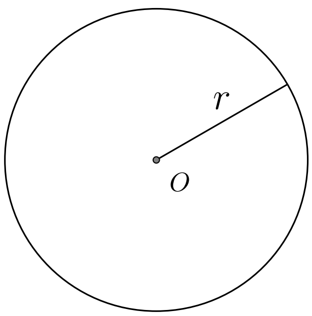
Definition: A **circle** is the set of all points in a plane that are equidistant from a given point called the **center** of the circle. We use the symbol $\odot$ to represent a circle.

The a line segment from the center of the circle to any point on the circle is a **radius** of the circle. By definition of a circle, all radii have the same length. We also use the term radius to mean the length of a radius of the circle.

To refer to a circle, we may refer to the circle with a given center and a given radius. For example, we can say circle O with radius r.



The **circumference** of a circle is the length around the circle.

A **central angle** of a circle is an angle that is formed by two radii of the circle and has the center of the circle as its vertex. In other words, a central angle always has its vertex as the center of the circle.

An **arc** is a connected portion of a circle. An arc that is less than half a circle is a **minor arc**. An arc that is greater than half a circle is a **major arc**, and an arc that's equal to half a circle is a **semi-circle**.

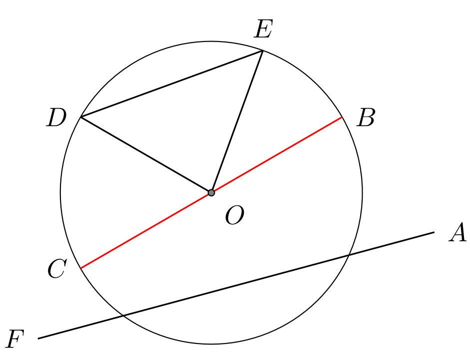
By definition, the degree measure of an arc is the central angle that **intercepts** the arc.

We use two letters with an arc symbol on top to refer to a minor arc, and three letters for a major arc.

A **chord** is an line segment that has any two points on the circumference as its end-points. A chord always lies inside a circle.

A **diameter** of a circle is a chord that contains the center of the circle.

A **secant** is a line that intersects the circle at two points.



For the circle above, $\angle EOB$ is a central angle. So is $\angle DOE$

$\widehat{DE}$ is a minor arc. The central angle $\angle DOE$ is the angle that *intercepts* this arc. The (degree) measure of $\widehat{DE}$ is the measure of $\angle DOE$.

$\widehat{DCB}$ is a major arc.

$\widehat{CB}$ is a semi-circle.

$\overline{CB}$ is a diameter.

$\overline{DE}$ is a chord.

$\overline{FA}$ is a secant.

By definition, two circles are congruent if their radii are congruent. Two arcs are congruent if they have the same degree measure and same length.

Postulates and/or facts: For circles that are congruent or the same:

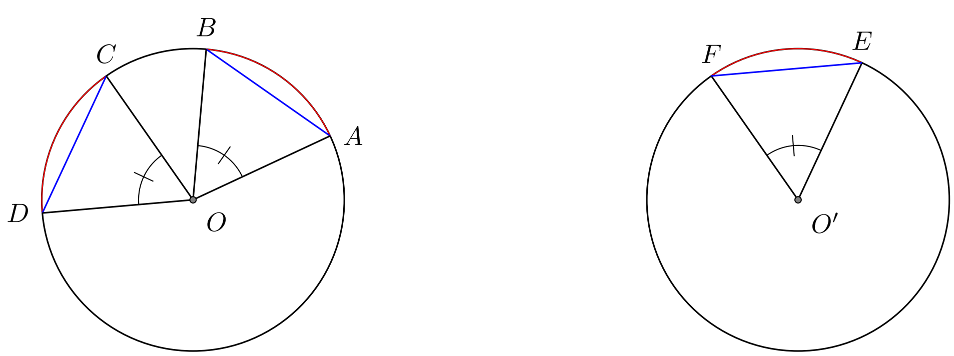
All radii are congruent

All diameters are congruent

A diameter of a circle divides the circle into two equal arcs (semi-circles). Conversely, **If a chord divides the circle into two equal arcs, then the chord is a diameter**

Congruent central angles intercept congruent arcs, and conversely, **congruent arcs are intercepted by congruent central angles.**

Congruent chords divide congruent arcs, and conversely, **Congruent arcs have congruent chords**.



In the picture above, assume $\odot O \cong \odot O'$.

If central angle $\angle AOB \cong \angle COD \cong \angle EO'F$, then

$\overline{AB} \cong \overline{CD} \cong \overline{EF}$, and $\widehat{AB} \cong \widehat{CD} \cong \widehat{EF}$

Conversely, if $\widehat{AB} \cong \widehat{CD} \cong \widehat{EF}$, then

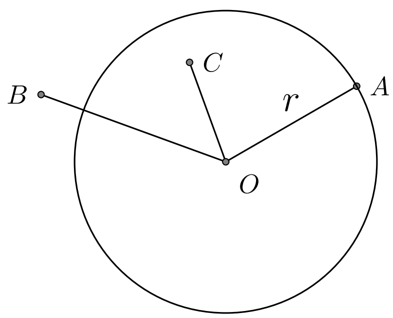
$\overline{AB} \cong \overline{CD} \cong \overline{EF}$, and $\angle AOB \cong \angle COD \cong \angle EO'F$

Let $\odot O$ be a circle with center O and radius r, and let P be a point.

If the distance between a point P and the center O of a circle is less then the radius of the circle, the point P is **inside** the circle.

If the distance between a point P and the center O of a circle is greater then the radius of the circle, the point is **outside** the circle.

If the distance between a point P and the center O of a circle is equal to the radius of the circle, the point is **on** the circle.



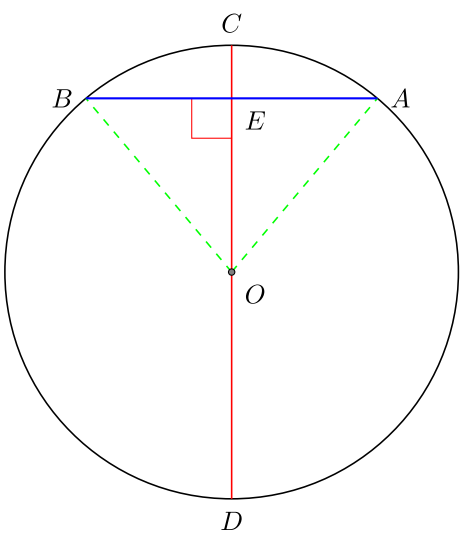
In the picture above, C is inside circle $\odot O$ since $\overline{OC} < r$. B is outside circle $\odot O$ since $\overline{OB} > r$, and A is on circle $\odot O$ since $\overline{OA} = r$.

Theorem: A diameter perpendicular to a chord bisects the chord and its arcs.

Proof: Given $\odot O$ and $\overline{AB}$ a chord, let $\overline{CD}$ be a diameter of the circle such that $\overline{CD}$ is perpendicular to $\overline{AB}$ at point E. We must prove that

$\overline{BE} \cong \overline{AE}$ and $\widehat{BC} \cong \widehat{AC}$

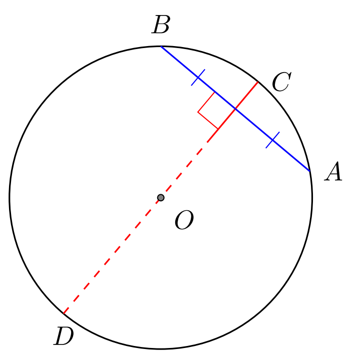
We will construct the radii $\overline{OA}$ and $\overline{OB}$



Statements	Reasons
1. Diameter $\overline{CD} \perp$ chord $\overline{AB}$ at E	1. given
2. $\overline{OA} \cong \overline{OB}$	2. Radii of same $\odot$ are $\cong$
3. $\overline{OE} \cong \overline{OE}$	3. Reflexive
4. $\triangle OBE \cong \triangle OAE$	4. HL
5. $\overline{BE} \cong \overline{AE}$; $\angle BOE \cong \angle AOE$	5. CPCTC
6. $\overparen{BC} \cong \overparen{AC}$	6. $\cong$ central $\angle$'s intersect $\cong$ arc

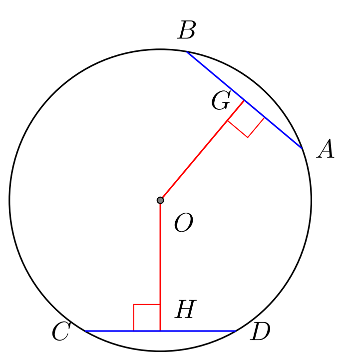
Theorem: A perpendicular bisector of a chord to a circle contains the center of the circle.

This theorem says that, if $\overline{CD}$ is a line that bisects chord $\overline{AB}$ and is perpendicular to $\overline{AB}$, then $\overline{CD}$ necessarily contains the center, O, of the circle.



Remember that the *distance* between a point and a line is the perpendicular distance. We have the following:

Theorem: In same or congruent circles, equal chords are equidistant from the center. Conversely, chords that are equidistant from the center are congruent to each other.

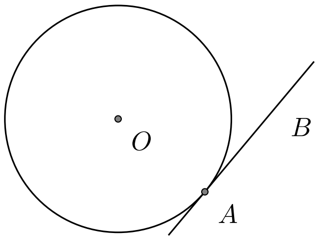


The theorem says that, if $\overline{AB} \cong \overline{CD}$, then $\overline{OG} \cong \overline{OH}$.

Conversely, if $\overline{OG} \cong \overline{OH}$, then $\overline{AB} \cong \overline{CD}$.

Tangent:

A **tangent** to a circle is a line that intersects the circle at only one point. The point where the tangent intersects the circle is called the **point of tangency**.

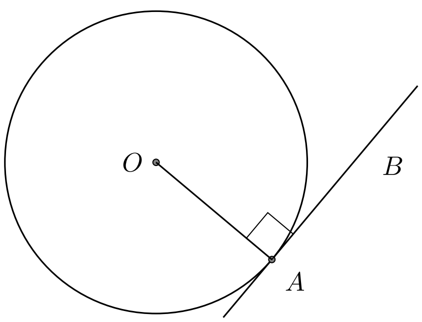


In the above, $\overline{AB}$ is the tangent to $\odot O$ at point A. Point A is the point of tangency.

Theorem:

The tangent to a circle is perpendicular to the radius of the circle at the point of tangency.

Conversely, if a line is perpendicular to a radius of a circle at a pont on the circle, then that line is a tangent to the circle.



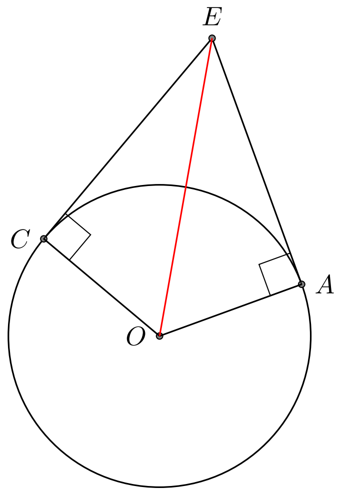
In picture above, if $\overline{AB}$ is a tangent to $\cdot O$, then $\overline{OA} \perp \overline{AB}$.

conversely, if $\overline{OA} \perp \overline{AB}$, then $\overline{AB}$ is a tangent to the circle at point A.

Theorem:

If two different tangents to the same circle at a common point, the distance between that point and the two points of tangency are the same.

Proof: Let $\overline{EC}$, $\overline{EA}$ be tangents to circle O at points of tangency C and A, and intersect at point E. We must prove that $\overline{EC} \cong \overline{EA}$



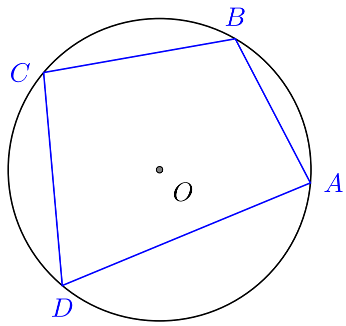
Statements	Reasons
1. $\overline{EC}$ and $\overline{EA}$ are tangents to $\odot O$ $\overline{EC}$ and $\overline{EA}$ intersects at E	1. given
2. $\overline{OA} \cong \overline{OC}$	2. Radii of same $\odot$ are $\cong$
3. $\overline{OE} \cong \overline{OE}$	3. Reflexive
4. $\overline{EA} \perp \overline{OA}$, $\overline{EC} \perp \overline{OC}$	4. Tangents are $\perp$ to radius at pt. of tangency
5. $\angle OCE$, $\angle OAE$ are right angles	5. Definition of perpendicular lines
6. $\triangle OCE \cong \triangle OAE$	6. HL
7. $\overline{EC} \cong \overline{EA}$	7. CPCTC

In proving the above theorem, notice that CPCTC allows us to say that $\angle CEO \cong \angle AEO$. In other words, $\overline{EO}$ is an angle bisector of $\angle CEA$. We have also proved the following:

Theorem: The line segment from the center of a circle to the common intersection of two tangents also bisects the angle that is formed by the two tangents.

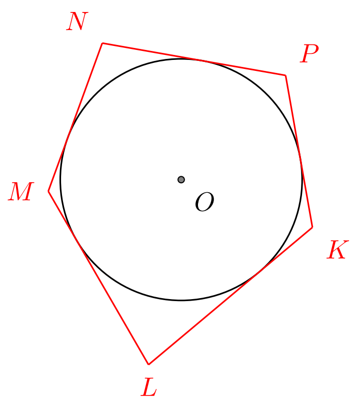
We say that a polygon is **inscribed** inside a circle if all of the vertices of the polygon are on the circumference of the circle. The circle is said to **circumscribe** the polygon.

In picture below, quadrilateral $ABCD$ is an inscribed polygon of circle O, and circle O circumscribes quadrilateral $ABCD$.



We say that a circle is **inscribed** by a polygon if each side of the polygon is a tangent of the circle, and the polygon **circumscribes** the circle.

In picture below, pentagon $KLMNP$ **circumscribes** circle O, and circle O is **inscribed** by pentagon $KLMNP$. Each side of the pentagon is a tangent of the circle.

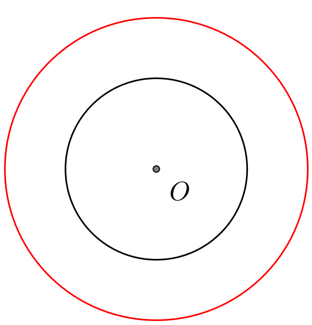


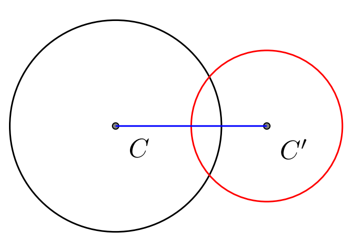
We say that two circles are **concentric** if they share the same center.

If two circles are ont concentric, we call the line segment connects their two centers the **line of centers** of the two circles.

In below left figure, the two circles are concentric. The inner (black) and outer (red) circle both have the same center O.

In below right figure, circles C and C' are not concentric. The segment $\overline{CC'}$ is the line of centers of the two circles.





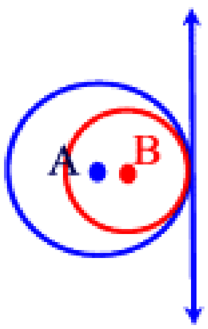
Concentric circles cannot have a common tangent, but two circles that are not concentric can share one or more common tangents.

We say that two circles are **internally tangent** to each other if they intersect at one point (the same line will be tangent to the two circles at their point of intersection) and one is inside another.

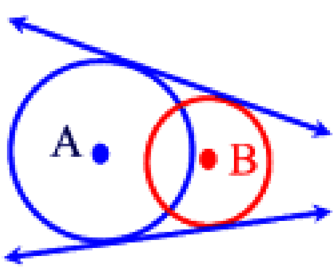
circles are **externally tangent** to each other if they intersect at one point (the same line will be tangent to the two circles at their point of intersection) but each is on the outside of another.

A tangent that is common to two circles is a **common internal tangent** if it intersects the *line of centers* of the two circles.

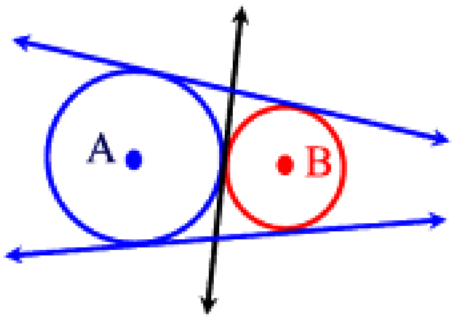
A tangent that is common to two circles is a **common external tangent** if it does *not* intersects the *line of centers* of the two circles.



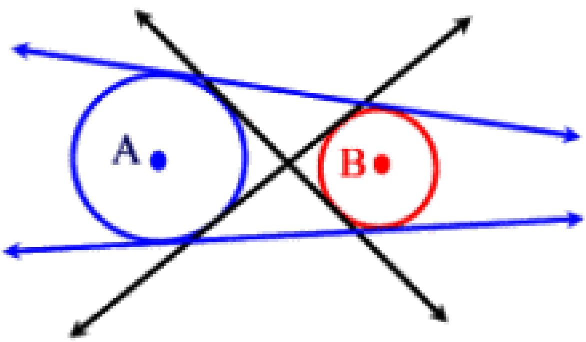
The two circles A and B above are **internally tangent** to each other. They share *one* common tangent. The common tangent they share is a **common external tangent**.



The two circles A and B above share *two* **common external tangents**.



The two circles A and B above are **externally tangent** to each other. The share one **common internal tangent** (black line), and two **common external tangents** (blue lines).



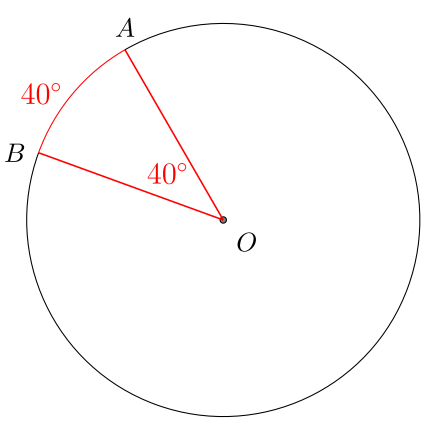
The two circles A and B above share two **common internal tangents** (black lines), and two **common external tangents** (blue lines).

Inscribed Angles

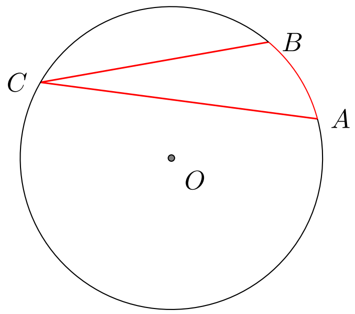
Remember that the (degree) measurement of an arc is by definition the (degree) measurement of the central angle that intercepts the arc.

In the picture below, the measure of central angle $\angle AOB$ is $40°$, so we say that the (degree) measure of arc $\widehat{AB}$ is also $40°$. In other words,

$m\angle AOB = m\widehat{AB} = 40°$



An **inscribed angle** to a circle is an angle whose vertex is on the circle and whose sides are chords of the circle.



In picture above, $\angle BCA$ is an inscribed angle. $\angle BCA$ is **inscribed by** arc $\widehat{BCA}$ and $\angle BCA$ **intercepts** arc $\widehat{BA}$

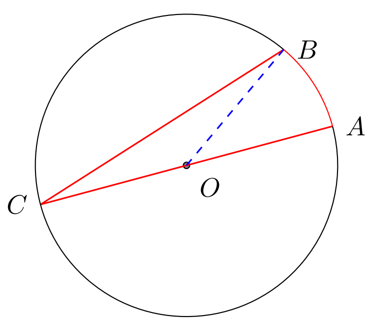
Inscribed Angle Theorem:

The degree measure of an inscribed angle is half the degree measure of its inscribed arc.

Proof: To prove this theorem, let $\angle ACB$ be an inscribed angle of $\odot O$. We must prove that

$$m\angle AOB = m\widehat{AB} = \frac{1}{2}\angle ACB$$

There are three possible cases we need to consider, the case where the center of the circle, O, is outside of the angle $\angle, ACB$; another case is where O is inside of $\angle ACB$. The third case is if center O is *on* one of the side of $\angle ACB$. We prove this third case:



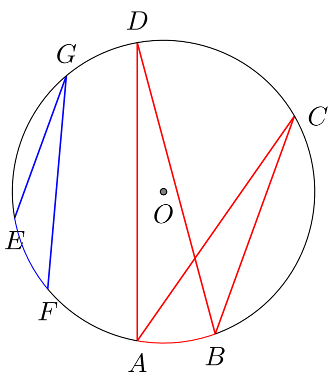
Statements	Reasons
1. $\angle ACB$ is an inscribed $\angle$; Center O is on $\overline{AC}$	1. given
2. $\overline{OC} \cong \overline{OB}$	2. Radii of same $\odot$ are $\cong$
3. $\angle OCB \cong \angle OBC$	3. Isoceles $\triangle$
4. $m\angle OCB + m\angle OBC = m\angle AOB$	4. Sum of int. $\angle$ of $\triangle$ = oppo. exterior $\angle$
5. $m\angle OCB + m\angle OCB = m\angle AOB$	5. Substitution
6. $m\angle OCB = \frac{1}{2}(m\angle AOB)$	6. Algebra
7. $m\angle OCB = \frac{1}{2}\left(m\widehat{AB}\right)$	7. Def. of measure of an arc

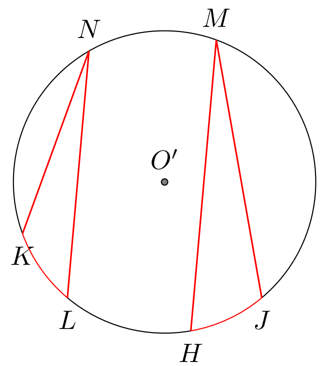
Theorem: In the same or congruent circles, inscribed Angles that intercept the same or congruent arcs are congruent.

Conversely, arcs that are intercepted by congruent angles are congruent.

So $\angle D \cong \angle C$ since they both intercept the same arc, $\widehat{AB}$. Furthermore, if $\widehat{AB} \cong \widehat{EF}$, then $\angle D \cong \angle G$.

If $\angle N \cong \angle M$, then $\widehat{KL} \cong \widehat{HJ}$





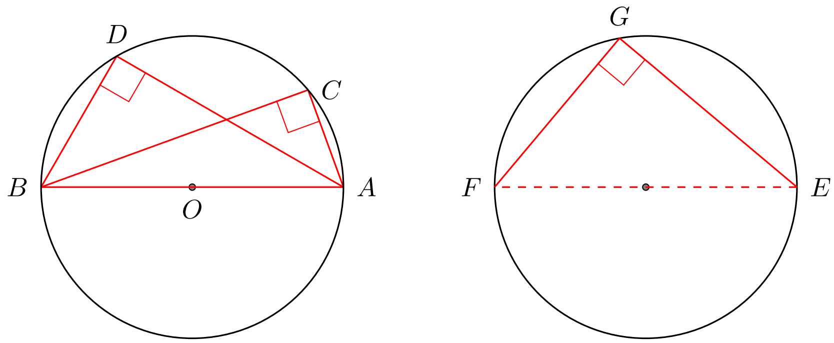
Thales Theorem: The diameter of a circle subtends a right angle to any point on the circle

In other words, an inscribed angle whose intercepted arc is a semi-circle is always a right angle.

Conversely, if an inscribed angle is right, then the chord that connects the two end-points of the intercepted arc is a diameter.

In $\odot O$, if $\overline{AB}$ is a diameter, then $\angle BDA$ and $\angle BCA$ are both right angles.

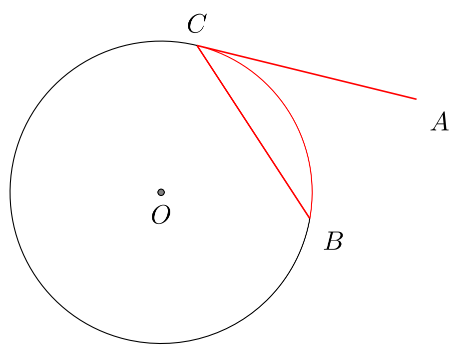
Conversely, in the circle to the right, if $\angle FGE$ is right, then $\overline{FE}$ is a diameter of the circle. In other words, $\overline{FE}$ contains the center of the circle.



Theorem An angle formed by a tangent and a chord of a circle is equal to *half* of the (degree) measurement of the intercepted arc.

$\overline{AC}$ is a tangent to $\odot O$, and $\overline{CB}$ is a chord. Therefore,

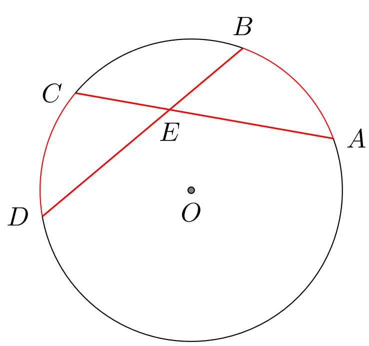
$$m\angle ACB = \frac{1}{2} m\widehat{CB}$$



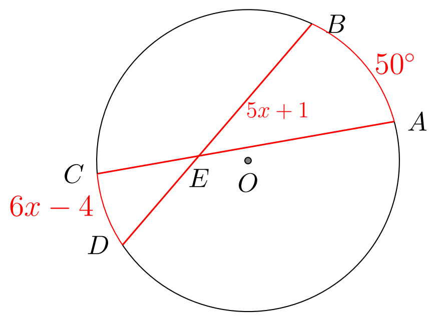
Theorem: The angle formed by the intersection of two chords is equal to *half* of the sum of the two intercepted arcs.

$\overline{AC}$, $\overline{BD}$ are chords of $\odot O$, therefore,

$$m\angle BEA = \frac{1}{2}\left(m\widehat{AB} + m\widehat{CD}\right)$$



Example: In $\odot O$ below, $m\widehat{AB} = 50°$, $m\angle BEA = 5x + 1$, $m\widehat{CD} = 6x - 4$, find the value of x.



Ans: According to the theorem, we have:

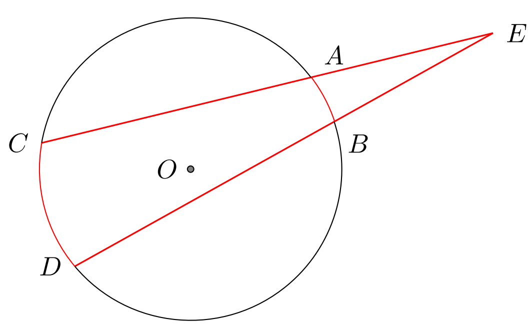
$$m\angle BEA = \frac{1}{2}\left(m\widehat{AB} + m\widehat{CD}\right) \Rightarrow 5x + 1 = \frac{1}{2}\left(50 + (6x - 4)\right) \Rightarrow$$

$$5x + 1 = \frac{1}{2}(6x + 46) \Rightarrow 5x + 1 = 3x + 23 \Rightarrow 2x = 22 \Rightarrow x = 11$$

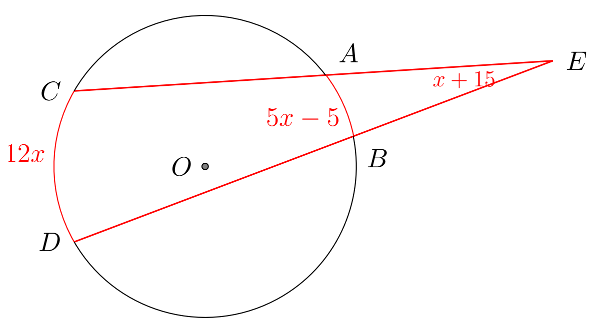
Theorem: An angle formed by two secants intersecting outside a circle is equal to *half* of the *difference* of the two intercepted arcs.

In picture below, $\overline{CA}$ and $\overline{DB}$ are secants that intersect at point E outside of $\odot O$, we have:

$$m\angle E = \frac{1}{2}\left(m\widehat{CD} - m\widehat{AB}\right)$$



Example: In picture below, $m\angle AEB = x + 15$, $m\widehat{CD} = 12x$, $m\widehat{AB} = 5x - 5$, find the value of x.



Ans: According to theorem,

$m\angle AEB = \dfrac{1}{2}\left(m\widehat{CD} - m\widehat{AB}\right) \Rightarrow x + 15 = \dfrac{1}{2}\left(12x - (5x - 5)\right) \Rightarrow$

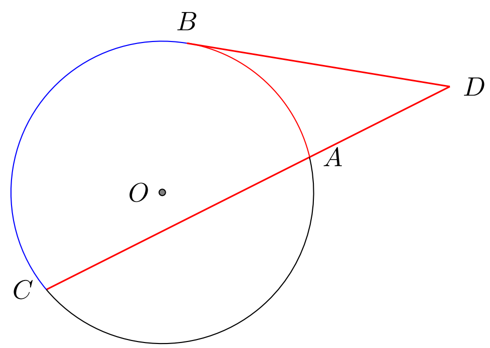
$x + 15 = \dfrac{1}{2}(12x - 5x + 5) \Rightarrow x + 15 = \dfrac{1}{2}(7x + 5) \Rightarrow 2(x + 15) = 7x + 5 \Rightarrow$

$2x + 30 = 7x + 5 \Rightarrow -5x = -25 \Rightarrow x = 5 \Rightarrow$

Theorem: The angle formed with a tangent and a secant intersecting at a point outside a circle is equal to *half* of the *difference* of the intercepted arcs.

In picture below, $\overline{BD}$ is a tangent and $\overline{CA}$ a secant to $\odot O$, intersecting at D, the theorem says that:

$m\angle D = \dfrac{1}{2}\left(m\widehat{BC} - m\widehat{BA}\right)$



Theorem: The angle formed with two tangents to a circle intersecting at a point outside the circle is equal to *half* of the *difference* of the intercepted arcs.

In picture below, $\overline{AC}$, $\overline{BC}$ are tangents to $\odot O$, the theorem says that:

$m\angle C = \dfrac{1}{2}\left(m\widehat{ADB} - m\widehat{AB}\right)$