4.3 * Second Derivative Test

Let f be a function such that f' and f'' exist at every point in an open interval (a,b) containing c, and suppose that $f'(c)=0$.

1) If $f''(c)<0$, then $f(c)$ is a local max val of f
2) If $f''(c)>0$, then $f(c)$ is a local min val of f

4.6 Mean Value Thm for Derivatives

If f is continuous on a closed interval $[a,b]$ and differentiable on its interior (a,b) then there is at least one number c in (a,b) where

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

OR equivalently where

$$f(b)-f(a) = f'(c)(b-a)$$

Thm B If $F'(x)=G'(x)$ $\forall x$ in (a,b) then there is a constant C such that

$$F(x)=G(x)+C.$$

4.3

(8) Identify the critical points & use the 1st & 2nd Derivative Tests to determine if the critical points are local maximums or minimums.

$$g(z) = \frac{z^2}{1+z^2} \qquad g'(z) = \frac{2z(1+z^2) - z^2 \cdot 2z}{(1+z^2)^2}$$

$$g'(z) = \frac{2z}{(1+z^2)^2}$$

critical pts: $g'(z)=0$ and $g'(z)=$ undefined.

$2z = 0$ $\qquad (1+z^2)^2 = 0$

$\boxed{z=0 \text{ critical point}}$ $\qquad$ Never

$$g''(z) = \frac{2(1+z^2)^2 - 2z(2(1+z^2)\cdot 2z)}{(1+z^2)^4}$$

$$= \frac{\cancel{(1+z^2)}\left[2(1+z^2) - 8z^2\right]}{\cancel{(1+z^2)^4}} = \frac{2-6z^2}{(1+z^2)^3}$$

$$g''(z) = \frac{2(1-3z^2)}{(1+z^2)^3}$$

Now test crit. pt $z=0$

$$g''(0) = \frac{2(1-3(0)^2)}{(1+0^2)^3} = \frac{2\cdot 1}{1} = 2$$ positive

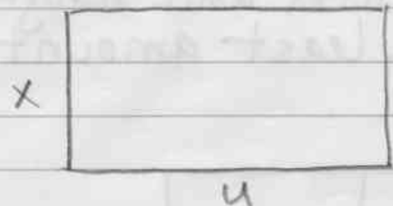
concave up

minimum

$\boxed{\therefore g(z) \text{ has local minimum at } z=0}$

4.4

(8) Show that for a rectangle of given perimeter K, the one with maximum area is a square



$$K = 2x + 2y$$
$$K - 2y = 2x$$
$$\frac{K-2y}{2} = x$$

$$\text{Area} = x \cdot y$$
$$= \left(\frac{K-2y}{2}\right) y$$
$$A = \frac{Ky - 2y^2}{2}$$

$$A' = \frac{K-4y}{2} = 0 \quad \text{when} \quad K - 4y = 0$$
$$K = 4y$$
$$\frac{K}{4} = y$$

Now Find x

$$x = \frac{K-2y}{2}$$
$$2x = K - 2y$$
$$2x = K - 2 \cdot \frac{K}{4}$$
$$2x = K - \frac{K}{2}$$
$$2x = \frac{K}{2}$$
$$x = \frac{K}{4}$$

$x = \frac{K}{4} \quad y = \frac{K}{4}$

$\Rightarrow x = y$
$\therefore$ Rectangle is square!

4.4

(14) A farmer wishes to fence off three identical adjoining pens, each with 300 square feet of area. What should the width and length of each pen be so that the least amount of fence is required?

300 square feet each $\Rightarrow$ $x \cdot y = 300$

$$y = \frac{300}{x}$$

Amount of Fence needed = Perimeter.
Minimize fencing = minimize Perimeter.

Perimeter $= 6x + 4y$

$$P = 6x + 4 \cdot \frac{300}{x}$$

$$P = 6x + 1200x^{-1}$$

$$P' = 6 - \frac{1200}{x^2} = 0 \quad \text{when} \quad 6 = \frac{1200}{x^2}$$

$$6x^2 = 1200$$

$$x^2 = 200$$

$$x = \sqrt{200}$$

$$x = 10\sqrt{2}$$

$$y = \frac{300}{x}$$

$$y = 15\sqrt{2}$$

$$\boxed{\begin{array}{l} x = 10\sqrt{2} \text{ ft} \\ y = 15\sqrt{2} \text{ ft} \end{array}}$$

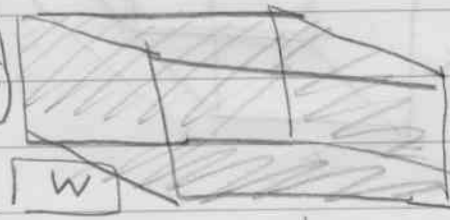
(21) A company wishes to manufacture a box with a volume of 20 cu ft that is open on top and is twice as long as it is wide. Find the width of the box that can be produced using the minimum amount of material. Round to nearest tenth if necessary

$V = 20 = L \cdot W \cdot H$

$20 = (2 \cdot W) \cdot W \cdot H$

$\frac{20}{2W^2} = H.$

$\boxed{\frac{20}{2W^2} = H}$



$\boxed{L = 2W}$

MINIMIZE "AMT OF MATERIAL FUNCTION"

Find width (w) that produces min value.

$$\text{Amt (SURFACE AREA)} = \underset{\text{BOT}}{\text{AREA}} + \underset{2\,\text{SIDES}\,(W)}{\text{AREA}} + \underset{2\,\text{SIDES}\,(L)}{\text{AREA}}$$

$$= L \cdot W + 2 \cdot W \cdot H + 2 \cdot L \cdot H$$

$$= 2W \cdot W + 2W \cdot \frac{20}{2W^2} + 2 \cdot (2W) \cdot \frac{20}{2W^2}$$

$$= 2W^2 + \frac{20}{W} + \frac{40}{W}$$

$$y = 2W^2 + 60W^{-1}$$

$$y' = 4W - 60W^{-2}$$

$$= 4W - \frac{60}{W^2}$$

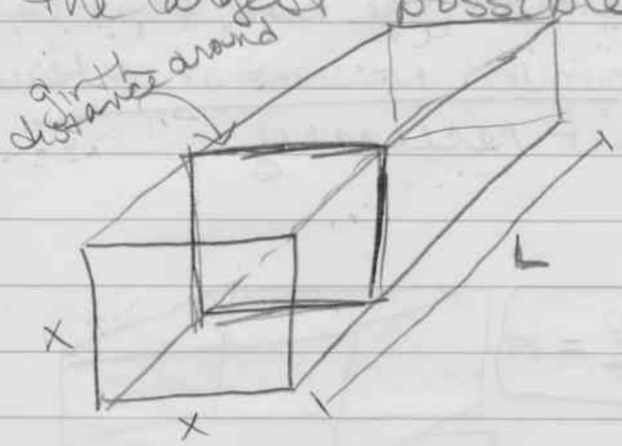
$$= \frac{4W^3 - 60}{W^2} = 0 \text{ when } 4W^3 - 60 = 0$$

$$4W^3 = 60$$

$$W^3 = 15$$

$$W^3 = \sqrt[3]{15}$$

(22) Private shipping company will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 120 in. What dimensions will give a box with a square end the largest possible volume?



$120 = \text{LENGTH} + \text{GIRTH}$

$120 = L + 4x$

$\boxed{120 - 4x = L}$

LARGEST VOLUME = MAXIMIZE VOL FUNCTION

ANSWER: DIMENSIONS FOR MAX VOL

$\boxed{\text{FIND } x \text{ and } L}$ FOR MAX VOL

$V = L \cdot W \cdot H$

$= L \cdot x \cdot x$

$= (120 - 4x)x^2$

$V = 120x^2 - 4x^3$

$V' = 240x - 12x^2 = 0$

$12x(20 - x) = 0$

$x = 0$ OR $20 - x = 0$

impossible
unreasonable

$\boxed{20 = x}$

$120 - 4x = L$

$120 - 4(20) = L$

$120 - 80 = L$

$40 = L$

$$\boxed{\begin{array}{ccccc} 20^{in} & \times & 20^{in} & \times & 40'' \\ x & \times & x & \times & L \end{array}}$$