1)(3^n+n)}$$

$$= \lim_{n\to\infty} \frac{3^{n+1}+n+1}{n3^n+n^2+3^n+n}$$

$$= \lim_{n\to\infty} \frac{\cancel{3^n}\left(3+\frac{n}{3^n}+\frac{1}{3^n}\right)}{\cancel{3^n}\left(n+\frac{n^2}{3^n}+1+\frac{n}{3^n}\right)}$$

L'Hopital's Rule shows $\lim_{n\to\infty} \frac{n}{3^n} = 0$

$\lim_{n\to\infty} \frac{n^2}{3^n} = 0.$

$3^n = e^{(\ln 3)\cdot n}$

$$\overset{*}{=} \frac{3+0+0}{\infty+0+1+0} = 0 < 1$$

as $n \to \infty$

$$\boxed{\therefore \text{ the series } \sum_{k=1}^{\infty} \frac{3^k+k}{k!} \text{ Converges.}}$$

9.4/ Positive Series: Other Tests.

16) Determine convergence or divergence.
State the test you use

$$\sum_{n=1}^{\infty} \frac{\ln n}{2^n}$$ Ratio Test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{\ln(n+1)}{2^{n+1}}\right)}{\left(\frac{\ln(n)}{2^n}\right)}$$

$$= \lim_{n\to\infty} \frac{2^n \ln(n+1)}{2^{n+1} \ln(n)} = \lim_{n\to\infty} \frac{\ln(n+1)}{2\ln(n)} = \frac{\infty}{\infty}$$

L'Hopital's Rule Applies

$$= \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{2}{n}} \quad \text{(LH Rule)}$$

$$= \lim_{n\to\infty} \frac{n}{2(n+1)} = \frac{1}{2} < 1$$

$$\boxed{\therefore \sum_{n=1}^{\infty} \frac{\ln(n)}{2^n} \text{ converges by the RATIO TEST}}$$

9.4/ Positive series: Other Tests.

(32) Determine convergence or divergence

$$\sum_{n=2}^{\infty}\left(1-\frac{1}{n}\right)^n$$

let $y=\left(1-\frac{1}{x}\right)^x$

$$\ln y = \ln\left(1-\frac{1}{x}\right)^x$$

$$= x\ln\left(1-\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} x\ln\left(1-\frac{1}{x}\right) = \infty \cdot 0$$

$$= \lim_{x\to\infty} \frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$$ L'Hopital's Rule applies

$$= \lim_{x\to\infty} \frac{\left(\frac{1}{1-\frac{1}{x}}\right)\cdot\frac{1}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} -\frac{1}{1-\frac{1}{x}} = -1$$

so since $\lim_{x\to\infty} \ln y = -1$ } exponentiate both sides

$$\lim_{x\to\infty} y = e^{-1}$$

Since $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = e^{-1} \neq 0$

series $\sum_{n=2}^{\infty}\left(1-\frac{1}{n}\right)^n$ diverges by the nth term test for Divergence