Find the work done in pumping all the oil (density $s = 50$ pounds per cubic foot) over the edge of a cylindrical tank that stands on one of its bases. Assume that the radius of the base is 4 feet, the height is 10 ft, and the tank is full of oil.

![Diagram of a cylindrical tank with dimensions labeled]

**Work = Force \cdot Distance**

Find Force $F(x)$ first.

Force needed to lift oil = weight = volume $\cdot s$

An $Ay$ slice is a circle. Volume of slice $= \pi r^2 Ay$

$= \pi (4)^2 Ay$

$= 16\pi Ay \cdot \frac{ft^3}{4}$

$S = 50 \frac{lbs}{ft^3}$

$F(x) = Vol \cdot s$

The $Ay$ slice is at height $= y$ and must be pumped just over top, so it must travel distance $10 - y$.

**Work = Force \cdot Distance**

$= 16\pi Ay \cdot s \cdot (10 - y)$ where $0 \leq y \leq 10$

$W = \int_0^{10} 16\pi s (10 - y) dy$

$= 16\pi s \int_0^{10} 10 - y dy$

$= 16\pi s \left[ 10y - \frac{y^2}{2} \right]_0^{10}$

$= 16\pi s \left( 100 - \frac{100}{2} \right)$

$= 16\pi s (50 - 50) = 125,164 ft-lbs$

$W = 125,164 ft-lbs.$
Finding Centroids $(\bar{x}, \bar{y})$

- Find slice $\Delta x$ or $\Delta y$
- Remember $m$ is moment wrt $y$-axis
- Distance to $y$-axis = $x$.
- $m_x$ is moment wrt $x$-axis
- Midpoint between $f(x)$ and $g(x)$

\[
\bar{x} = 0 \quad \text{and} \quad \bar{y} = \frac{\frac{1}{2} \int_a^b \left[ (f(x))^2 - (g(x))^2 \right] \, dx}{\int_a^b [f(x) - g(x)] \, dx} \quad \text{center of mass of rectangle slice = geom center}
\]

- Use $\Delta x$:

\[
y = \frac{\frac{1}{2} \int_a^b \left[ (f(x))^2 - (g(x))^2 \right] \, dx}{\int_a^b [f(x) - g(x)] \, dx}
\]

- If you have $x$-axis symmetry:

\[
\bar{y} = 0 \quad \text{and} \quad \bar{x} = \frac{M_y}{m} = \frac{\frac{1}{2} \int_c^d \left[ (f(y))^2 - (g(y))^2 \right] \, dy}{\int_c^d [f(y) - g(y)] \, dy}
\]

- Mass = Density $\cdot$ Area or mass = Density $\cdot$ Volume depending on density units.

Centroids = center of mass of plane region

- It doesn't depend on density or mass
- Really because $S$ factors and ultimately cancels out of $\frac{M_y}{m}$ and $\frac{m}{m}$

\[
\bar{x} = \frac{M_y}{m} = \frac{\text{Total Moment wrt y-axis}}{\text{Total mass}} \quad \bar{x} \quad \text{and} \quad \bar{y}
\]

\[
\bar{y} = \frac{M_x}{m} = \frac{\text{Total Moment wrt x-axis}}{\text{Total mass}}
\]
Find the centroid of the region bounded by the given curves. Make a sketch and use symmetry where possible.

\[ y = \frac{1}{2} (x^2 - 10), \quad y = 0, \text{ and between } x = -2 \text{ and } x = 2 \]

\[ y - \text{axis symmetry} \]

1. Choose \( \Delta x \)
2. \( \bar{y} = \frac{m_x}{m} = \frac{\frac{1}{2} \int_{-2}^{2} 0^2 - \left(\frac{1}{2}(x^2 - 10)\right)^2 \, dx}{\frac{1}{2} \int_{-2}^{2} (x^2 - 10) \, dx} \]
3. \( \frac{1}{8} \int_{-2}^{2} (x^2 - 10)^2 \, dx \)
4. \( \frac{1}{4} \int_{-2}^{2} x^4 - 20x + 100 \, dx \)

\[ \bar{y} = \frac{1148}{15} = 76.53 \quad \text{and} \quad \bar{y} = \frac{287}{130} \]

\[ \frac{1}{4} \int_{-2}^{2} x^4 - 20x + 100 \, dx = \frac{1}{4} \left[ \frac{x^5}{5} - 10x^3 + 100x \right]_{-2}^{2} \]

\[ = \frac{1}{4} \left( \frac{32}{5} - 160 + 200 + \frac{32}{5} \right) = \frac{1}{5} \left( \frac{64}{5} - \frac{320}{5} + \frac{64}{5} \right) = \frac{1}{5} \left( \frac{128}{5} - \frac{320}{5} + \frac{64}{5} \right) \]

\[ = \frac{1}{5} \left( \frac{128 - 320 + 64}{5} \right) = \frac{1}{5} \left( \frac{128 - 320 + 64}{5} \right) = \frac{1}{5} \left( \frac{128 - 320 + 64}{5} \right) \]

\[ = \frac{1}{5} \left( \frac{128 - 320 + 64}{5} \right) = \frac{1}{5} \left( \frac{128 - 320 + 64}{5} \right) = \frac{1}{5} \left( \frac{128 - 320 + 64}{5} \right) \]

\[ = \frac{1}{5} \left( \frac{128 - 320 + 64}{5} \right) = \frac{1}{5} \left( \frac{128 - 320 + 64}{5} \right) = \frac{1}{5} \left( \frac{128 - 320 + 64}{5} \right) \]
6.6 Center of mass of plane region

Find the centroid of the region bounded by the given curves. Make a sketch and use symmetry where possible.

\[ x = y^2, \quad x = 2 \]

\( f(y), g(y) \)

To find the points of intersection solve

\[ y^2 = 2 \quad \text{(set boundary functions equal)} \]

\[ y = \pm \sqrt{2} \]

Center of mass \( \overline{x} = \frac{My}{m} = \frac{\text{Total moment}}{\text{Total mass}} \)

\[ \Delta y = \overline{x} \cdot \Delta \text{mass} \]

\[ \Delta \text{mass} = \int [f(y) - g(y)] \, dy \]

\[ \Delta y = \left( \frac{f(y) + g(y)}{2} \right) \cdot \int [f(y) - g(y)] \, dy \]

\[ \Delta y = \frac{1}{2} \left[ f(y)^2 - g(y)^2 \right] \, dy \]

\[ \overline{x} = \frac{My}{m} = \frac{\int \Delta y}{\int \Delta m} = \frac{\frac{1}{2} \left[ \int_{-\sqrt{2}}^{\sqrt{2}} y^4 \, dy \right]}{\frac{1}{2} \left[ \int_{-\sqrt{2}}^{\sqrt{2}} 2 - y^2 \, dy \right]} \]

\[ \overline{y} = 0 \quad \text{by symmetry of plane region} \]

Centroid = \((\overline{x}, \overline{y})\)
6.6
Prove Pappus's Theorem by assuming that the region of area $A$ in Figure 20 is to be revolved about the $y$-axis. Hint: $V = 2\pi \int_a^b x h(x) \, dx$
and $\overline{x} = \frac{\int_a^b (x h(x)) \, dx}{A}$

$\Delta V$ can be found using shell method where $r = x$ and $h = h(x)$

$\Delta V = 2\pi \times h \times \Delta x = 2\pi \times h(x) \Delta x \quad \Rightarrow \quad \text{Volume} = 2\pi \int_a^b x h(x) \, dx$.

$\Delta M \times h(x) \Delta x \quad \Rightarrow \quad \Delta M = \int_a^b h(x) \, dx = A.$

$\Delta M_y \approx x \times h(x) \Delta x \quad \Rightarrow \quad M_y = \int_a^b x h(x) \, dx$


centroid

$\overline{x} = \frac{M_y}{M} = \frac{\int_a^b x h(x) \, dx}{\int_a^b h(x) \, dx} = \frac{\int_a^b x h(x) \, dx}{A}$

The distance travelled by centroid around $y$-axis is $2\pi \overline{x}$.

and from above, multiplying both sides by $2\pi A$ we get

$2\pi A \overline{x} = 2\pi \int_a^b x h(x) \, dx$

Pappus's Theorem

$\Rightarrow \quad \text{Volume of solid} = (2\pi \overline{x}) A = \text{Volume of solid}$

distance \cdot \text{area} \quad \text{travels by centroid around y-axis}$