What is Calculus?

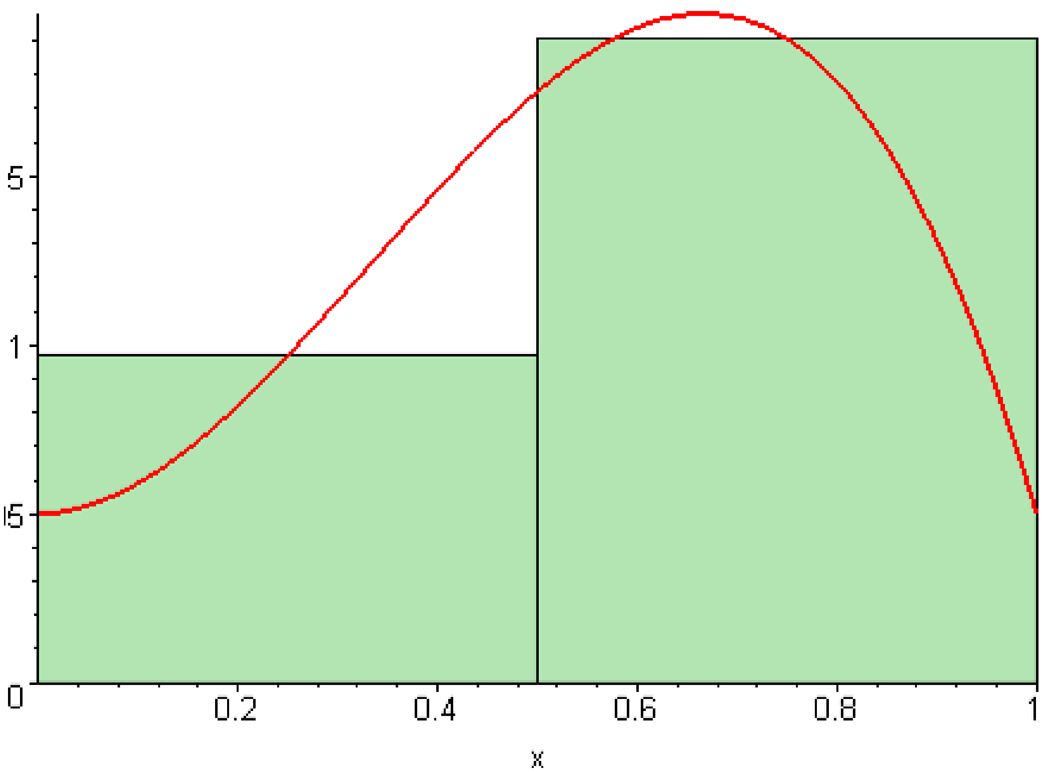
Calculus is motivated by two main problems. The first is the **area problem**. It is a well known fact that the area of a rectangle with length l and width w is given by $A = wl$.

From this fact (or definition) one can readily verifies that the area of a triangle with base b and height h is $A = \frac{1}{2}bh$

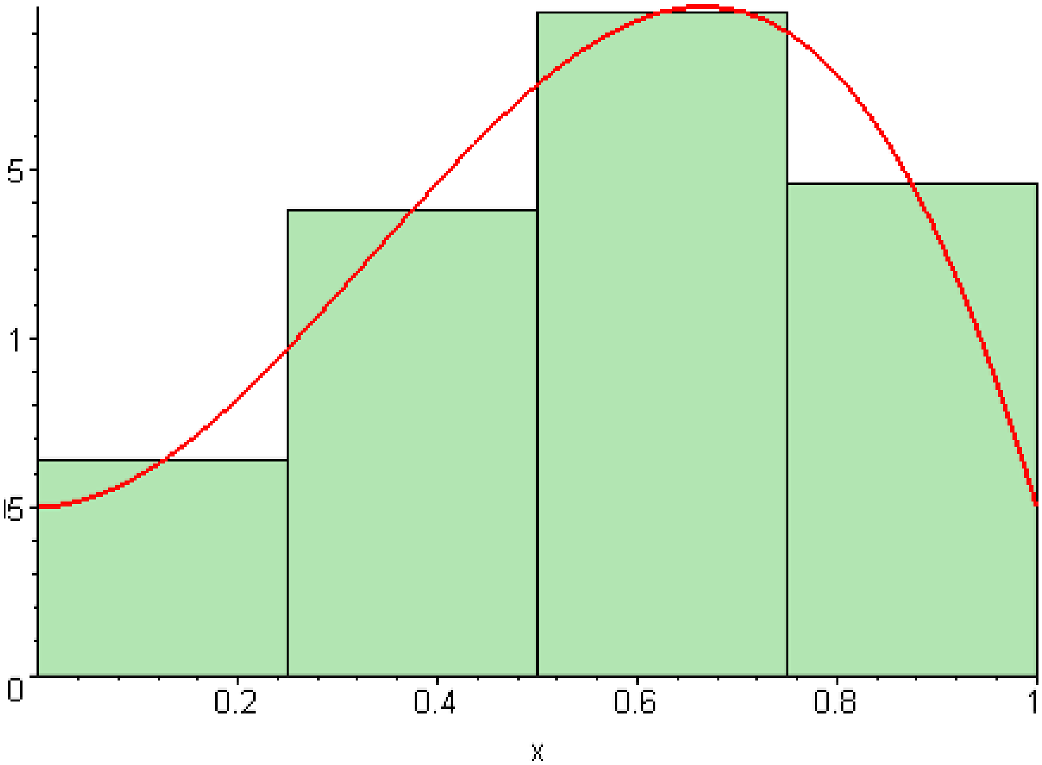
The formula for area of triangles can be used to find the area of any polygon. We just need to divide the polygon into triangles and find the sum of the area of the triangles.

The problem comes when we want to find the area of a figure with curved edges. For example, how do we find the area of a circle?

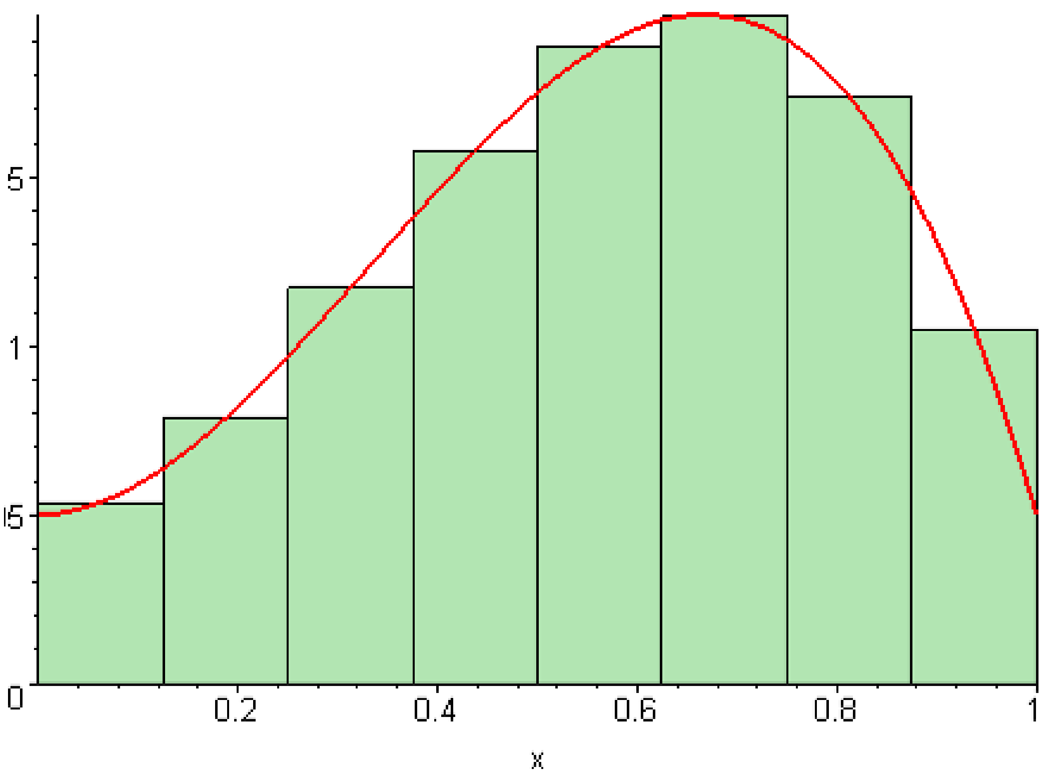
One intuitive idea is to divide the curved figure into rectangles, and estimate the area of the curved figure by the sum of the area of each rectangle.

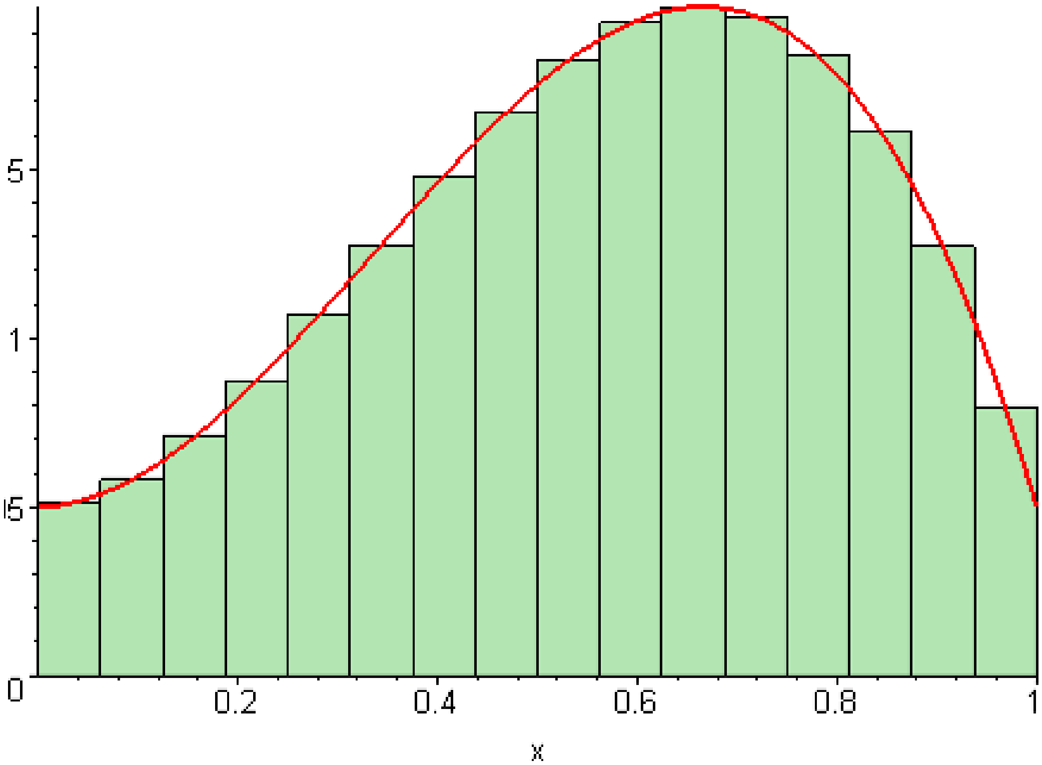


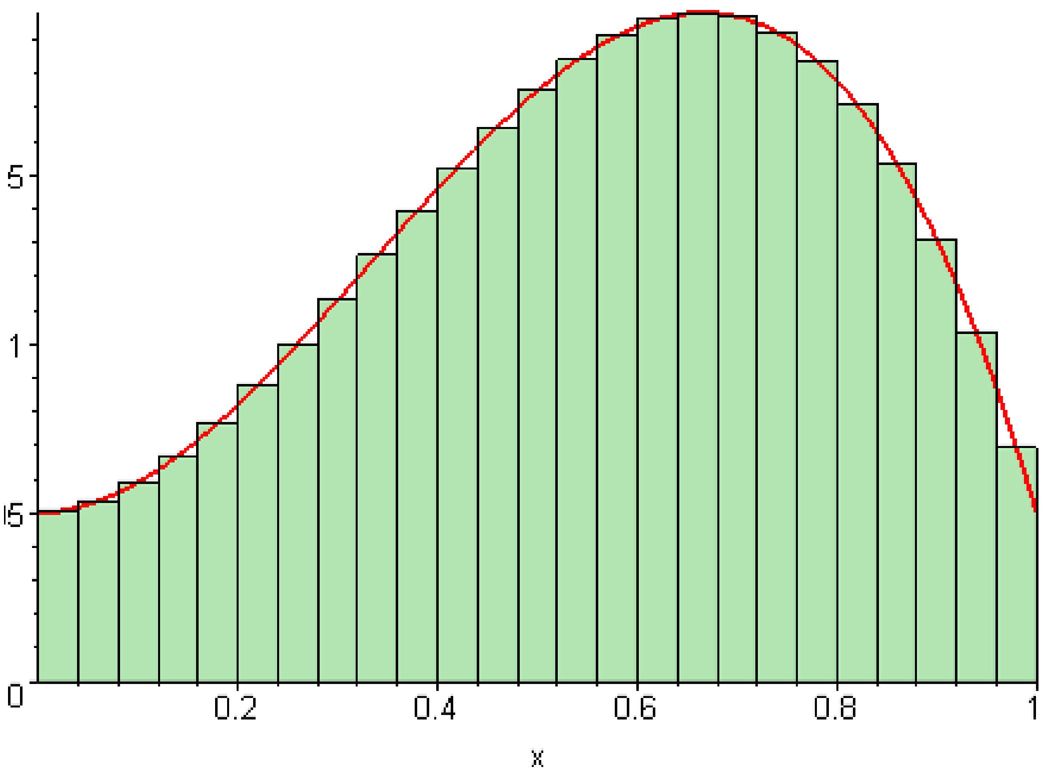
This is just an estimate. What can we do to get a more accurate result? Intuition tells us that if we make the rectangles smaller, the approximation should be better.



Can we get a still better result? Yes, by making the rectangles still smaller. As we make the rectangles smaller and smaller, the approximation (of the curved area by the rectangles) become better and better.

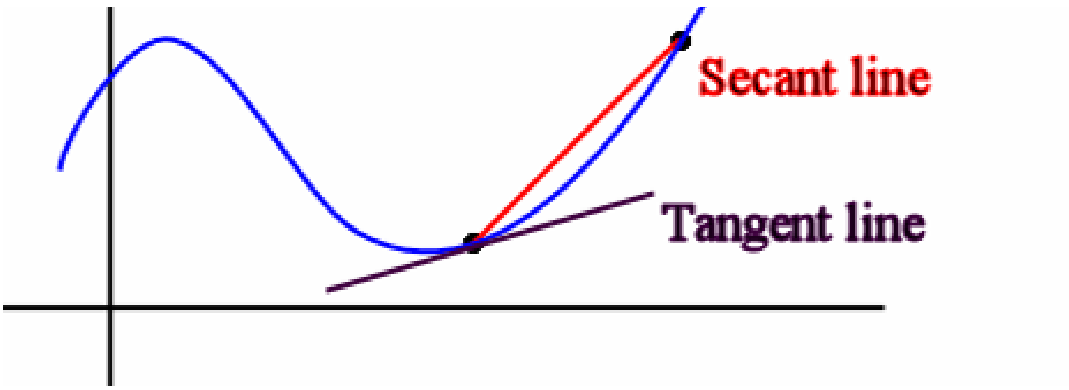




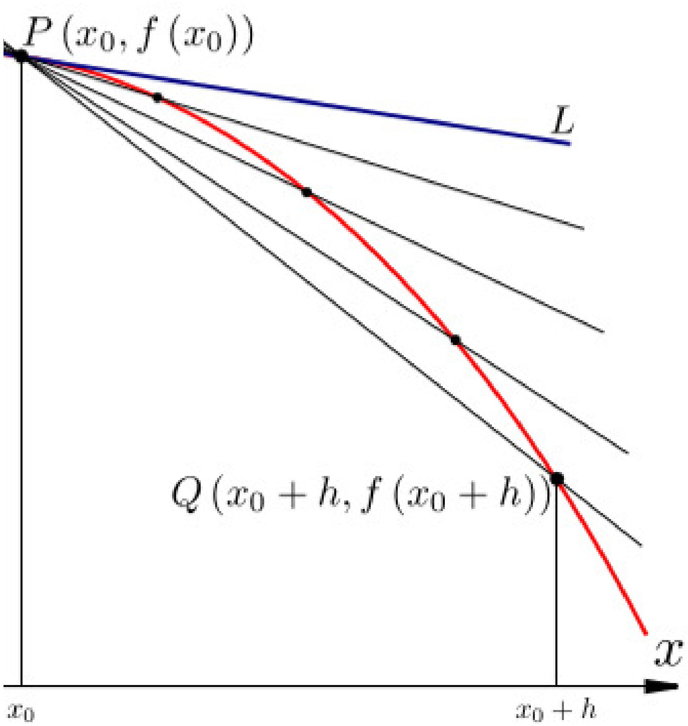


Now, what if we want the **exact** value? To do this, imagine that we make the triangles (in particular, the width of the rectangles) so small that it **practically** has zero length, then if we are still able to measure the area of the rectangles, this should give us an **exact** value of the curved figure. That is, in the *limit* case (this word will become meaningful in a while), when the width of the rectangle becomes **infinitesimally** small, we get the *exact* value of the area of the curved figure. One thing to keep in mind, too, is that when the *width* of each rectangle becomes extremely small, the number of rectangles that are used to approximate the figure must necessarily increase. In fact, if the width of the rectangles is to become infinitesimally small, we should expect that we will need to use *infinitely* many rectangles to fill the curved figure.

Another problem of calculus is the **Tangent** problem. We have a curve defined by a function $f(x)$, and we want to find the slope of the line *tangent* to f at a given point $(x_0, f(x_0))$ where x_0 is a constant. What we meant by a line tangent to a curve is the line that *best approximates the curve at the given point.* Think of the tangent to a circle. The tangent to a function is a generalized concept of the tangent to a circle.



From algebra we know that in order to find the slope of a line, we need two points. We have only one point, namely $(x_0, f(x_0))$. We can try to approximate the slope of the line by choosing another point on the curve that is near the point in question, say $(x_0 + h, f(x_0 + h))$, where h is a **small** number.



The slope of the line (called the secant line) that contains these two points $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$ is given by:

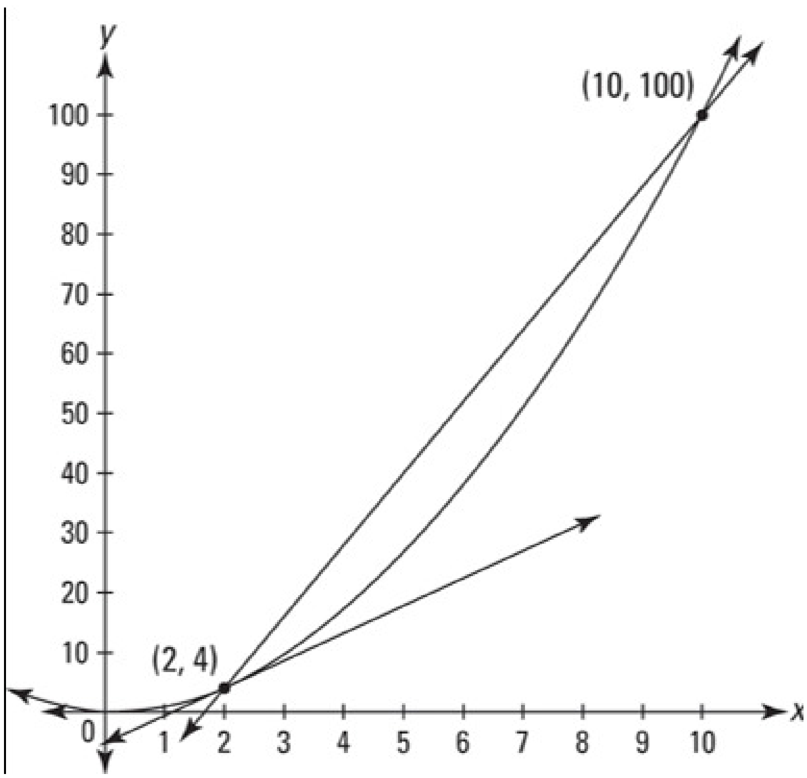
$$\text{slope of secant line} = m_{sec} = \frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0} = \frac{f(x_0+h) - f(x_0)}{h}$$

As mentioned, the above formula gives an *approximation* of the slope of the tangent line. Intuitively, if we make the two points **P** and **Q** closer to each other, we get a better approximation. Making **P** and **Q** closer is the same as making h smaller, or **close to zero**.

At the **limit** case, when h is **arbitrarily close to** 0, or when h is **infinitesimally small**, we should expect the approximation to be exact.

What does the slope of the tangent to a function represent?

Suppose you are going on a one-direction, 10 hours trip. Let the x-axis represent the time (in hours) since you have started your trip, and the y-axis represent the distance (in miles) you are from your origin.



What does the slope of the line between $(2, 4)$ and $(10, 100)$ represent?

The slope of the secant line, $m = \frac{100 - 4}{10 - 2} = 12$ represents the **average velocity** of your car between time $t = 2$ and $t = 10$. That is, on average, you are driving at 12 miles per hour between the 2nd hour to the 10th hour of your travel. In general, if we want to find the average velocity of the car between any two times t_1 and t_2, we just need to find the slope $m = \frac{d_2 - d_1}{t_2 - t_1}$. But what if we want to find the velocity of your car at **exactly** $t = 2$? This is called the **instantaneous velociy** of the car at $t = 2$ and is what the slope of the tangent line represents.

In general, the slope between any two points of a function represents the **average**

rate of change of the function between the two given points, while the slope of the tangent line of a function at a given point represents the **instantaneous rate of change** of the function at that given point.

Limits

In the two problems that we mentioned, the area problem and the tangent problem, a very important concept is common between the two. In the case with the area, we have to take the width of the rectangles very close to zero; in the case of the tangent problem, we also have to take h very close to 0. Both of these give rise to the concept of **limit**. What do we mean when we say the value of one variable is **close to** a particular number? What is the behavior of a function $f(x)$ when x **approaches** a particular number?

Informal definition of limit:

Let f be a function and a and L constants, We say

$$\lim_{x\to a} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L by choosing x close to a. (We use the notation $x \to a$ to mean x *approaches* a)

E.g. What is

$$\lim_{x\to 2} x + 1 = ?$$

That is, what does the value of the expression $x+1$ get **close** to when x is **close** to 2?

We may try a few values of x that are close to 2 and see if we observe any trend here:

x	$x+1$
2.1	3.1
2.05	3.05
2.01	3.01
2.001	3.001
1.9	2.9
1.95	2.95
1.99	2.99
1.999	2.999

From the table, we can see that the value of the expression $x+1$ **approaches** 3 as x **approaches** 2. It should not surprise you to find that

$$\lim_{x\to 2} x + 1 = 3$$

Note that we could have found the value of the limit by directly substituting the value of 2 into the expression $x+1$ to find the value of 3. Occasionally, directly substituting the limiting value into the expression may allow us to find the value of the limit (and we will see under what cercumstances we may do this), but there are cases such method will fail and we need more sophisticated methods. Consider:

$$\lim_{x \to 3} \frac{x^2-9}{x-3}$$

If we substitute the value of 3 into the expression $\frac{x^2-9}{x-3}$ we get the **indeterminate form** $\frac{0}{0}$. Let us try some values again:

x	$\frac{x^2-9}{x-3}$
3.1	6.1
3.05	6.05
3.01	6.01
3.001	6.001
2.9	5.9
2.95	5.95
2.99	5.99
2.999	5.999

It seems that as x **approaches** 3, $\frac{x^2-9}{x-3}$ **approaches** 6, so we would guess that

$$\lim_{x \to 3} \frac{x^2-9}{x-3} = 6$$

This is a correct guess. Notice that in order to evaluate the limit of the function at a point a, we do **not** need to require the function to be **defined** at a. In the above example, the expression $\frac{x^2-9}{x-3}$ is not defined at 3, yet its limit is defined at 3.

Let's look at another example:

$$\lim_{x \to 1} \frac{|x-1|}{x-1} =?$$

Notice that if $x=1$, the expression gives the $\frac{0}{0}$ indeterminate form, so we try some values as before:

x	$\dfrac{\lvert x-1\rvert}{x-1}$
1.1	1
1.05	1
1.01	1
1.001	1
0.9	-1
0.95	-1
0.99	-1
0.999	-1

So what is the limit? Is it 1 or -1?

Since we cannot find a "trend", i.e., since the value of the expression $\frac{|x-1|}{x-1}$ does not tend to any particular value, we cannot come to any conclusion as to what the value of the expression *should* be, we say that the limit $\lim_{x\to 1} \dfrac{|x-1|}{x-1}$ **does not exist**

The previous example illustrates that limits do not have to exist. In the above example, the value of the expression tends to one value (-1) when x approaches 1 but x is always less than 1 (from the left hand side), and the expression tends to another value (1) when x approaches 1 but x is always greater than 1 (from the right hand side). Since the two values do not agree, we cannot make a proper guess, so the limit does not exist.

Trying to find the limit by finding values of $f(x)$ on the left and right hand side of a gives rise to the concept of **left** and **right**-handed limits:

Definition:

We say that

$$\lim_{x\to a^-} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L by choosing x close to a **and is always less than** a (x is on the *left* hand side of a). The limit $\lim_{x\to a^-} f(x) = L$ is called the **left-handed limit of** $f(x)$ **as** x **approaches a**.

We similarily define the **right-handed limit of** $f(x)$:

We say that

$$\lim_{x\to a^+} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L by choosing x close to a **and is always greater than** a (x is on the *right* hand side of a).

Notice that while a function may not have a limit, it may have a left and/or a right-handed limit. In the previous example, $\lim_{x\to 1} \dfrac{|x-1|}{x-1}$ does not exist, but the left $\lim_{x \to 1^-} \frac{|x-1|}{x-1} = -1$ and right-handed limit $\lim_{x \to 1^+} \frac{|x-1|}{x-1} = 1$ both exist.

What is the relationship between the limit (sometimes called a *two-sided limit*) of a function and its *left* and *right-handed* limits?

Theorem:

a function has a limit if and only if it has a right and left-handed limit and they are equal to each other.

E.g.

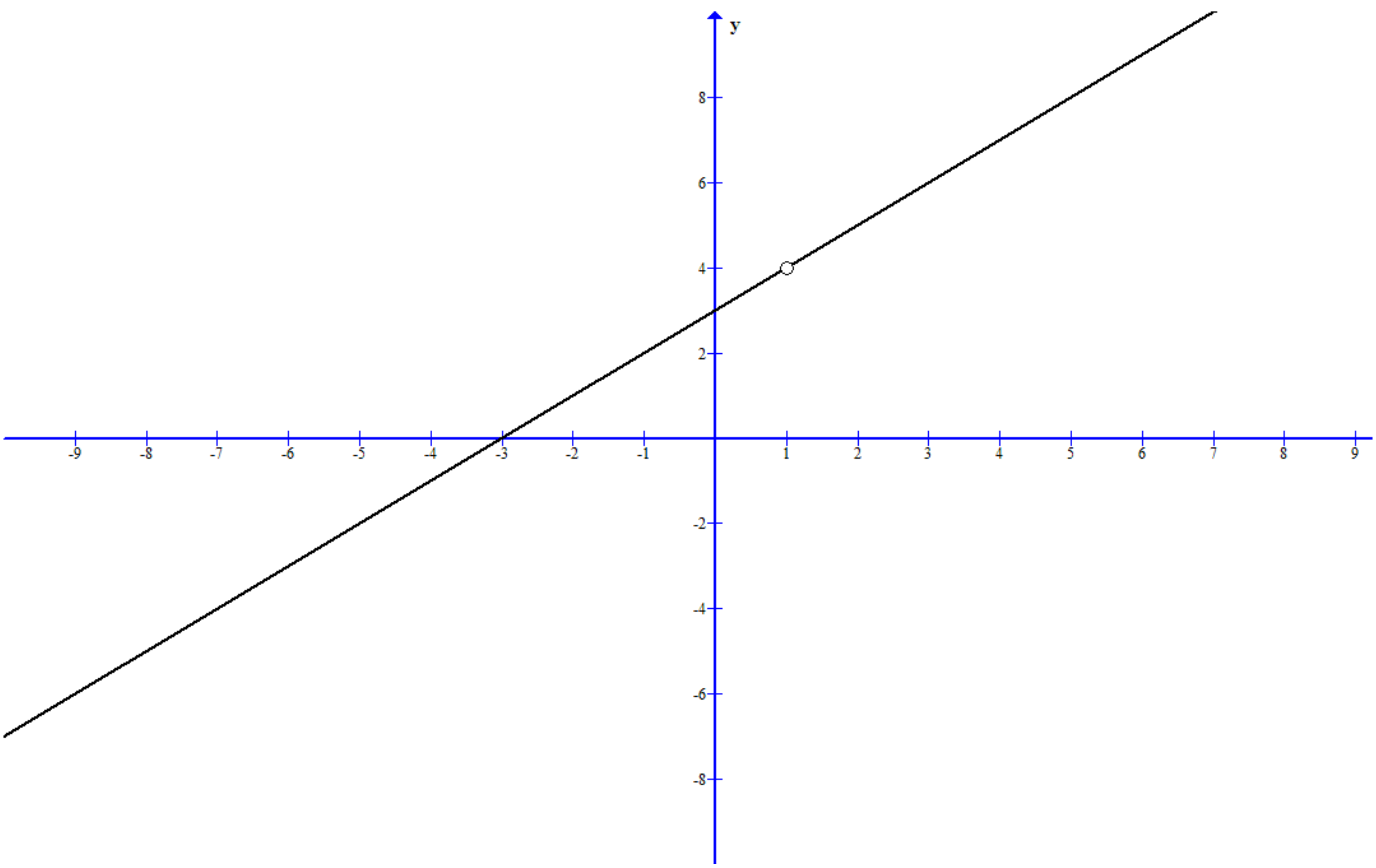
$$\lim_{x\to 1^-} \frac{x^2+2x-3}{x-1} = 4$$

and

$$\lim_{x\to 1^+} \frac{x^2+2x-3}{x-1} = 4$$

so

$$\lim_{x\to 1} \frac{x^2+2x-3}{x-1} = 4$$



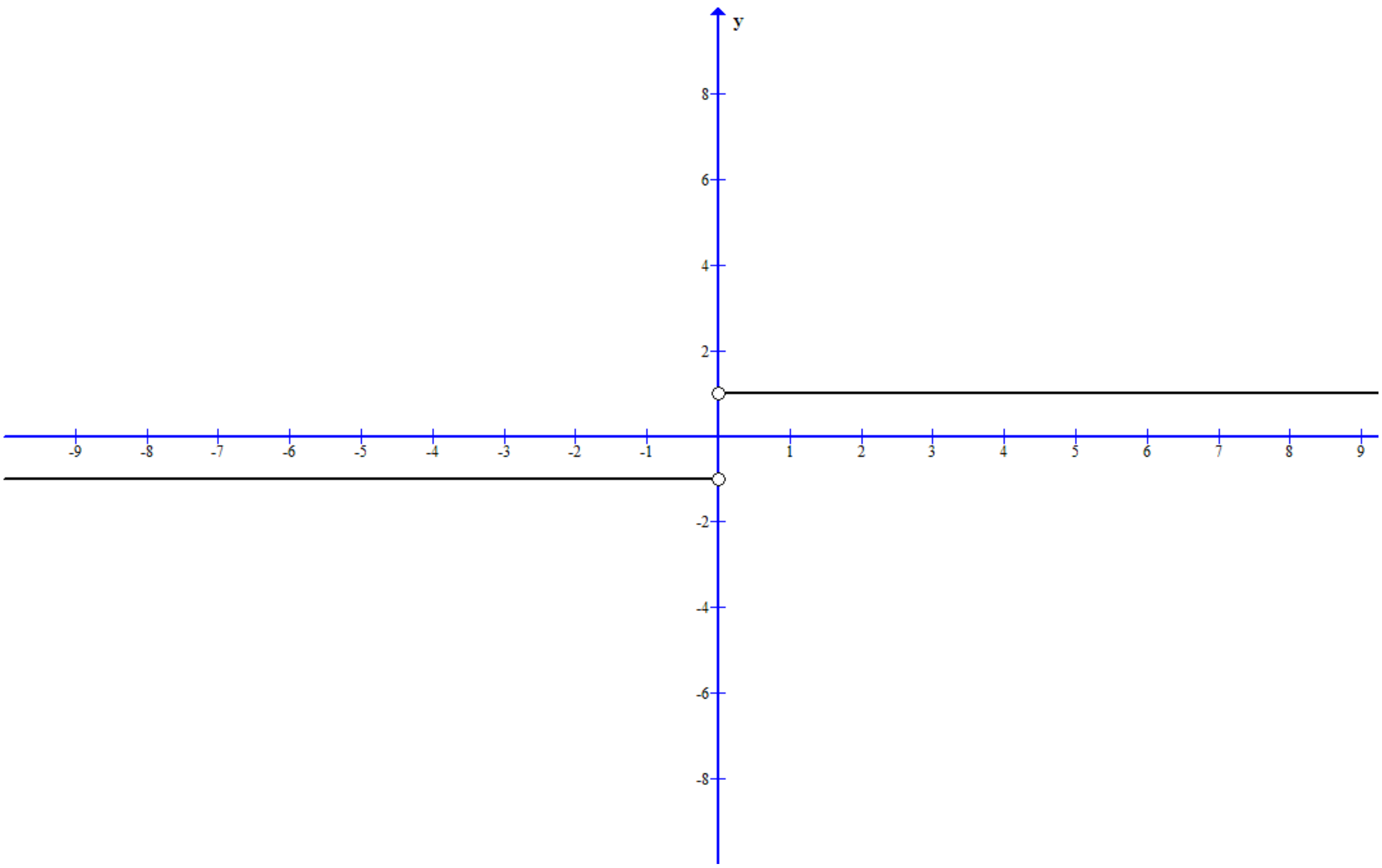
E.g.

$$\lim_{x\to 0^-} \frac{|x|}{x} = -1$$

but

$$\lim_{x\to 0^+} \frac{|x|}{x} = 1$$

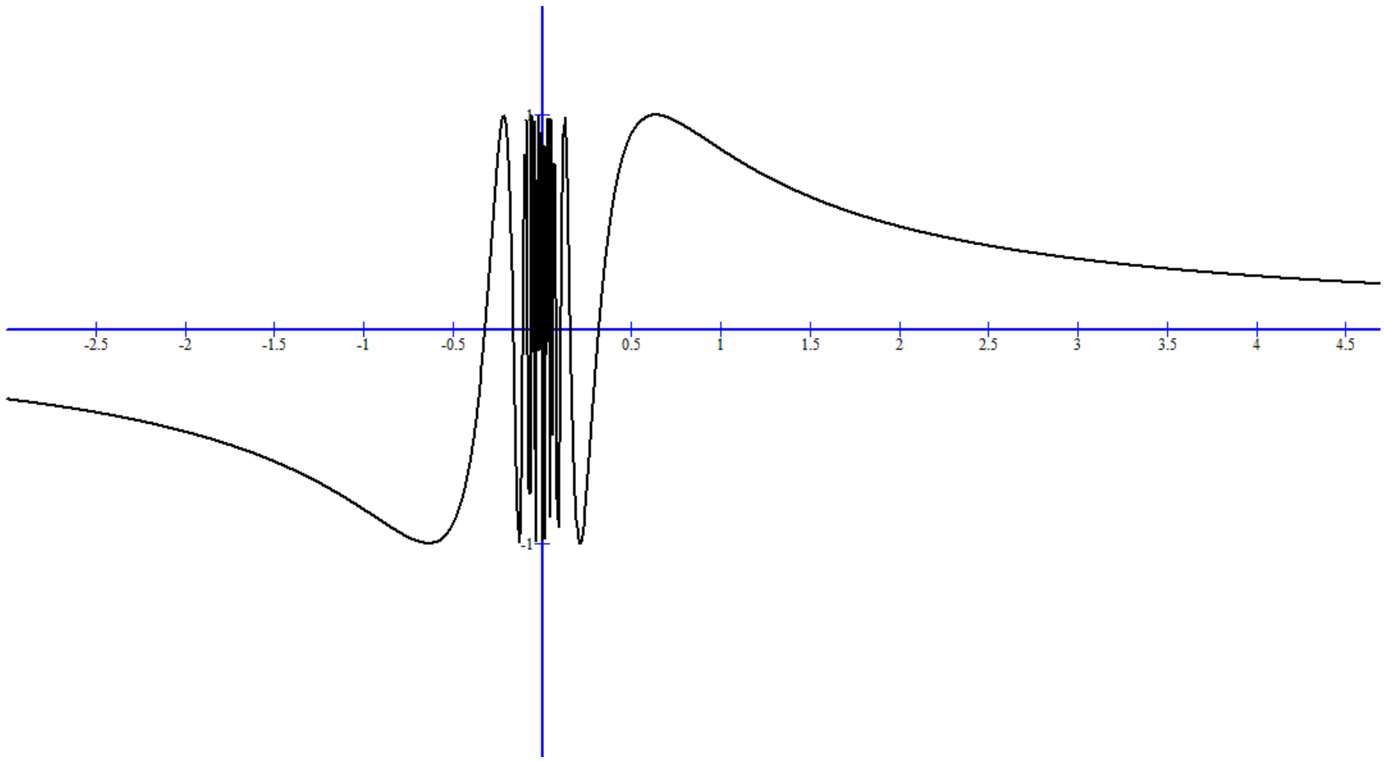
so $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.



It is important to note that even the left and/or right-handed limit of a function need not exist. Consider the limit:

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

This limit does not exist and neither does the left or right-handed limit. The value of the function osculates between -1 and 1 as x approaches 0.



Infinite limit:

Consider the limit:

$$\lim_{x\to 0}\frac{1}{x^2}$$

Let's try some values of x close to 0.

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

We see that the closer x is to 0 (from both sides), the larger the value of the expression, but it never tends to any particular value.

In this example, we use the *notation*

$$\lim_{x\to 0}\frac{1}{x^2}=\infty$$

to mean that, as x approaches 0, the value of $\frac{1}{x^2}$ increases without bound.

In general, we say

$$\lim_{x\to a}f(x)=\infty$$

if the value of f increases without bound as x approaches a.

It is important to point out that ∞ is **not** a number, so when we say that $\lim_{x\to a}f(x)=\infty$ we are **not** suggesting that the limit of the function exists at point a. We are simply using the notation to mean that the value of f increases without bound, we are **not** saying that the limit exists.

Similarily, we define

$$\lim_{x\to a}f(x)=-\infty$$

to mean that the value of f decreases without bound (f goes to *negative infinity*) as x approaches a.

Note that we may use the infinity symbol for the limit only if the value of the function increases without bound as x approaches a from **both sides** of a. Consider:

$$\lim_{x\to 0}\frac{1}{x}$$

In this case,

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

but

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

so the limit $\lim_{x \to 0} \frac{1}{x}$ simply does not exist, and we may not say the limit is ∞ or $-\infty$.

In the case when $\lim_{x \to a} f(x) = \infty$ or $\lim_{x \to a} f(x) = -\infty$, we say that the line $x = a$ is a **vertical asymptote** of f.

How do we evaluate limits?

Rules of limit: Let c be a constant and that

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

both exist, then:

$$\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

i.e. limit of the sum = sum of the limits

$$\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

i.e. limit of the difference = difference of the limits

$$\lim_{x \to a}[cf(x)] = c \lim_{x \to a} f(x)$$

i.e. limit of a constant times a function = constant times the limit of the function

$$\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

i.e. limit of the product = product of the limits

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \qquad \text{provided that} \qquad \lim_{x \to a} g(x) \neq 0$$

i.e. limit of the quotient = quotient of the limits

if n is a positive integer, then

$$\lim_{x \to a}[f(x)]^n = [\lim_{x \to a} f(x)]^n$$

i.e. limit of power = power of limit

$$\lim_{x \to a} c = c$$

i.e. limit of a constant is itself

if $f(x)$ is a polynomial, then

$$\lim_{x \to a}[f(x)] = f(a)$$

if n is any positive integer,

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$$

We also introduce two limits that would be useful later on. We cannot (easily) prove these two limits at this point, we'll just accept it for the moment:

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Some examples on using the limit rules:

Evaluate

$$\lim_{x \to -1} \frac{x^2 + 8x + 7}{x + 1}$$

To solve this problem, notice that if we just plug in the value of $x = -1$ into the expression, we get the **indeterminate form** $\frac{0}{0}$. We could try what we did before, which is to use various values of x close to -1, and try to see if there's a tendency. However, as we mentioned that method is tedious and also may not be accurate. We use the rules of limit instead:

Factoring the numerator we have:

$$\lim_{x \to -1} \frac{x^2 + 8x + 7}{x + 1} = \lim_{x \to -1} \frac{(x + 7)(x + 1)}{x + 1}$$

If we know that the two limits $\lim_{x \to -1} x + 7$ and $\lim_{x \to -1} \frac{x + 1}{x + 1}$ exist, then we may use the limit rules to say that

$$\lim_{x \to -1} \frac{(x + 7)(x + 1)}{x + 1} = \lim_{x \to -1} x + 7 \cdot \lim_{x \to -1} \frac{x + 1}{x + 1}$$

but we know that

$$\lim_{x \to -1} x + 7 = 6$$

and

$$\lim_{x \to -1} \frac{x + 1}{x + 1} = 1$$

so

$$\lim_{x \to -1} \frac{(x + 7)(x + 1)}{x + 1} = \lim_{x \to -1} x + 7 \cdot \lim_{x \to -1} \frac{x + 1}{x + 1} = 6 \cdot 1 = 6$$

Why is $\lim_{x \to -1} \frac{x + 1}{x + 1} = 1$? If we just plug in the value of -1 into the expression $\frac{x+1}{x+1}$, we get the indeterminate $\frac{0}{0}$ form. However, the definition of limit tells us that, in order to evaluate the value of the limit, we only need to know the behavior of the expression for x *close* to -1, not when $x = -1$. Hence, even though the expression is undefined at $x = -1$, but since $\frac{x+1}{x+1} = 1$ for all values of $x \neq -1$, we have

$$\lim_{x \to -1} \frac{x + 1}{x + 1} = \lim_{x \to -1} 1 = 1$$

We could actually have solved the problem a little more cleanly:

$$\lim_{x\to -1}\frac{x^2+8x+7}{x+1}=\lim_{x\to -1}\frac{(x+7)(x+1)}{x+1}=\lim_{x\to -1}x+7=-1+7=6$$

Let's try another example:

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=?$$

If we just plug in the value of $x=0$ into the expression we get the indeterminate form $\frac{0}{0}$ again. So we will need to do a little more work. The trick for this one is to multiply by the conjugate of the numerator:

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)}$$

$$=\lim_{x\to 0}\frac{1+x-1}{x(\sqrt{1+x}+1)}=\lim_{x\to 0}\frac{x}{x(\sqrt{1+x}+1)}=\lim_{x\to 0}\frac{1}{\sqrt{1+x}+1}$$

The reason we are able to cancel the x on the numerator and denominator is same as before. Since we are finding the limit as $x\to 0$, x does not need to be equal to 0, hence we can cancel the common factor of x in the fraction.

At this point, even if we plug in $x=0$, the expression is defined:

$$\lim_{x\to 0}\frac{1}{\sqrt{1+x}+1}=\frac{1}{\sqrt{1+0}+1}=\frac{1}{2}$$

These two examples show that when evaluating limits, upon encountering the indeterminate form $\frac{0}{0}$ we must try to change the expression so that we can have an expression that is defined. Here's another example:

$$\lim_{x\to 2}\frac{x-2}{x^2-4x+4}$$

Once again if we plug in $x=2$, we have $\frac{0}{0}$, so we try some algebra:

$$\lim_{x\to 2}\frac{x-2}{x^2-4x+4}=\lim_{x\to 2}\frac{x-2}{(x-2)(x-2)}=\lim_{x\to 2}\frac{1}{x-2}$$

The limit

$$\lim_{x\to 2}\frac{1}{x-2}$$

is undefined since $\lim_{x\to 2^-}\frac{1}{x-2}=-\infty$ and $\lim_{x\to 2^+}\frac{1}{x-2}=+\infty$

Let's try another one:

$$\lim_{x\to -3}\frac{x+3}{|x+3|}=?$$

To evaluate this limit, we need to understand how the absolute value function behaves. Remember that

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

For the above expression, $x + 3 < 0$ if $x < -3$, and $x + 3 \geq 0$ if $x \geq -3$, so

$$\lim_{x \to -3^-} \frac{x+3}{|x+3|} = \lim_{x \to -3^-} \frac{x+3}{-(x+3)} = \lim_{x \to -3^-} -1 = -1$$

while

$$\lim_{x \to -3^+} \frac{x+3}{|x+3|} = \lim_{x \to -3^+} \frac{x+3}{x+3} = \lim_{x \to -3^+} 1 = 1$$

Since the right-handed limit and left-handed limit do not agree,

$$\lim_{x \to -3} \frac{x+3}{|x+3|} \qquad \text{does not exist.}$$

Summary of limit evaluation:

If the value of the expression is defined when directly substitute the value, and the function is *not* conditionally defined, just plug in the value. E.g.

$$\lim_{x\to 1}\frac{x-3}{x+1}=\frac{1-3}{1+1}=\frac{-2}{2}=-1$$

If the function *is* conditionally defined, evaluate the left- and right-handed limits and see if they are equal.

E.g.

Let

$$f(x)=\begin{cases} x^2-1 & \text{if} \quad x\geq 2 \\ 4x & \text{if} \quad x<2 \end{cases}$$

Evaluate $\lim_{x\to 2} f(x)$

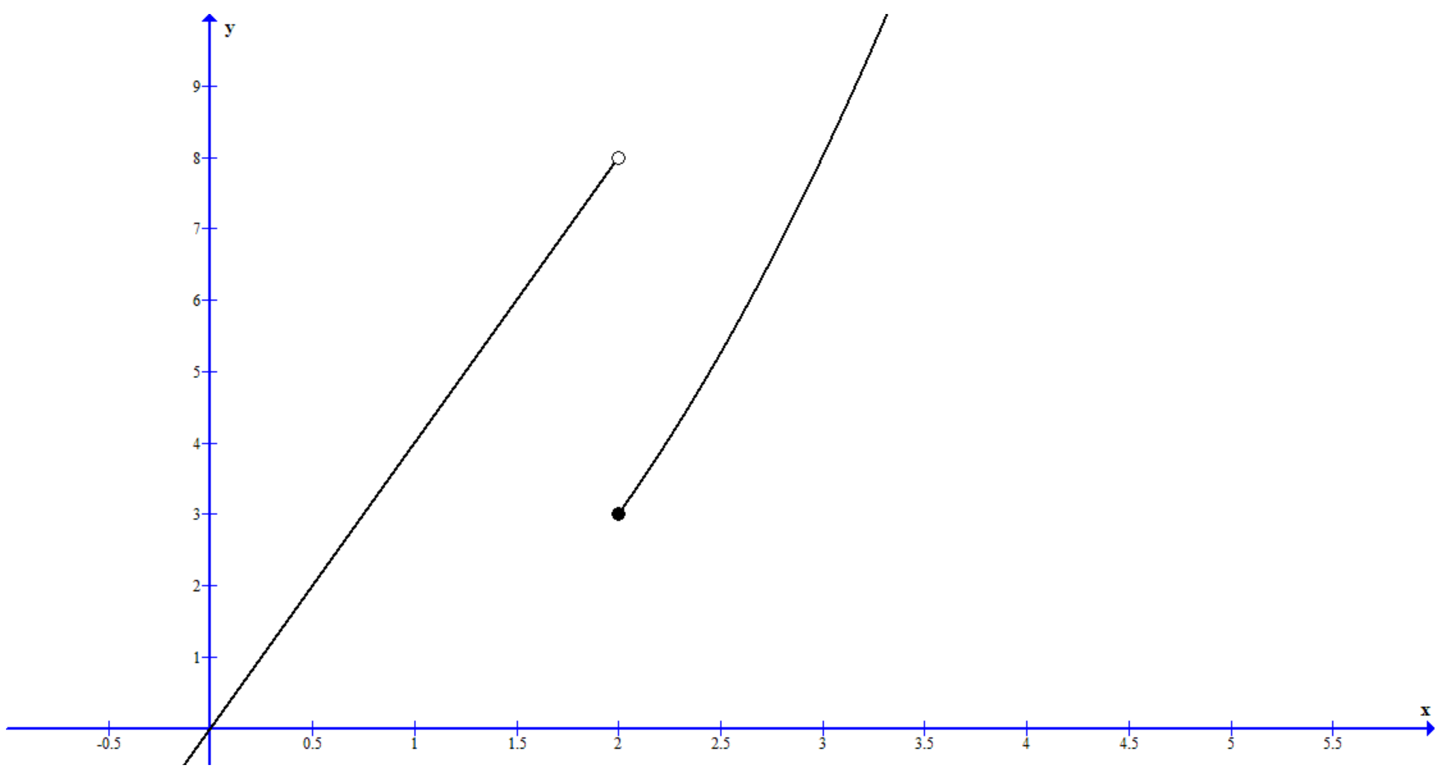
Since

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-} 4x=4(2)=8$$

but

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+} x^2-1=2^2-1=3$$

The left and right-handed limits are not equal, so $\lim_{x\to 2} f(x)$ does not exist



If the expression is in the form $\frac{f(x)}{g(x)}$, and upon substitute the limiting value a, if $f(a) \neq 0$ but $g(a) = 0$, then the limit does not exist. E.g.

$$\lim_{x \to 1} \frac{x^2 + 2 + 5}{x^2 + 3x - 4}$$

Since the denominator $= 0$ but the numerator $\neq 0$ when $x = 1$, the limit does not exist.

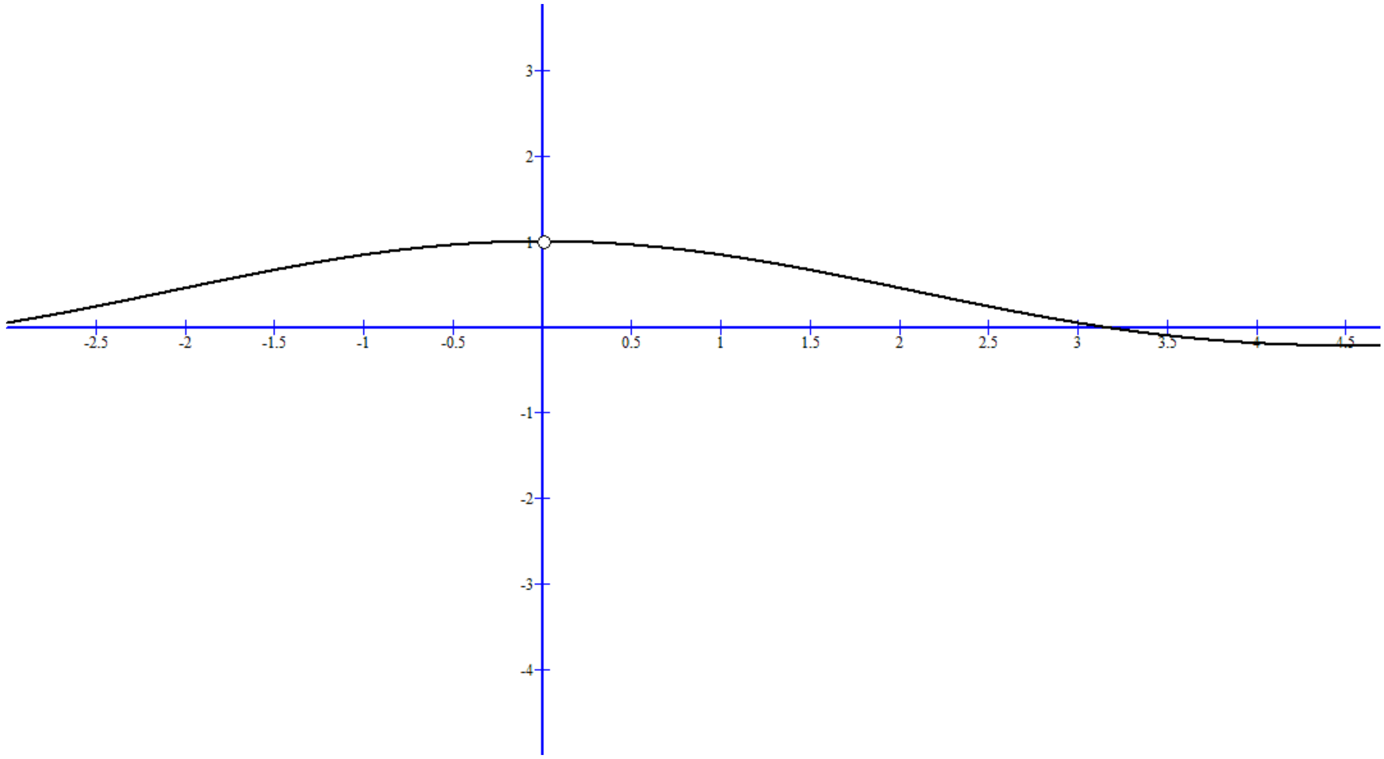
If the expression is the indeterminate form $\frac{0}{0}$ when we substitute a into the expression, more work needs to be done. E.g.

$$\lim_{x \to 2} \frac{x - 2}{x^2 + 3x - 10}$$

Note that the limit may or may not exist for the indeterminate case $\frac{0}{0}$. Even if the limit exists, it may take more than just simple algebra to find the limit or show that it does not exist. Consider for example:

$$\lim_{x \to 0} \frac{\sin x}{x}$$

This limit exists and is equal to 1, but there's no simple algebraic method that we may use to find the limit.



3
2
1
-1
-2
-3
-4
-2.5
-2
-1.5
-1
-0.5
0.5
1
1.5
2
2.5
3
3.5
4
4.5

Squeeze Theorem (aka Sandwich Lemma)

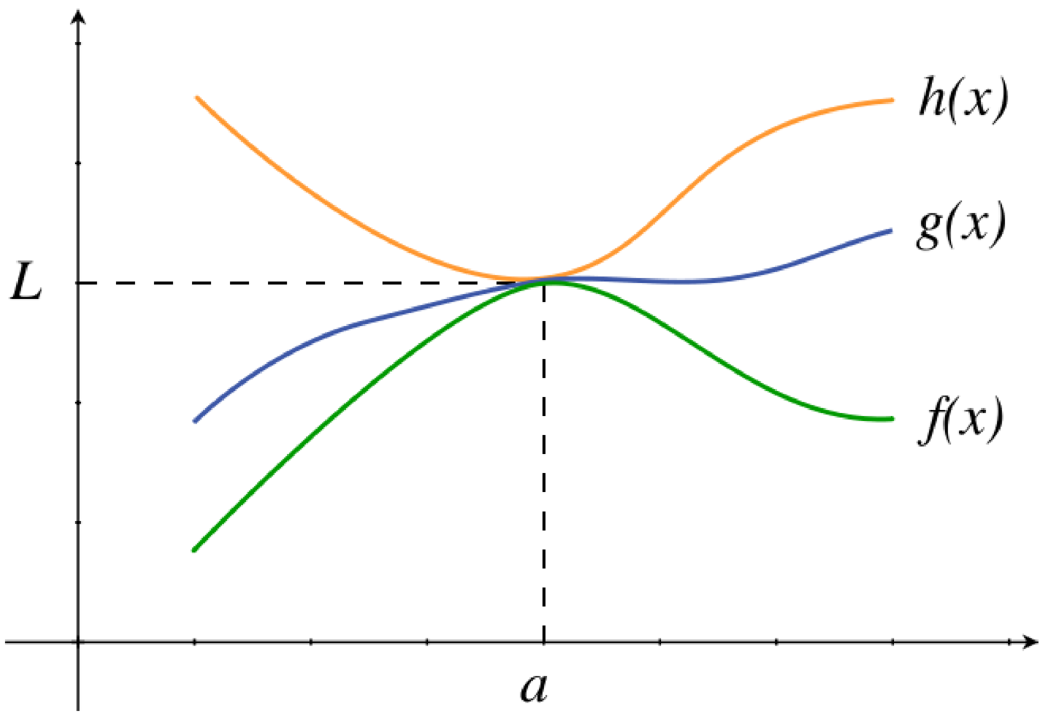
Let a be a constant, Suppose there is an open interval containing a such that $f(x) \leq g(x) \leq h(x)$ for all x inside that open interval, and further more

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

What the Squeeze theorem says is that, if a function $g(x)$ is (locally) squeezed between two other functions $f(x)$, $h(x)$, and the two functions $f(x)$, $h(x)$ both go to L when x approaches a, then g must also approach L as x approaches a



Formal Definition of Limit

So far we have used the concept of limit rather casually. We say that $\lim_{x \to a} f(x) = L$ if $f(x)$ is *close* to L when x *approaches* a. While these terms suffices for an intuitive understanding of limits, they are too vague and inexact to be used in formal mathematics. What do you mean by x *approaching* a number? How close is *close*? We must make these concepts precise if we want to *prove* that $\lim_{x \to 1} x + 1 = 2$. After all, how can you say that when $x \to 1$, $x + 1$ *approaches* 2? Why can't I say that $\lim_{x \to 1} x + 1 = 2.001$? It seems to me that as $x \to 1$, $x + 1$ is very close to 2.001 too.

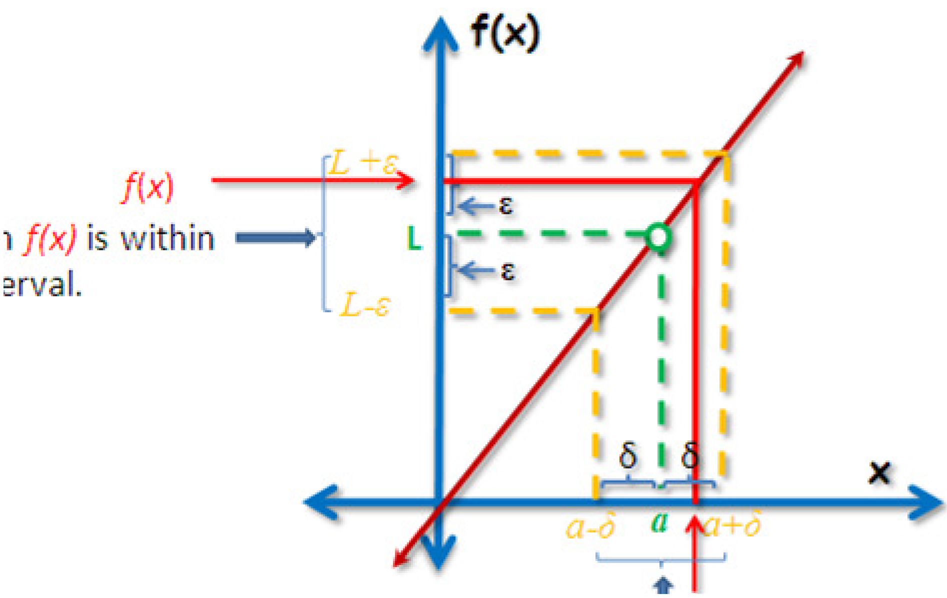
Formal Definition of Limit: We say that

$$\lim_{x \to a} f(x) = L$$

if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \qquad \text{whenever} \qquad |x - a| < \delta$$

What the definition says is that, regardless of how *close* (as close as ϵ) you want $f(x)$ to L, I can always "beat" you when I choose x close enough to a (as close as δ). As long as the difference between x and a is less than δ ($|x - a| < \delta$), then the difference between $f(x)$ and L will be less than ϵ ($|f(x) - L| < \epsilon$).



Example: Prove that

$$\lim_{x \to 2} x^2 = 4$$

To *prove* such a statement, we must use the formal limit definition. It is *not* enough just to try some x close to 2 and say that the result x^2 is close to 4. What you need to prove is this: How close do you want? You want x^2 and 4 to be so close that their difference is less than ϵ, so you want $|x^2 - 4| < \epsilon$. That is, you get to choose however small you want ϵ to be, as long as ϵ is greater than 0. I must now choose δ so that, as long as x is within δ distance of 2, that is, as long as $|x - 2| < \delta$, then I will have $|x^2 - 4| < \epsilon$. The choice for δ is not unique. As long as you find a δ that works, that will be fine. In this case here, let's have $\delta = \min\{1, \frac{\epsilon}{10}\}$. If $0 < |x - 2| < \delta$, then since $\delta \leq 1$, we have $|x - 2| < \delta \leq 1 \Rightarrow |x - 2| < 1 \Rightarrow |x| < 3 \Rightarrow |x + 2| < 5$. Since we also have $|x - 2| < \frac{\epsilon}{10}$, together we have:

$$|x^2 - 4| = |(x - 2)(x + 2)| = |x - 2||x + 2| < \frac{\epsilon}{10} \cdot 5 < \frac{\epsilon}{2} < \epsilon$$

Notice that a different choice of δ may also have worked. We are not required to find *all* δ that works, just one.

Formal definition of infinite Limit: We say

$$\lim_{x \to a} f(x) = \infty$$

if for every $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $|x - a| < \delta$.

The idea behind the infinite limit definition is the same as that of a finite limit. We are able to say that $f(x)$ *approaches* infinity if $f(x)$ grows without bound. What does it mean to *grow without bound*? If whatever the bound you provide (M) I can always beat you $(f(x) > M)$ then that means $f(x)$ grows without bound.

Limits at Infinity

So far we have studied the limit of a function f only at a particular number c. What if we want to study the *long term behavior* of a function f? That if, what can we tell about a function $f(x)$ when the values of x become tremendously large ($x \to \infty$) or tremendously negative ($x \to -\infty$)?

Let's look at the behavior of the function

$$f(x) = \frac{x-1}{x+1}$$

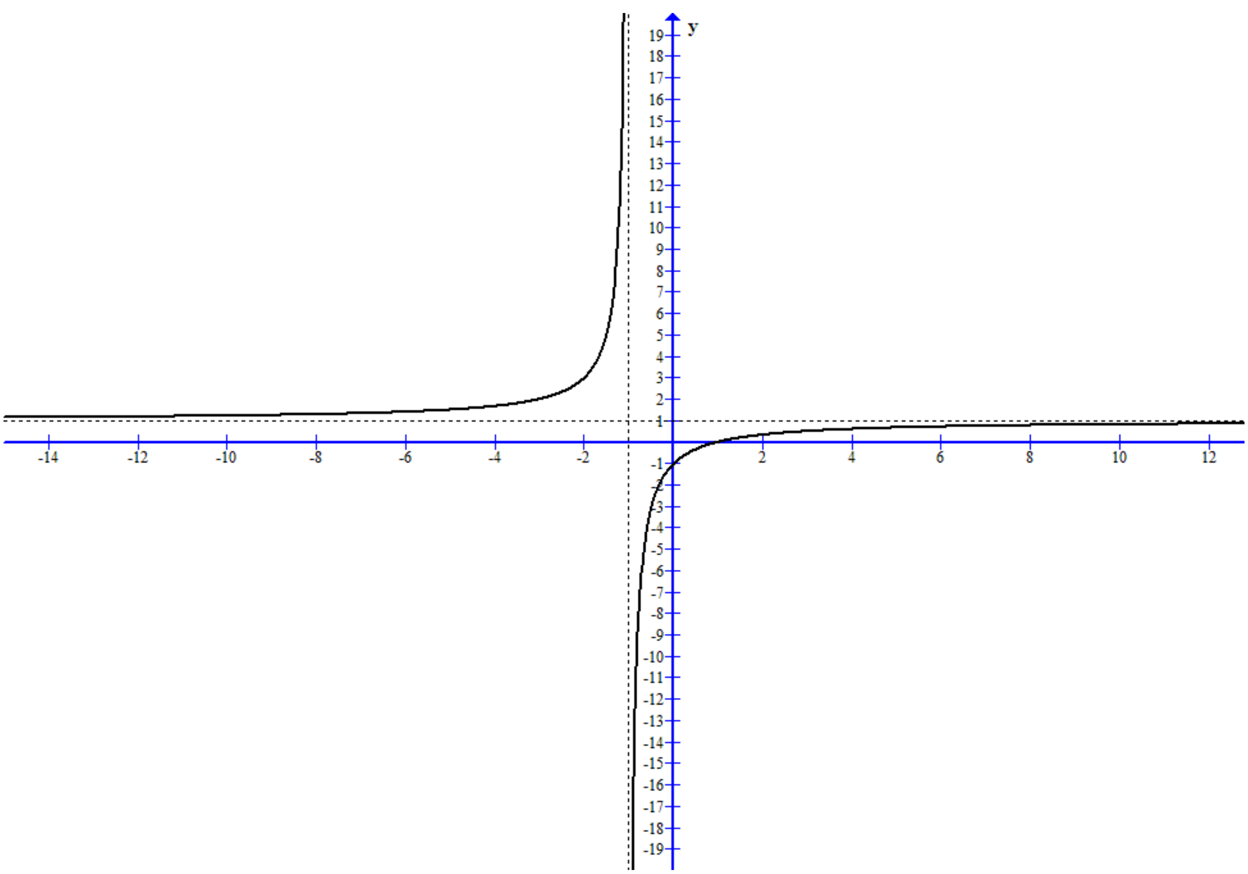
when x becomes very large:

x	$f(x)$
1000	0.998
10000	0.9998
100000	0.99998
1000000	0.999998
10000000	0.9999998

As we can see, the value of f approaches 1 as the value of x gets larger and larger. We use the notation:

$$\lim_{x\to\infty} \frac{x-1}{x+1} = 1$$

to denote this fact.



The meaning of the limit says that, the value of f approaches 1 as x approaches infinity. As we already said, infinity is *not* a number. So when we say x approaches infinity, we are *not* saying that x approaches any particular number. When we say x approaches infinity, we are saying that the value of x increases without bound.

What happens to the function f if x approaches negative infinity? That is, what happens when the values of x decreases without bound? We make a similar table like the one above:

x	$f(x)$
-1000	1.002
-10000	1.0002
-100000	1.00002
-1000000	1.000002
-10000000	1.0000002

So f approaches 1 as x decreases without bound. We use the notation:

$$\lim_{x \to -\infty} \frac{x-1}{x+1} = 1$$

to denote this fact.

Once again, when we say x approaches negative infinity, we mean x decreases without bound. We are not suggesting that x approaches any particular number.

Definition:

We use the notation

$$\lim_{x \to \infty} f(x) = L$$

to mean that, when the value of x grows without bound, the value of $f(x)$ approaches the number L.

We use similar notation for negative infinity:

$$\lim_{x \to -\infty} f(x) = L$$

to mean that, when the value of x decreases without bound, the value of $f(x)$ approaches the number L.

If the value of f approaches a particular number L when x approaches infinity (or negative infinity), the graph of f looks like a horizontal line at the tail end (when x is extremely large or extremely negative).

Definition:

The horizontal line $y = L$ is called a **horizontal asymptote** of f if

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$

The function we just considered, $f(x) = \frac{x-1}{x+1}$, has a horizontal asymptote $y = 1$.

If a function has a horizontal asymptote $y = L$, that means f behaves pretty much like the constant function of $y = L$ at large values of x.

How do we find the limit of a function at infinity? Let us see some simple cases:

$$\lim_{x\to\infty} \frac{1}{x}$$

As x gets tremendously large, we are dividing 1 by a tremendously large number, the result is a number very close to zero.

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

In general, whenever we have a constant divided by an expression that keeps on growing to infinity, then the result is always zero. We state a slightly simpler version here:

If $r > 0$ and c is a constant then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0$$

This makes sense because, since $r >$, as x approaches infinity, x^r approaches infinity, and when you divide a constant by something that is tremendously large, the result is a number that is very close to 0.

If $r > 0$, and r is a rational number, and c is a constant then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0$$

if x^r is defined for negative numbers x

E.g

Evaluate

$$\lim_{x\to\infty} \frac{x^3 - 4x^2 + x - 1}{-2x^3 + x + 5}$$

In this case here, both the numerator and the denominator goes to infinity as x grows without bound. In order to evaluate this limit formally, we need to express the fraction in terms that we can evaluate the value when x approaches infinity. The trick here is to divide the numerator and denominator by the highest power of x. In this example here, we divide the numerator and denominator by x^3:

$$\lim_{x\to\infty} \frac{x^3 - 4x^2 + x - 1}{-2x^3 + x + 5}$$

$$= \lim_{x\to\infty} \frac{\frac{x^3-4x^2+x-1}{x^3}}{\frac{-2x^3+x+5}{x^3}}$$

$$= \lim_{x\to\infty} \frac{1-\frac{4}{x}+\frac{1}{x^2}-\frac{1}{x^3}}{-2+\frac{1}{x^2}+\frac{5}{x^3}}$$

As $x \to \infty$, all the expressions that have a constant on the numerator and x to a power in the denominator will become 0, and this allows us to evaluate the limit:

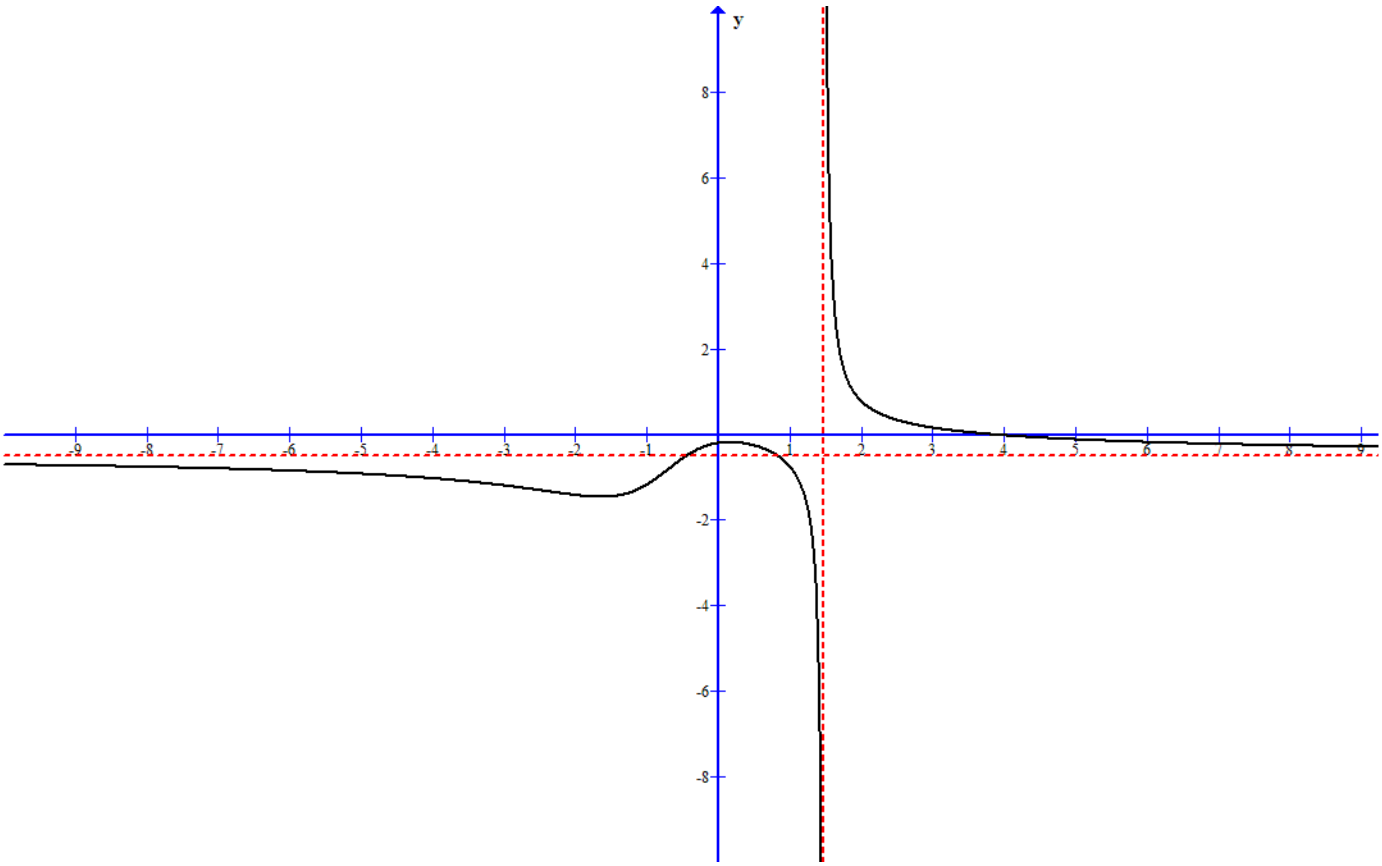
$$= \lim_{x\to\infty} \frac{1-\frac{4}{x}+\frac{1}{x^2}-\frac{1}{x^3}}{-2+\frac{1}{x^2}+\frac{5}{x^3}}$$

$$= \frac{\lim_{x\to\infty}\left(1-\dfrac{4}{x}+\dfrac{1}{x^2}-\dfrac{1}{x^3}\right)}{\lim_{x\to\infty}\left(-2+\dfrac{1}{x^2}+\dfrac{5}{x^3}\right)}$$

$$= \frac{\lim_{x\to\infty} 1 - \lim_{x\to\infty}\dfrac{4}{x} + \lim_{x\to\infty}\dfrac{1}{x^2} - \lim_{x\to\infty}\dfrac{1}{x^3}}{\lim_{x\to\infty} -2 + \lim_{x\to\infty}\dfrac{1}{x^2} + \lim_{x\to\infty}\dfrac{5}{x^3}}$$

$$= \frac{1-0+0-0}{-2+0+0}$$

$$= -\frac{1}{2}$$



y
8
6
4
2
-2
-4
-6
-8
-9
-8
-7
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6
7
8
9

There is a easier, though less formal method to solve a limit problem of this sort. Notice that in the above limit, both the numerator and denominator are polynomials. We can argue that when x becomes tremendously large, *the highest power of x in the polynomial will dominate.* Consider the polynomial $f(x) = x^3 - 4x^2 + x - 1$. When the value of x becomes tremendously large, the x^3 term is going to be much larger than the $-4x^2$ term and the x term. i.e,

$$x^3 - 4x^2 + x - 1 \approx x^3 \text{ as } x \to \infty$$

Similarly, as x becomes tremendously large,

$$-2x^3 + x + 5 \approx -2x^3 \text{ as } x \to \infty$$

Therefore, the limit:

$$\lim_{x\to\infty} \frac{x^3 - 4x^2 + x - 1}{-2x^3 + x + 5}$$

may be evaluated like this:

$$\lim_{x\to\infty} \frac{x^3 - 4x^2 + x - 1}{-2x^3 + x + 5} = \lim_{x\to\infty} \frac{x^3}{-2x^3} = \lim_{x\to\infty} \frac{1}{-2} = -\frac{1}{2}$$

It is important to note that the above argument works for polynomials (and power functions in general) only if x approaches infinity or negative infinity. If we are to evaluate

$$\lim_{x\to 5} \frac{x^3 - 4x^2 + x - 1}{-2x^3 + x + 5}$$

then the previous argument does not work, since x does not approach infinity, so we may *not* ignore the other terms.

E.g.

Evaluate

$$\lim_{x\to-\infty} \frac{\sqrt{x^2 + 1}}{x - 1}$$

To evaluate this formally, we divide by the highest power of x. In this case here, the highest power of the denominator is x. How about the numerator? Since we are taking the radical of x^2, the highest power of x in the numerator is also x. (This is generally true. If we have $\sqrt{x^4 + 3x + 1}$, then the highest power of x is x^2 since we are taking the square root of x^4. If we have $\sqrt[3]{x^7 - 4x^3 + 3}$ then the highest power of x is $x^{7/3}$.)

$$\lim_{x\to-\infty} \frac{\sqrt{x^2 + 1}}{x - 1}$$

$$= \lim_{x\to-\infty} \frac{\dfrac{\sqrt{x^2+1}}{x}}{\dfrac{x-1}{x}}$$

$$= \lim_{x\to-\infty} \frac{\dfrac{\sqrt{x^2+1}}{-\sqrt{x^2}}}{\dfrac{x-1}{x}}$$

Why does the x turned into $-\sqrt{x^2}$? The reason is that we are taking the limit as x goes to **negative infinity**. As x approaches negative infinity, x is a negative number. If we just turn x into $\sqrt{x^2}$, then this would have turned x into a positive number since $\sqrt{x^2}$ is a positive number for any value of x. Therefore, we must put the negative sign in front of the expression if we were to take the limit as x goes to negative infinity.

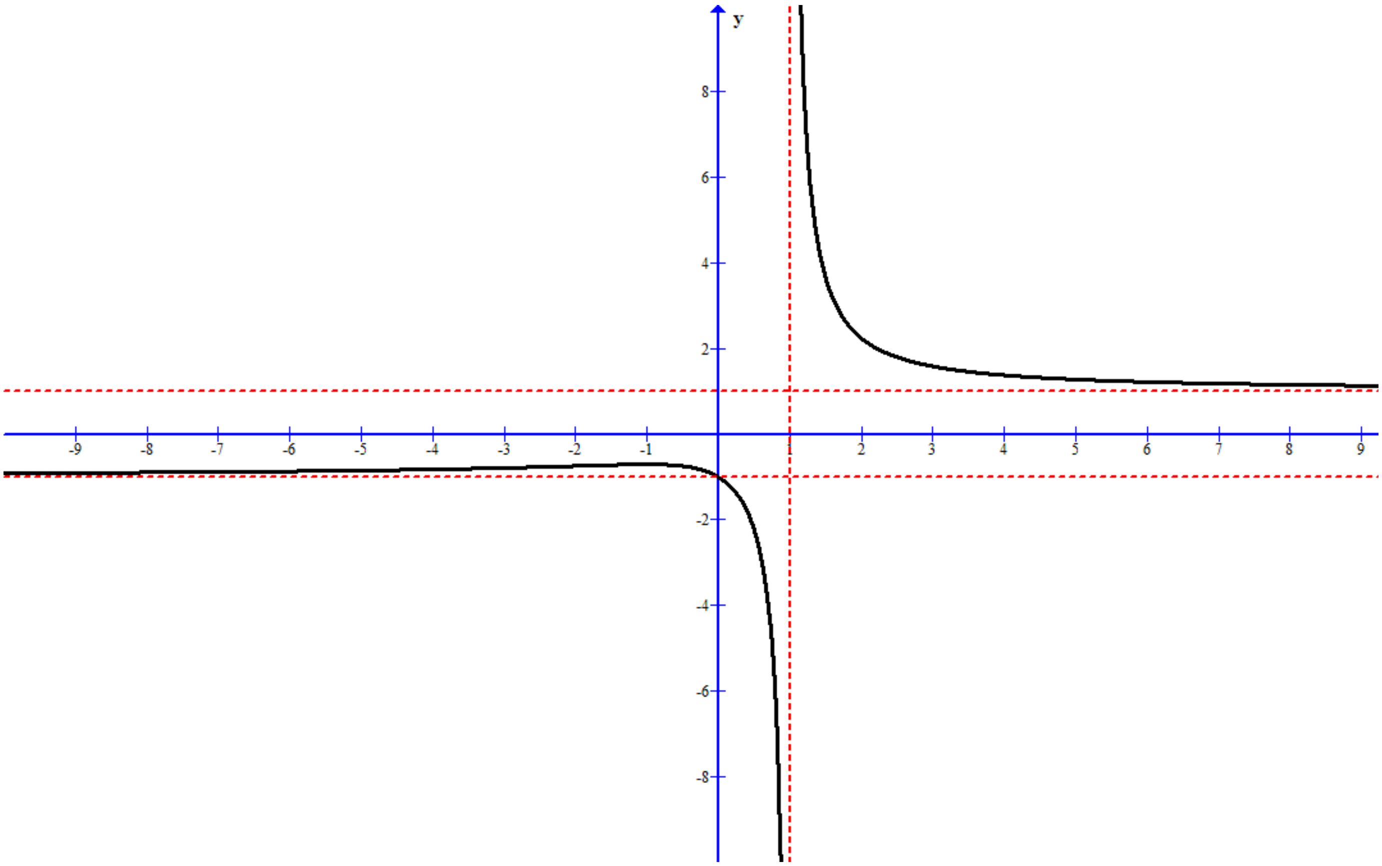
$$\lim_{x\to-\infty} \frac{\dfrac{\sqrt{x^2+1}}{-\sqrt{x^2}}}{\dfrac{x-1}{x}}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{\dfrac{x^2+1}{x^2}}}{\dfrac{x}{x}-\dfrac{1}{x}}$$

$$= \frac{\lim_{x\to-\infty} -\sqrt{1+\dfrac{1}{x^2}}}{\lim_{x\to-\infty} 1-\dfrac{1}{x}}$$

$$= \frac{-\sqrt{\lim_{x\to-\infty} 1 + \lim_{x\to-\infty} \dfrac{1}{x^2}}}{\lim_{x\to-\infty} 1 - \lim_{x\to-\infty} \dfrac{1}{x}}$$

$$= \frac{-\sqrt{1+0}}{1-0} = -1$$



We could have solved the problem more easily by arguing that, as x approaches $-\infty$, the highest power of x is going to dominate, so $\sqrt{x^2+1} \approx \sqrt{x^2}$ and $x-1 \approx x$ when x is very negatively large. We have

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x-1}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^2}}{x}$$

$$= \lim_{x\to-\infty} \frac{|x|}{x}$$

Since x goes to **negative infinity**, the numerator will be positive while the denominator will be negative, so their ratio will be -1:

$$\lim_{x\to-\infty} \frac{|x|}{x} = -1$$

E.g. Evaluate the limit:

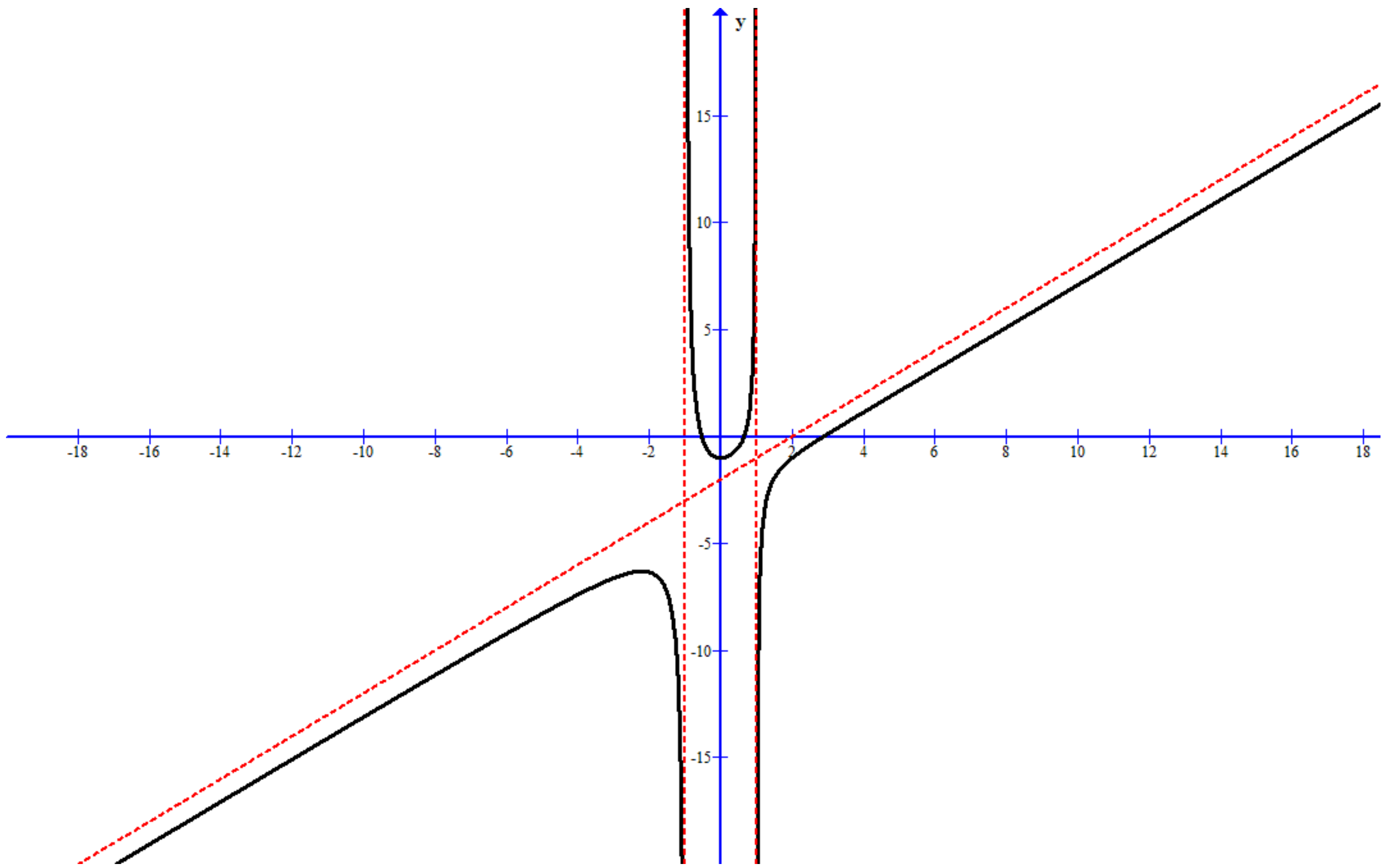
$$\lim_{x\to\infty} \frac{x^3 - 3x^2 + 1}{x^2 - 1}$$

We divide by the highest power of x, which is x^3.

$$\begin{aligned}
&\lim_{x\to\infty} \frac{x^3 - 3x^2 + 1}{x^2 - 1} \\
&= \lim_{x\to\infty} \frac{\dfrac{x^3 - 3x^2 + 1}{x^3}}{\dfrac{x^2 - 1}{x^3}} \\
&= \frac{\lim\limits_{x\to\infty} \dfrac{x^3 - 3x^2 + 1}{x^3}}{\lim\limits_{x\to\infty} \dfrac{x^2 - 1}{x^3}} \\
&= \frac{\lim\limits_{x\to\infty} \left(1 - \dfrac{3}{x} + \dfrac{1}{x^3}\right)}{\lim\limits_{x\to\infty} \left(\dfrac{1}{x} - \dfrac{1}{x^3}\right)} \\
&= \frac{\lim\limits_{x\to\infty} 1 - \lim\limits_{x\to\infty} \dfrac{3}{x} + \lim\limits_{x\to\infty} \dfrac{1}{x^3}}{\lim\limits_{x\to\infty} \dfrac{1}{x} - \lim\limits_{x\to\infty} \dfrac{1}{x^3}} \\
&= \frac{1 - 0 + 0}{0 + 0} = \frac{1}{0}
\end{aligned}$$

We have division by zero, which is undefined. So

$$\lim_{x\to\infty} \frac{x^3 - 3x^2 + 1}{x^2 - 1} \text{ does not exist}$$



What happens here is that, since the degree of numerator is higher than the degree of the denominator, the numerator increases much faster than the denominator as x goes to infinity, so the result is a tremendously large number.

Sometimes we use the notation

$$\lim_{x \to \infty} f(x) = \infty$$

to mean that as x increases without bound, the value of $f(x)$ also increases without bound. Once again remember that **infinity is not a number**, so this notation is not saying that the limit exists. It is simply a notation used to denote the behavior of the function f as x increases without bound.

Some general rules about limits at infinity of a rational function:

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$ and $q(x) = b_m x^m + b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \cdots + b_2x^2 + b_1x + b_0$ are two polynomials of degree n and m, respectively. Then

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = 0 \text{ if } n < m$$

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \frac{a_n}{b_m} \text{ if } n = m$$

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} \text{ does not exist if } n > m$$

This is true because, if the numerator has a lower degree than the denominator, then as x increases without bound, the denominator increases much faster than the numerator, so the fraction approaches 0.

If the numerator and the denominator has the same degree, then they increase at about the same rate, and the limit will be doniminated by the highest power of x, which in this case is x^n (or x^m), and the result is

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m}$$

If the numerator has a higher degree than the denominator, the numerator will increase faster than the denominator as x goes to ∞, and the whole fraction will grow without bound.

E.g. Evaluate

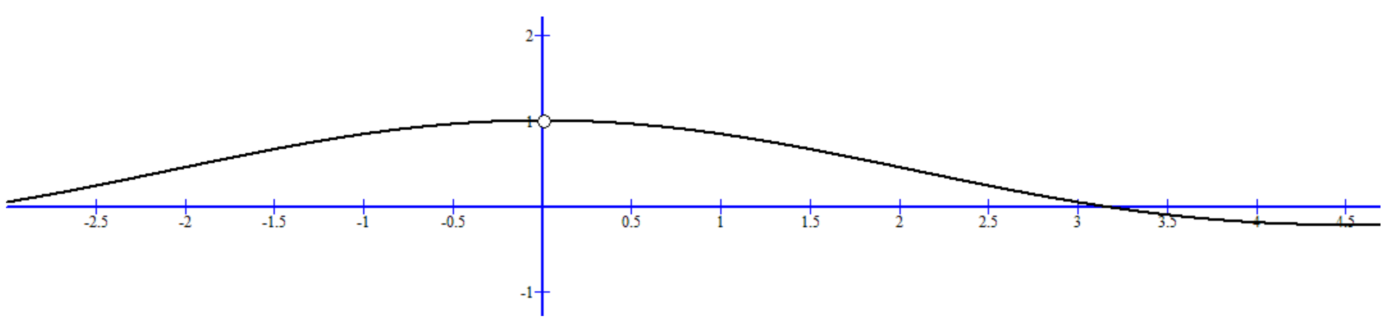
$$\lim_{x\to\infty} \frac{\sin x}{x}$$

We cannot evaluate this limit directly. However, notice that $-1 \leq \sin x \leq 1$, so

$$\lim_{x\to\infty} -\frac{1}{x} \leq \lim_{x\to\infty} \frac{\sin x}{x} \leq \lim_{x\to\infty} \frac{1}{x}$$

Since both limits $\lim_{x\to\infty} -\frac{1}{x}$ and $\lim_{x\to\infty} \frac{1}{x}$ are 0, using the squeeze theorem, we know that

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$



By definition, $f(x) = \dfrac{\sin x}{x}$ has a horizontal asymptote $y = 0$. Notice that the graph of f intercepts its horizontal asymptote many many times (in fact, infinitely many times). This should clarify a misconception that the graph of a function does not intercept its horizontal asymptote.

Slant Asymptote

For the function

$$f(x) = \frac{x^2}{x+1}$$

the degree of the numerator is greater than the degree of the denominator, so we know that

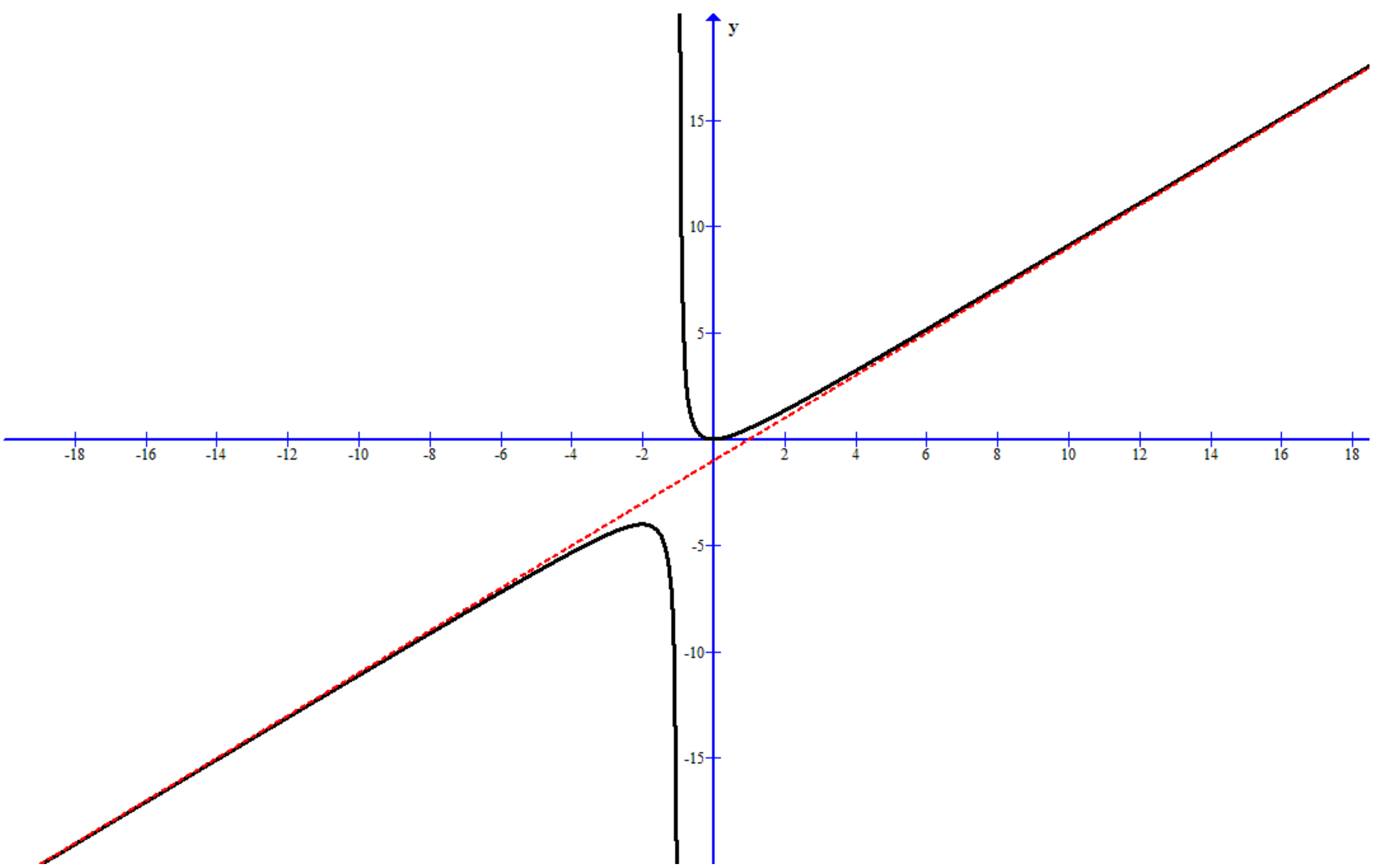
$$\lim_{x\to\infty} \frac{x^2}{x+1} \text{ does not exist}$$

However, if we perform the long division and express f as:

$$f(x) = \frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$$

Then we see that, as $x \to \infty$, the fraction $\frac{1}{x+1}$ approaches 0. This means that the expression $x - 1 + \frac{1}{x+1}$ approaches $x - 1$ as $x \to \infty$. What this means is that, as $x \to \infty$, $f(x)$ behaves very much like the line $y = x - 1$. i.e

$$\frac{x^2}{x+1} \approx x - 1 \,\text{as}\, x \to \infty$$



Since f approaches the line $y = x - 1$ as $x \to \infty$, we say that the line $y = x - 1$ is a **slant asymptote** of f. The line is not a horizontal line. Instead it is a line with a slope, and that's why the name *slant*

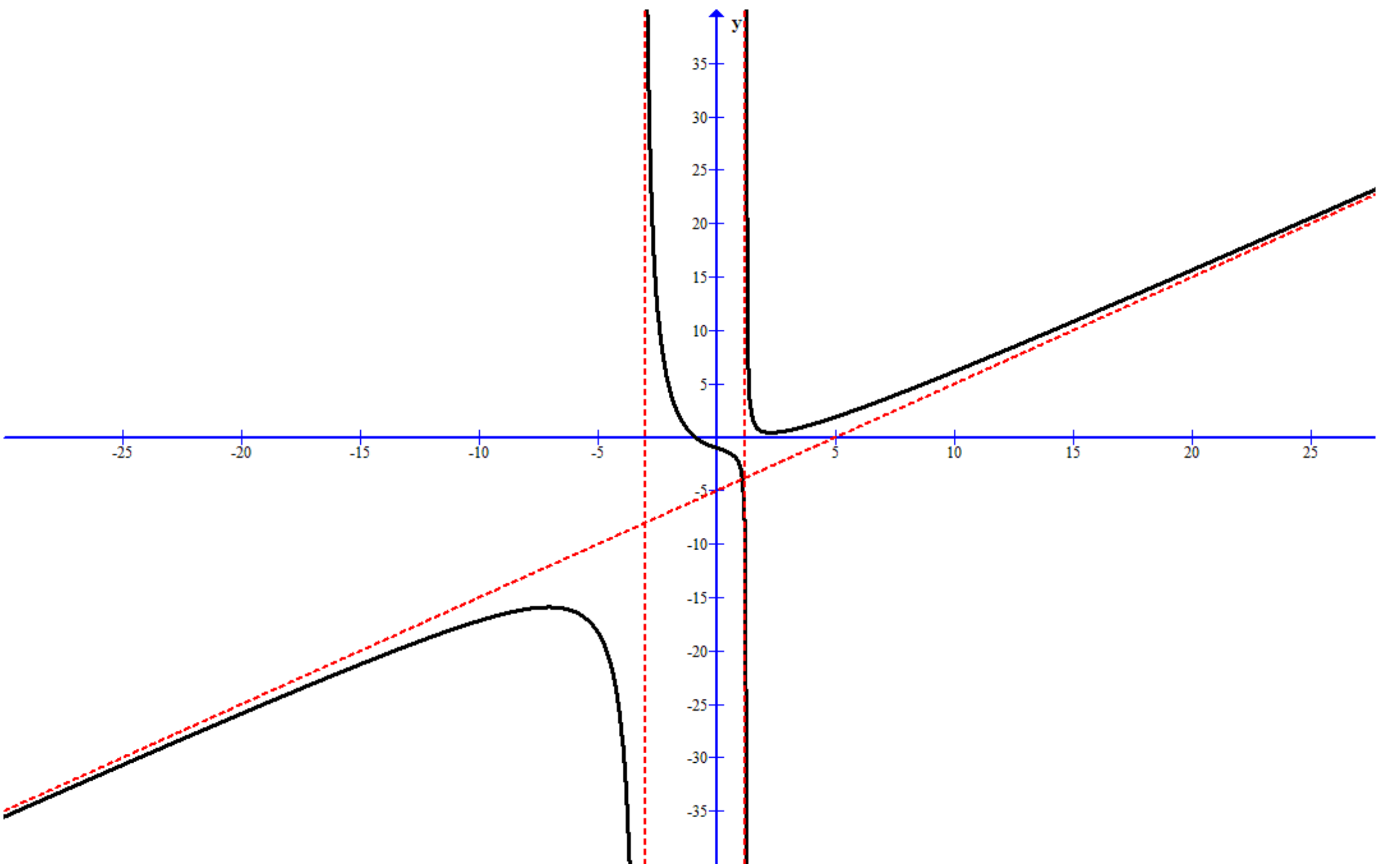
In general, a rational function will have a slant asymptote if the degree of the numerator is one greater than the degree of the denominator. In order to find the slant asymptote, we perform the long division to find the quotient. The quotient will be a line and that is the slant asymptote.

E.g. Find the slant asymptote of $f(x) = \dfrac{x^3 - 3x^2 + x + 4}{x^2 + 2x - 4}$.

Performing the long division we see that

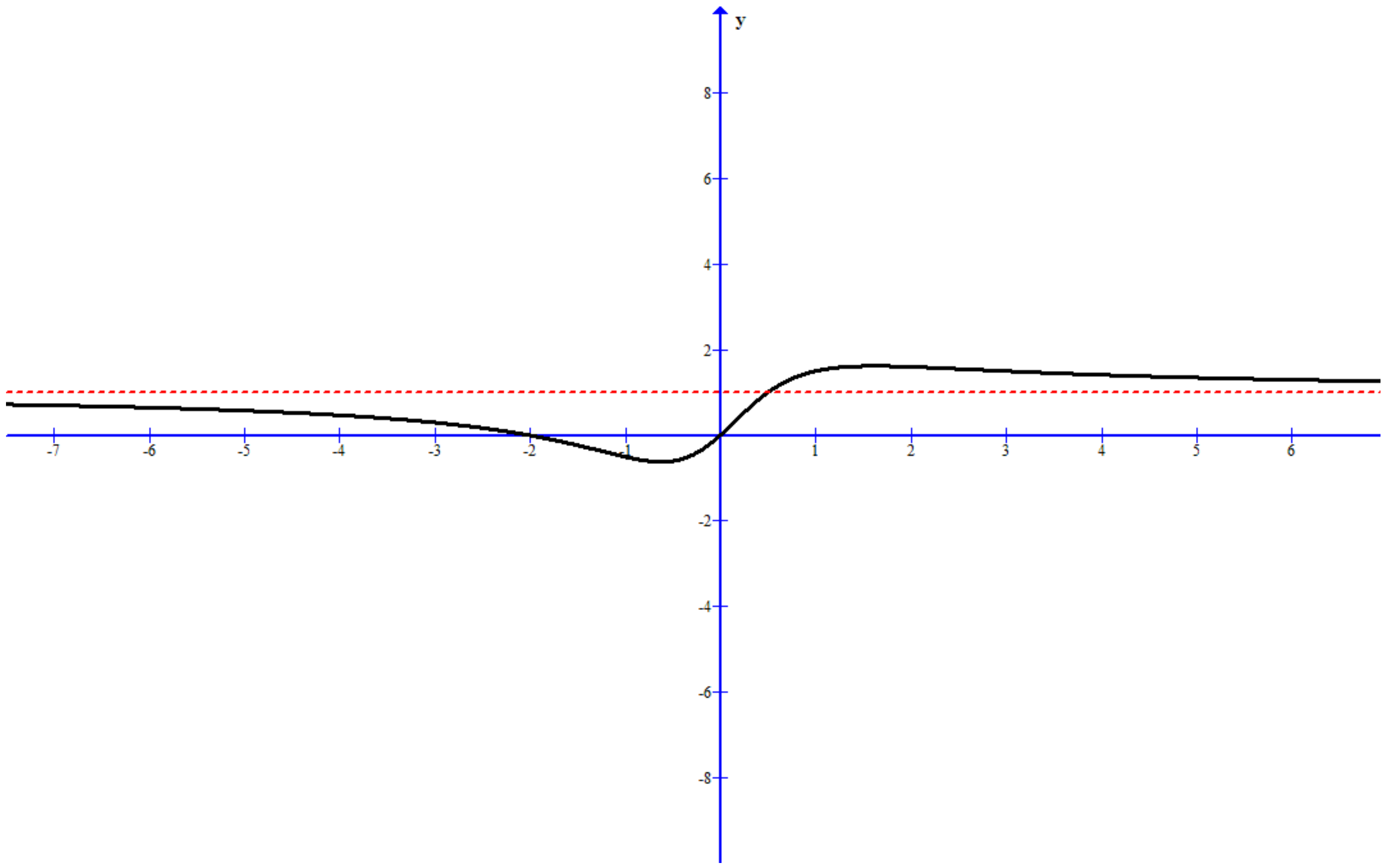
$$\frac{x^3 - 3x^2 + x + 4}{x^2 + 2x - 4} = x - 5 + \frac{15x - 15}{x^2 + 2x - 4}$$

Therefore, the slant asymptote for f is $y = x - 5$



E.g. Find the slant asymptote of $f(x) = \dfrac{x^2 + 2x}{x^2 + 1}$.

Since the degree of the numerator is equal to the degree of the denominator, f has *no* slant asymptote. f, though, has a horizontal asymptote of $y = 1$.



Continuous Functions

We have seen that for some functions, we can find its limit at some points by simply substitute the value. What type of functions can we do that? And what are their special properties? It turns out that the functions that we can find their limit by just substitution are very useful and exhibit very regular behavior that they deserve special attention.

Definition: Let a be a real number and $f(x)$ a function. We say that f is **continuous at** a if

$$\lim_{x \to a} f(x) = f(a)$$

A function is said to be **continuous** if it is continuous at every point in its domain. A function that is *not* continuous is said to be **discontinuous**.

What the definition says is that a function is continuous at a point a if three things happen, 1, if $f(a)$ is defined, 2, if the limit of the function exists at a, and 3, the limit is equal to the value of the function.

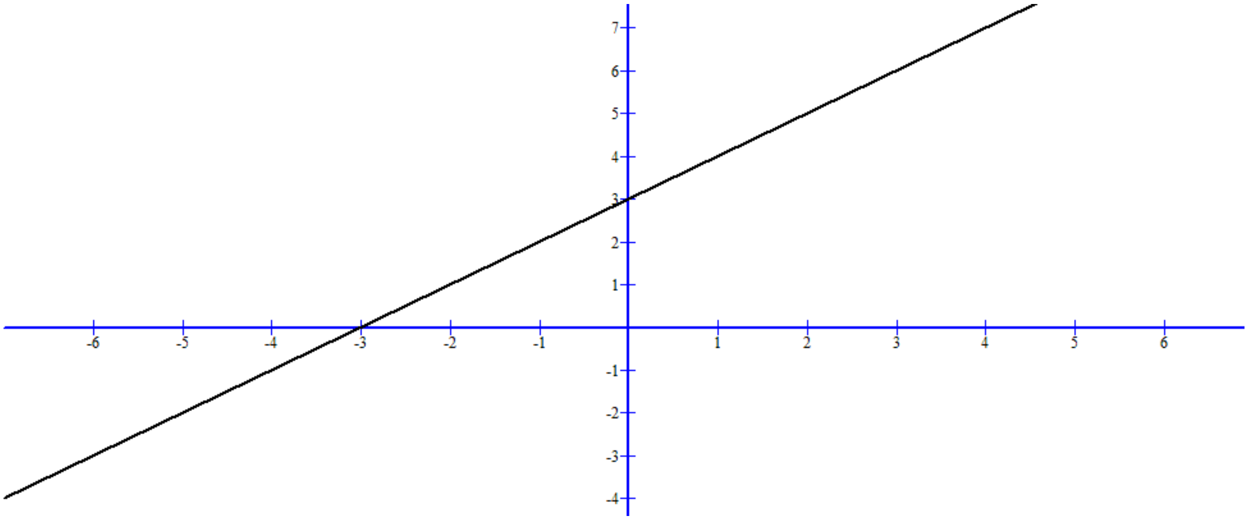
You may think of the limit of a function at a point a as what we *believe* the value of the function *should* be, according to what we have seen from the behavior of the function near a. So when we say $\lim_{x \to -1} \frac{x^2 + 2x + 1}{x + 1} = 0$, we are saying that the value of the expression $\frac{x^2 + 2x + 1}{x + 1}$ *should* be 0 when x is equal to -1, according to what we have seen from values of the function near 0.

The value of the function, $f(a)$, is the *reality*. i.e. $f(a)$ is *really* what happens to f at a. $f(a)$ may be undefined, it may be equal to a value that differs from the limit. If the reality, i.e. $f(a)$, is equal to what we think should be, i.e. $\lim_{x \to a} f(x)$, then we say the function is continuous.

E.g.

$f(x) = x + 3$

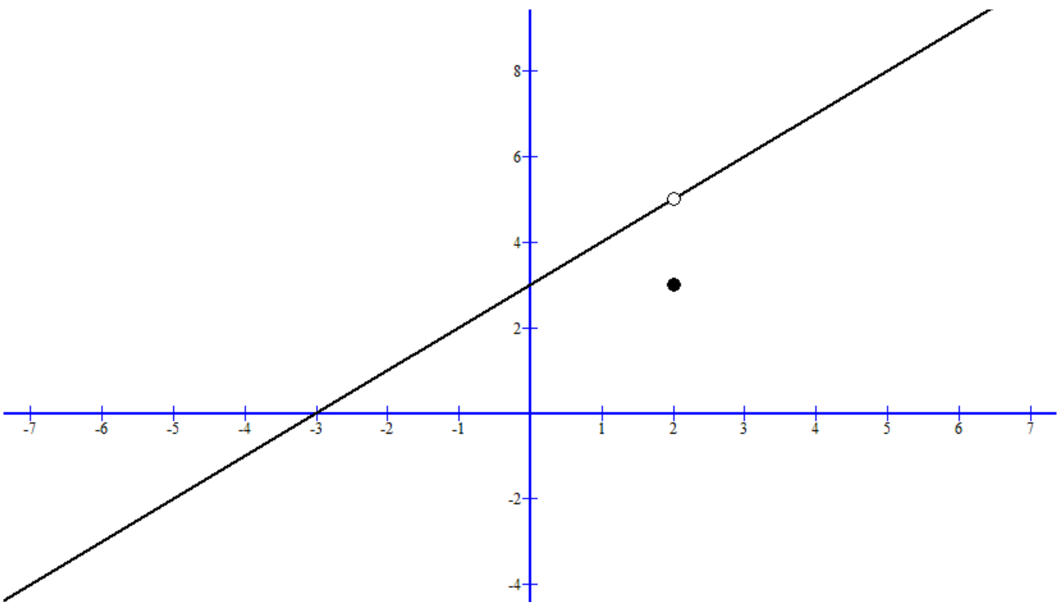
f is *continuous* at $x = 1$ since the limit $\lim_{x \to 1} x + 3 = 4$ and function $f(1) = 4$ are both equal to 4.



E.g.

$$f(x) = \begin{cases} \frac{x^2+x-6}{x-2} & \text{if} \quad x \neq 2 \\ 3 & \text{if} \quad x = 2 \end{cases}$$

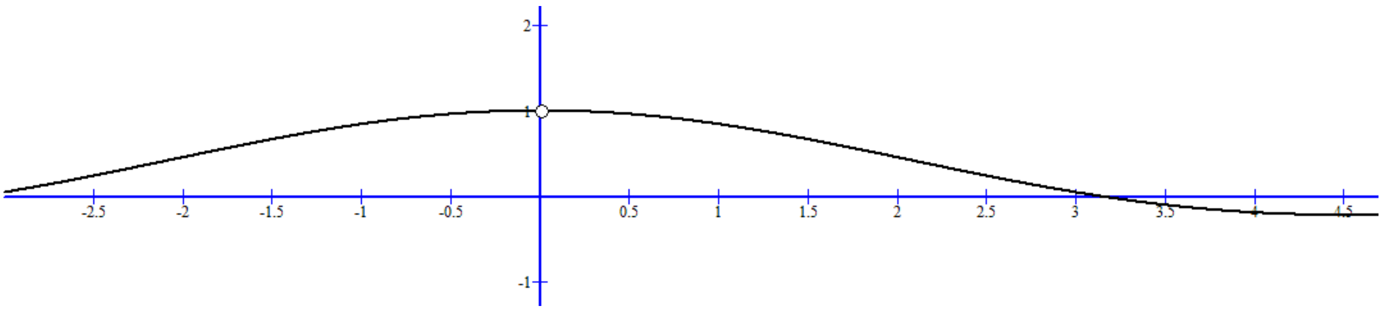
f is *discontinuous* at $x = 2$ because the limit $\lim_{x \to 2} f(x) = 5$ is **not** equal to the value of the function, $f(2) = 3$.



E.g.

$$f(x) = \frac{\sin x}{x}$$

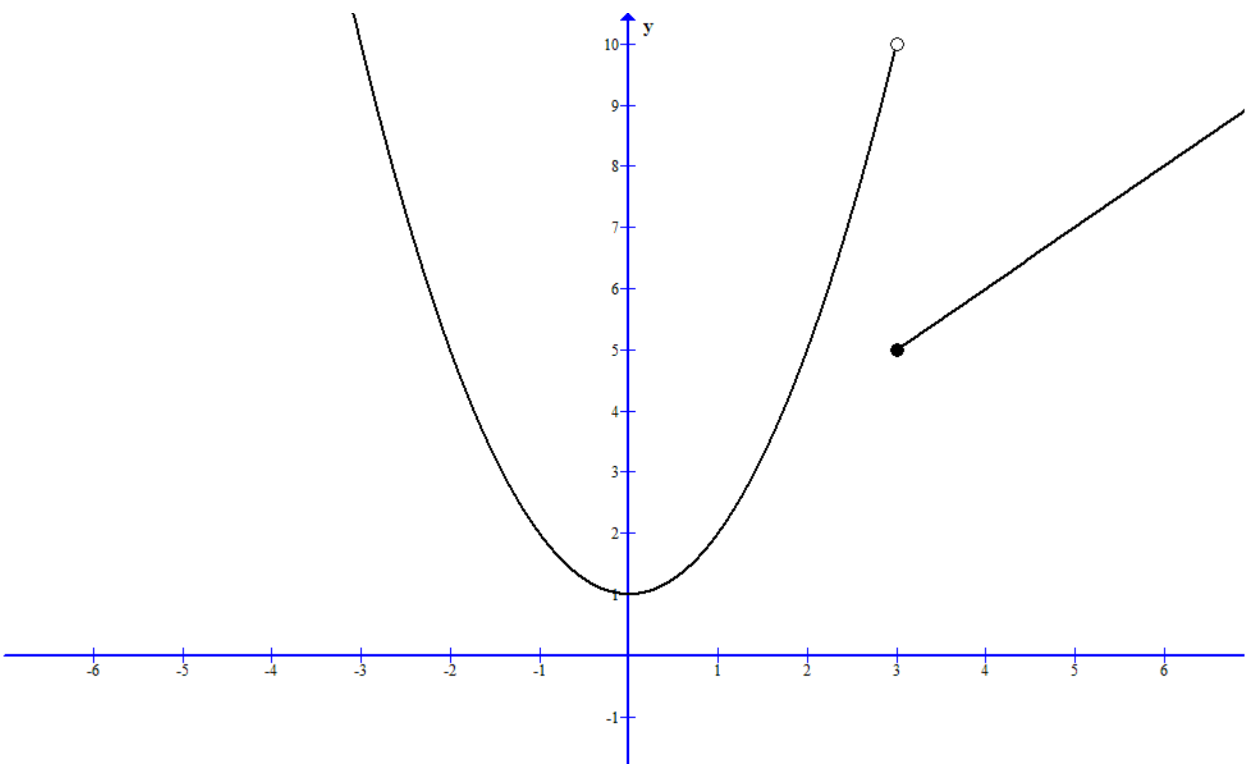
f is *discontinuous* at 0 because f is undefined at 0. $\lim_{x\to 0} \frac{\sin x}{x}$ exists, but f is not defined at 0, therefore it is discontinuous at 0.



E.g.

$$f(x) = \begin{cases} x^2 + 1 & \text{if} \quad x < 3 \\ x + 2 & \text{if} \quad x \geq 3 \end{cases}$$

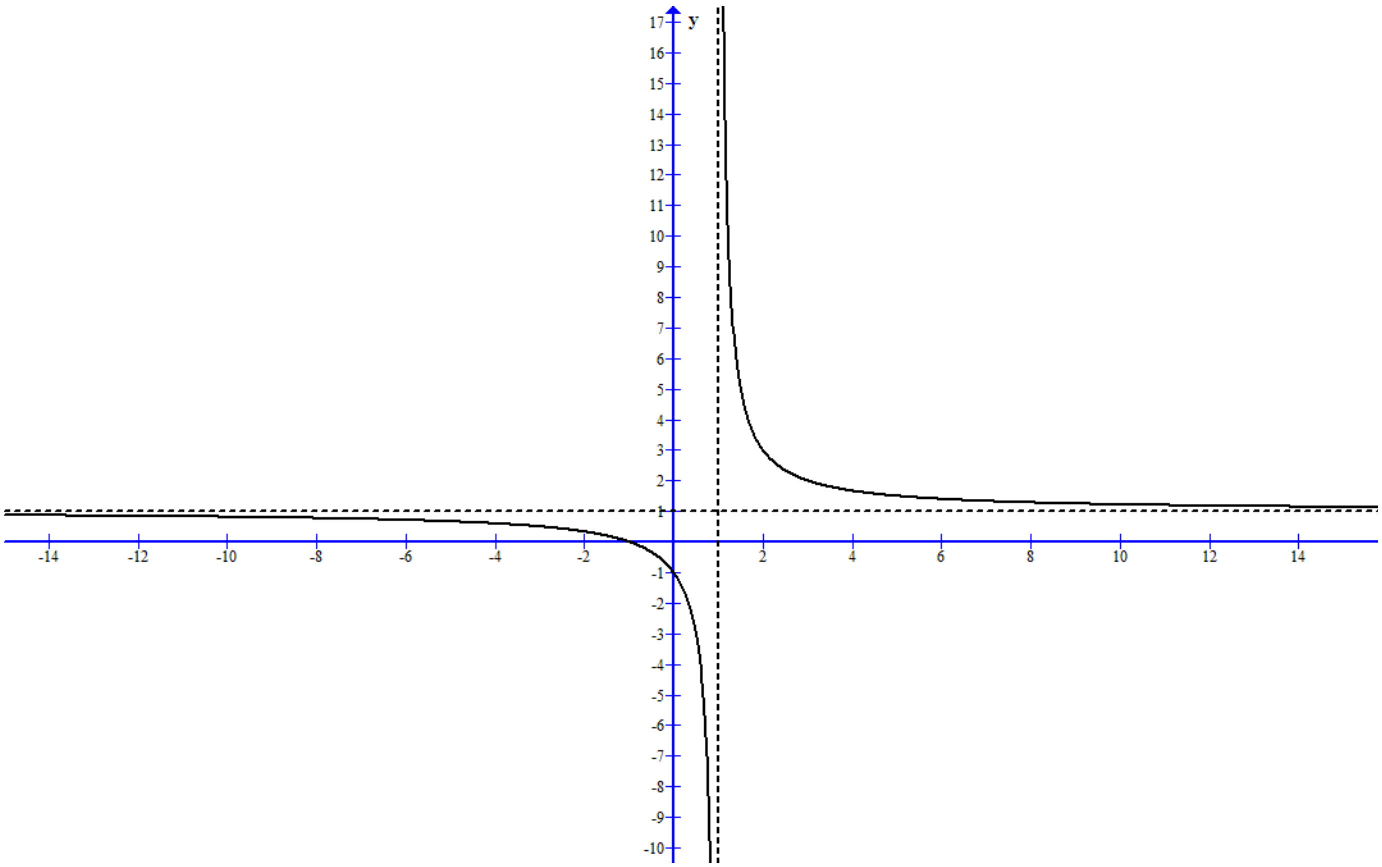
f is *discontinuous* because even though $f(3) = 5$ is defined, $\lim_{x\to 3} f(x)$ does not exist.



E.g.

$$f(x) = \frac{x+1}{x-1}$$

f is *discontinuous* at $x = 1$ because f is undefined at 1 and $\lim_{x \to 1} \frac{x+1}{x-1}$ does not exist either.



By definition, continuity is stronger than having a limit. This means that, if a function f is continuous at a, then f must have a limit at a (in fact, its limit at a must be equal to $f(a)$.

As some of the previous examples showed, a function having a limit at a does not necessarily have to be continuous at a.

Definition:

If f is discontinuous at a, if f has a limit at a, then we say that $x = a$ is a **removable discontinuity** of f. If f does not have a limit at a, then $x = a$ is a **non-removable discontinuity** of f.

Graphically, the graph of a function with a removable discontinuity at a has a hole at a, but no jump. We can **remove** the point of discontinuity by simply define the value of the function at a to be equal to its limit.

The graph of a function with a non-removable discontinuity at a necessarily has a jump or a vertical asymptote, or some kind of oscillating behavior that causes the function of jump around in values near a.

The graph of a continuous function does not have any holes or jumps.

E.g.

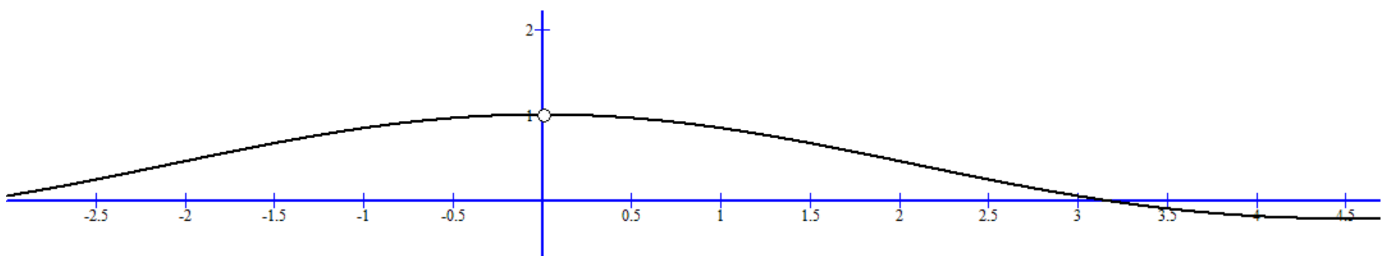
$f(x) = \frac{\sin x}{x}$

f is discontinuous at $x = 0$ since $f(0)$ is undefined.

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

Since f has a limit at 0, this discontinuity is removable. We can remove the point of discontinuity (the hole) by redefining f to be:

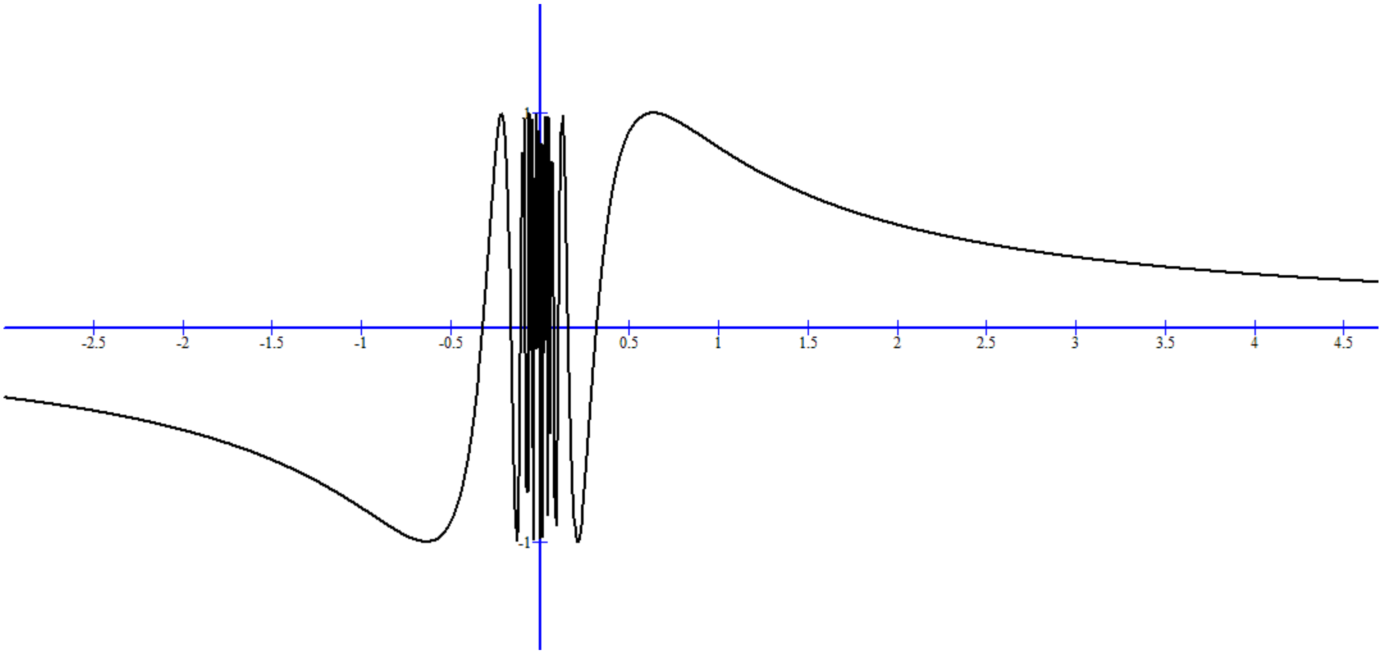
$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$



E.g.

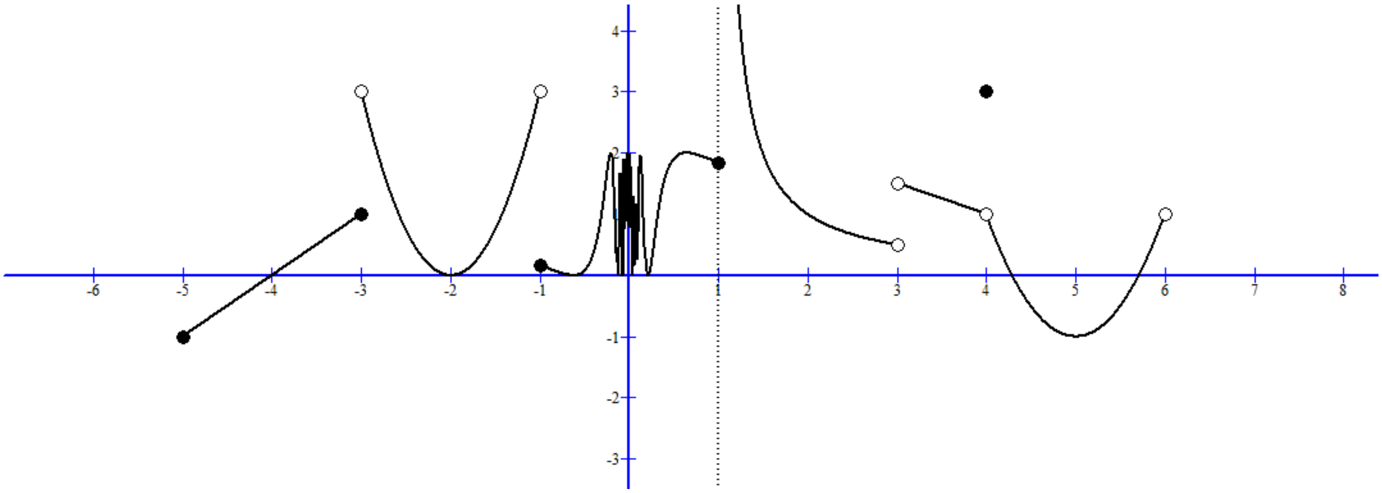
$f(x) = \sin\left(\frac{1}{x}\right)$

Limit of f at $x = 0$ does not exist, f has a non-removable discontinuity at $x = 0$. It does not matter how we define the value f at 0, f still will not be continuous at that point.



-2.5
-2
-1.5
-1
-0.5
0.5
1
1.5
2
2.5
3
3.5
4
4.5
-1

E.g.



For the function f with the above graph,

$f(-3) = 1$ $\quad \lim_{x \to -3^-} f(x) = 1$ $\quad \lim_{x \to -3^+} f(x) = 3$ $\quad \lim_{x \to -3} f(x)$ does not exist

f has a non-removable discontinuity at $x = -3$

$f(-1) \approx 0.2$ $\quad \lim_{x \to -1^-} f(x) = 3$ $\quad \lim_{x \to -1^+} f(x) \approx 0.2$ $\quad \lim_{x \to -1} f(x)$ does not exist

f has a non-removable discontinuity at $x = -1$

$f(0)$ is undefined. $\quad \lim_{x \to 0^-} f(x)$ does not exist $\quad \lim_{x \to 0^+} f(x)$ does not exist
$\lim_{x \to 0} f(x)$ does not exist

f has a non-removable discontinuity at $x = 0$

$f(1)$ is undefined. $\quad \lim_{x \to 1^-} f(x) = 2$ $\quad \lim_{x \to 1^+} f(x) = \infty$ $\quad \lim_{x \to 1} f(x)$ does not exist

f has a non-removable discontinuity at $x = 1$

$f(3)$ is undefined. $\quad \lim_{x \to 3^-} f(x) \approx 0.5$ $\quad \lim_{x \to 3^+} f(x) \approx 1.5$ $\quad \lim_{x \to 3} f(x)$ does not exist

f has a non-removable discontinuity at $x = 3$

$f(4) = 3$ $\quad \lim_{x \to 4^-} f(x) = 1$ $\quad \lim_{x \to 4^+} f(x) = 1$ $\quad \lim_{x \to 4} f(x) = 1$

f has a removable discontinuity at $x = 4$

The following functions are continuous functions in their respective domain:

Polynomials, rational functions, exponential functions, logarithmic functions, trigonometric functions, root functions.

Theorem: If f and g are continuous at a then the following functions are continuous at a:

$$f+g \qquad f-g \qquad fg \qquad \frac{f}{g} \qquad \text{provided that} g(a) \neq 0$$

In other words, sum, difference, product, and quotient of continuous functions are continuous.

Theorem: If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$$

What this theorem says is that if a function is continuous at a point a, and that the limit exists for another function g at a, then we can interchange the limit and function symbol. This theorem will be useful when you get to the part where L'Hospital's rule is needed.

The above theorem gives us the following:

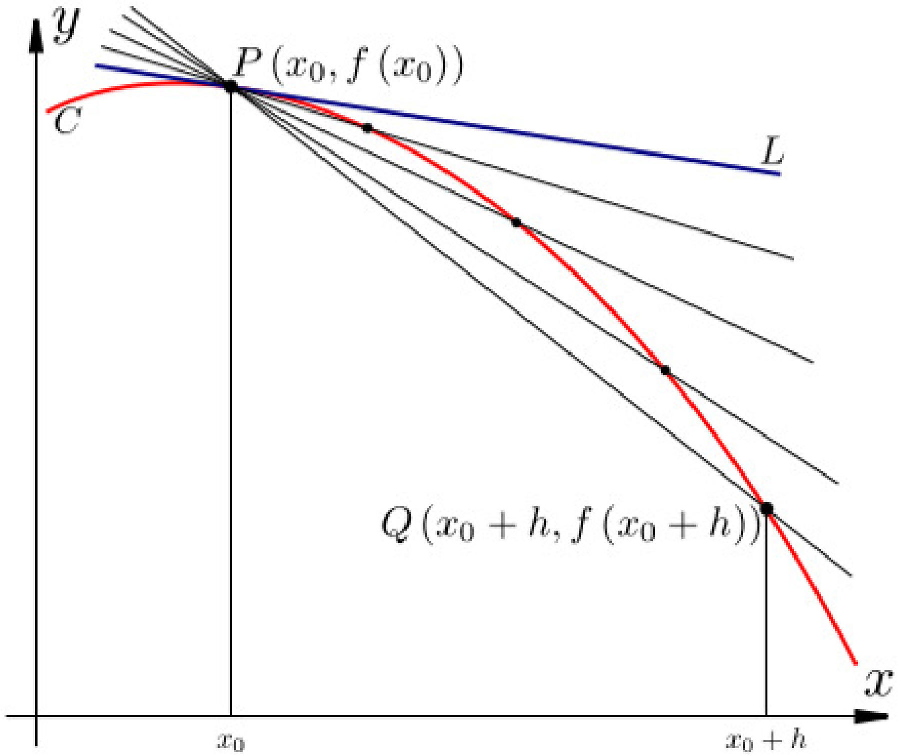
If f is continuous at a and g is continuous at $f(a)$, then $g(f)$ is continuous at a In other words, composition of continuous functions is continuous.

Since the graph of a continuous function does not have any holes or jumps, we may expect that if a continuous function f were to go from point $f(a)$ to $f(b)$, it must go through all the points between $f(a)$ and $f(b)$, and this is indeed the case.

Intermediate Value Theorem: Let f be a continuous function on the closed interval $[a, b]$. Without lost of generality, let's say $f(a) \leq f(b)$. Let N be any number such that $f(a) \leq N \leq f(b)$, then there exists a number c, $a \leq c \leq b$ such that $f(c) = N$

The intermediate value theorem says that, if you were to go from point a to point b, without jumping or skipping, then you must go through all the points between a and b.

Now that we have studied the concept of limits and how to find them, we are ready to discuss the tangent problem that we mentioned. Remember that given a function $f(x)$, we want to find the slope of its tangent line at a point $(x_0, f(x_0))$.



We said that we can try to approximate the slope of this line by finding the slope of a secant line with a nearby point $(x_0 + h, f(x_0 + h))$. According to algebra, the slope of this secant line is given by:

$$\text{slope of secant line} = m_{sec} = \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}$$

This formula gives an **approximation** of the slope of the tangent line to f at $x = x_0$, if h is small (close to 0). Our intuition tells us that, the smaller (closer to 0) h is, the better the slope of the secant line approximates the slope of the tangent line. With the concept of limit, we **define** the slope of the tangent line to be the limit of the secant lines:

Definition of the slope of a tangent line:

The **slope** m of the tangent line to a function f at the point $(x_0, f(x_0))$ is given by:

$$\text{slope of tangent line} = m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

if the limit exists.

E.g.: Find the slope of the line tangent to $f(x) = x^2 + 1$ at $x = 2$.

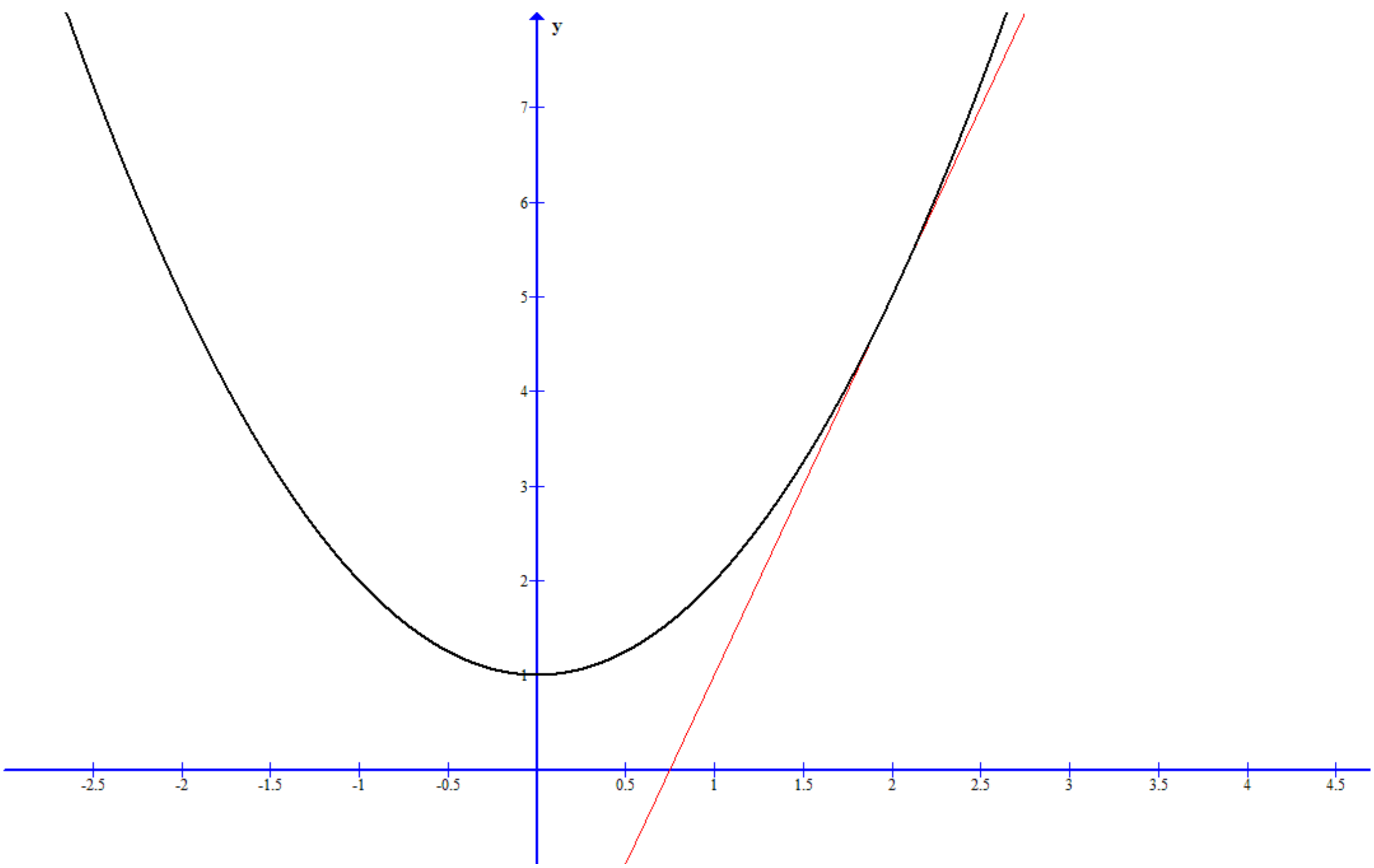
To find the slope of the tangent line, we use the definition just mentioned, with $x_0 = 2$:

$$m = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\left[(2+h)^2 + 1\right] - (5)}{h}$$

Evaluating the limit gives:

$$\begin{aligned} &\lim_{h \to 0} \frac{[4 + 4h + h^2 + 1] - 5}{h} \\ &= \lim_{h \to 0} \frac{4h + h^2}{h} \\ &= \lim_{h \to 0} 4 + h = 4 \end{aligned}$$

The slope of the tangent line to $f(x) = x^2 + 1$ at $x = 2$ is 4.



E.g. Find the equation of the line tangent to $f(x) = \sqrt{x}$ at $x = 4$

$$m = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} = \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h}$$

To evaluate this limit we multiply by the conjugate of the numerator:

$$\lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} = \lim_{h \to 0} \frac{(\sqrt{4+h} - 2)(\sqrt{4+h} + 2)}{h(\sqrt{4+h} + 2)}$$

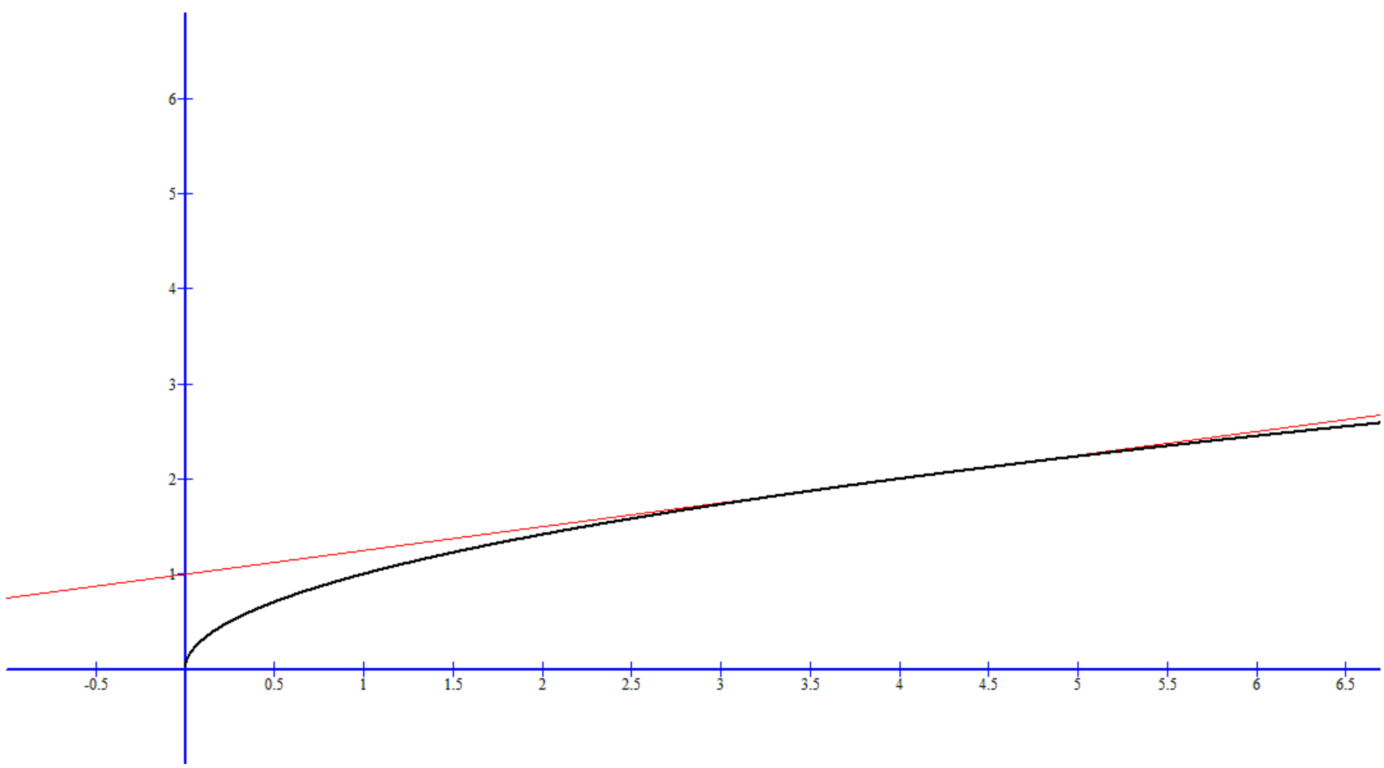
$$= \lim_{h \to 0} \frac{4 + h - 4}{h(\sqrt{4+h} + 2)}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{4+h} + 2)} = \lim_{h \to 0} \frac{1}{\sqrt{4+h} + 2}$$

$$= \frac{1}{\sqrt{4+0} + 2} = \frac{1}{2+2} = \frac{1}{4}$$

We now have the slope of the tangent line, $\frac{1}{4}$, and we have a point, $(4, 2)$, so we may use the point-slope form of the equation of a line:

$$y - 2 = \frac{1}{4}(x - 4) \Rightarrow y = \frac{1}{4}x + 1$$



What does the slope of the tangent line represent? A slope between two points represents a *rate of change* between two quantities. For example, if the x-axis represents the time a car has spent in travelling along a straight line, and the y-axis represents the *displacement* (how far from origin) of the car, then the slope between any two points of the graph represents the *average* velocity of the car between the two times.

The slope of the tangent line, then, represents an **instantaneous rate of change** between two quantities. That is, given the distance *vs* time graph of ours, the slope of the *tangent line* at any given point t represents the *instantaneous velocity* of the car at that given moment in time. In general, for any function $y = f(x)$, the slope of the tangent line at $x = a$ represents the *instantaneous rate of change* of y *with respect to* x at a. For example, if x represents the time and y represents the population, then the slope of the tangent line at a point in time represents how fast the population is increasing (or decreasing) at that particular moment in time.