Tests for Convergence (or Divergence) of $\Sigma_{n=1}^{\infty} a_n$

Name	*Brief Formulation*	*When Used*	*Example*
1. *n*th-term test for divergence	If a_n does *not* approach 0 as $n \to \infty$, $\Sigma_{n=1}^{\infty} a_n$ diverges.	When you suspect a_n does not approach 0.	$\sum_{n=1}^{\infty} \frac{n}{2n+1}$ diverges.
2. Integral test	If $f(x) > 0$ decreases and $\int_1^{\infty} f(x)\,dx$ converges, $\Sigma_{n=1}^{\infty} a_n$ converges. If $\int_1^{\infty} f(x)\,dx$ diverges, $\Sigma_{n=1}^{\infty} a_n$ diverges.	When you have a *positive decreasing* series $a_n = f(n)$ and $\int f(x)\,dx$ is easy to calculate.	$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, since $\int_1^{\infty} x^{-2}\,dx$ is convergent.
3. Comparison test	If $0 \le a_n \le c_n$ and $\Sigma_{n=1}^{\infty} c_n$ converges, so must $\Sigma_{n=1}^{\infty} a_n$. If $a_n \ge d_n \ge 0$ and $\Sigma_{n=1}^{\infty} d_n$ diverges, so must $\Sigma_{n=1}^{\infty} a_n$.	When you have a *positive* series that can be compared to a series known to converge or diverge.	$\sum_{n=1}^{\infty} \frac{n+1}{n+2}\frac{1}{2^n}$ converges, since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ does.
4. Limit-comparison test	If $a_n/c_n \to$ nonzero limit, then $\Sigma_{n=1}^{\infty} a_n$ converges if $\Sigma_{n=1}^{\infty} c_n$ does and diverges if $\Sigma_{n=1}^{\infty} c_n$ does. (For a more general statement, see Theorem 2, Sec. 10.4.)	When you have a positive series very much like a series known to converge or diverge.	$\sum_{n=1}^{\infty} \frac{1+(-\frac{1}{2})^n}{2^n}$ converges, since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.
5. Ratio test		See "absolute-ratio" test, which is more useful.	$\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges.
6. Decreasing-alternating-series test	A decreasing alternating series whose *n*th term $\to 0$ converges.	When you have an *alternating series* whose terms diminish in absolute value from some point on and approach 0.	$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.
7. Absolute-convergence test	If $\Sigma_{n=1}^{\infty} \lvert a_n \rvert$ converges, so does $\Sigma_{n=1}^{\infty} a_n$.	When you feel that the series would converge even if its terms were all made positive.	$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges.
8. Absolute-ratio test	If $\lvert a_{n+1}/a_n \rvert \to L < 1$, $\Sigma_{n=1}^{\infty} a_n$ converges (absolutely). If $\lvert a_{n+1}/a_n \rvert \to L > 1$, $\Sigma_{n=1}^{\infty} a_n$ diverges. If $L = 1$, no information.	Especially suitable for power series.	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for all x.
9. Root test	If $\sqrt[n]{\lvert a_n \rvert} \to L < 1$, $\Sigma_{n=1}^{\infty} a_n$ converges.	If something like n^n appears, try root test.	$\sum_{n=1}^{\infty} \frac{3^n}{n^n}$ converges.

Estimating the Error

Assume that $\Sigma_{n=1}^{\infty} a_n = S$. Let $\Sigma_{i=1}^{n} a_i = S_n$. The error, or remainder, R_n is defined to be the difference $S - S_n$.

If $\Sigma_{n=1}^{\infty} a_n$ is a decreasing alternating series whose *n*th term approaches 0, then $\lvert R_n \rvert < \lvert a_{n+1} \rvert$. If $\Sigma_{n=1}^{\infty} a_n$ is a positive series to which the integral test applies, then $\int_{n+1}^{\infty} f(x)\,dx < R_n < \int_n^{\infty} f(x)\,dx$. If $\Sigma_{n=1}^{\infty} a_n$ is a convergent geometric series, $a_n = ar^{n-1}$, then $R_n = ar^n/(1-r)$.

In the case of the Maclaurin series associated with the function f, the error $R_n(x; 0)$ is the difference between $f(x)$ and the sum of the first $n + 1$ terms:

$$R_n(x; 0) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)x^k}{k!}.$$