1. Find $f'(x)$, or $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, as appropriate:

a. $f(x) = 3$

Ans: $f'(x) = 0$

b. $f(x) = 4x + 1$

Ans: $f'(x) = 4$

c. $f(x) = -2x^2 - 3x + 1$

Ans: $f'(x) = -4x - 3$

d. $f(x) = \dfrac{3x^3}{4}$

Ans: $f'(x) = \dfrac{9}{4}x^2$

e. $f(x) = \sqrt{x} - 1$

Ans: $f'(x) = \dfrac{1}{2\sqrt{x}}$

f. $f(x) = -\sqrt{x^5} + 3\sqrt{x^3} + 7\sqrt{x}$

Ans: $f'(x) = -\dfrac{5}{2}x^{3/2} + \dfrac{9}{2}\sqrt{x} + \dfrac{7}{2\sqrt{x}}$

g. $y = \sin x$

Ans: $f'(x) = \cos x$

h. $f(x) = \sin\dfrac{3\pi}{4}$

Ans: $f'(x) = 0$

i. $f(x) = e^x$

Ans: $f'(x) = e^x$

j. $f(x) = e$

Ans: $f'(x) = 0$

k. $f(x) = 3xe^x$

Ans: $f'(x) = 3e^x + 3xe^x$

l. $f(x) = 3x\sin x + \sin x\cos x$

Ans: $f'(x) = 3\sin x + 3x\cos x + \cos^2 x - \sin^2 x$

m. $f(x) = 3x^2e^x + 4x\tan x$

Ans: $f'(x) = 6xe^x + 3x^2e^x + 4x\sec^2 x + 4\tan x$

n. $f(x) = \sin x\cos xe^x$

Ans: $f'(x) = \sin x \cos x e^x - \sin^2 x e^x + \cos^2 x e^x$

o. $y = 3x^2 \sin x \cos x e^x$

Ans: $\dfrac{dy}{dx} = 6x \sin x \cos x e^x + 3x^2 \cos^2 x e^x - 3x^2 \sin^2 x e^x + 3x^2 \sin x \cos x e^x$

p. $y = \dfrac{3e^x - \sin x}{\cos x - 4e^x}$

Ans: $\dfrac{dy}{dx} = \dfrac{(\cos x - 4e^x)(3e^x - \cos x) - (3e^x - \sin x)(-\sin x - 4e^x)}{(\cos x - 4e^x)^2}$

$$= \frac{(\cos x - 4e^x)(3e^x - \cos x) + (3e^x - \sin x)(\sin x + 4e^x)}{(\cos x - 4e^x)^2}$$

q. $y = \dfrac{\sqrt{x} + 1}{3x - 4}$

Ans: $\dfrac{dy}{dx} = \dfrac{(3x-4)(\frac{1}{2\sqrt{x}}) - (\sqrt{x}+1)(3)}{(3x-4)^2} = \dfrac{-3x - 6\sqrt{x} - 4}{(3x-4)^2(2\sqrt{x})}$

r. $y = \dfrac{3e^x - \sin x}{\cos x - 4e^x}$

Ans: $\dfrac{dy}{dx} = \dfrac{(\cos x - 4e^x)(3e^x - \cos x) - (3e^x - \sin x)(-\sin x - 4e^x)}{(\cos x - 4e^x)^2} =$

$$\frac{(\cos x - 4e^x)(3e^x - \cos x) + (3e^x - \sin x)(\sin x + 4e^x)}{(\cos x - 4e^x)^2}$$

s. $y = \dfrac{9x^2 e^x + 4x \cos x}{2 \sin x + \sqrt{x} e^x}$

Ans: $\dfrac{dy}{dx} = \dfrac{(2\sin x + \sqrt{x}e^x)[9(2xe^x + x^2 e^x) + 4(\cos x - x \sin x)]}{(2\sin x + \sqrt{x}e^x)^2}$

$$- \frac{(9x^2 e^x + 4x \cos x)(2\cos x + \frac{e^x}{2\sqrt{x}} + \sqrt{x}e^x)}{(2\sin x + \sqrt{x}e^x)^2}$$

t. $y = \dfrac{e^x \sin x - x \cos x}{3x \cos x + 4xe^x}$

Ans: $\dfrac{dy}{dx} = \dfrac{(3x\cos x + 4xe^x)(e^x \cos x + e^x \sin x - \cos x + x \sin x)}{(3x \cos x + 4xe^x)^2}$

$$- \frac{(e^x \sin x - x \cos x)[3(\cos x - x \sin x) + 4e^x + 4xe^x]}{(3x \cos x + 4xe^x)^2}$$

u. $y = (4x + 3)^3$

Ans: $\dfrac{dy}{dx} = 3(4x+3)^2(4) = 12(4x+3)^2$

v. $y = (x^2 - 3x - 2)^5$

Ans: $\dfrac{dy}{dx} = 5(x^2 - 3x - 2)^4(2x - 3)$

w. $f(x) = \sqrt{4x^3 - 2x^2 + 4x + 5}$

Ans: $f'(x) = \dfrac{12x^2 - 4x + 4}{2\sqrt{4x^3 - 2x^2 + 4x + 5}} = \dfrac{6x^2 - 2x + 2}{\sqrt{4x^3 - 2x^2 + 4x + 5}}$

x. $f(x) = \sin(4x)$

Ans: $f'(x) = 4\cos 4x$

y. $f(x) = e^{3x-1}$

Ans: $f'(x) = 3e^{3x-1}$

z. $f(x) = \sin(3x^2 + 1)$

Ans: $f'(x) = 6x\cos(3x^2 + 1)$

aa. $f(x) = \cos(\sin(4x - 2))$

Ans: $f'(x) = -4\sin(\sin(4x - 2))(\cos(4x - 2))$

bb. $y = \sin(e^{3x})$

Ans: $\dfrac{dy}{dx} = \cos\left(e^{3x}\right) 3e^{3x}$

cc. $y = e^{x^2}$

Ans: $\dfrac{dy}{dx} = 2xe^{x^2}$

dd. $y = \sin^3 x$

Ans: $\dfrac{dy}{dx} = 3\sin^2 x\cos x$

ee. $y = \cos^4(x^2 + 1)$

Ans: $\dfrac{dy}{dx} = -8x\cos^3(x^2 + 1)(\sin(x^2 + 1))$

ff. $y = e^{\cos^2 x - 1}$

Ans: $\dfrac{dy}{dx} = -2\sin x\cos xe^{\cos^2 x - 1}$

gg. $f(x) = \tan^3\left(e^{2x-1}\right)$

Ans: $f'(x) = 6\tan^2(e^{2x-1})(\sec^2(e^{2x-1}))(e^{2x-1})$

hh. $f(x) = \cos(2x^3 - 4)e^{\cos x}$

Ans: $f'(x) = e^{\cos x}(-\sin(2x^3 - 4)(6x^2)) + \cos(2x^3 - 4)(e^{\cos x})(-\sin x)$

ii. $f(x) = x\tan(x^2 + 1)e^{3x}$

Ans: $f'(x) = \tan(x^2 + 1)e^{3x} + x[\sec^2(x^2 + 1)(2x)e^{3x} + 3e^{3x}\tan(x^2 + 1)]$

jj. $f(x) = x^2 e^{\sin x^2} + e^x \sin(\cos(3x-1))$

Ans: $f'(x) = 2xe^{\sin(x^2)} + x^2(e^{\sin(x^2)})(\cos(x^2))(2x) + e^x(\cos(\cos(3x-1)))(-\sin(3x-1)(3)) + \sin(\cos(3x-1))e^x$

kk. $y = \dfrac{e + 3xe^{x^2-2}}{e - 4\sin x \cos^2 x}$

Ans: $\dfrac{dy}{dx} = \dfrac{(e - 4\sin x\cos^2 x)[3xe^{x^2-2}(2x) + 3e^{x^2-2}]}{(e - 4\sin x\cos^2 x)^2} - \dfrac{(e + 3xe^{x^2-2})[-4\cos^3 x + 8\sin^2 x\cos x]}{(e - 4\sin x\cos^2 x)^2}$

ll. $y = \dfrac{2x\cos(x^2 - 4x) + e^{3x^2-1}}{3x^2 - 4e^{3x\cos^2(4x+1)} + \sin^3(5x+1)}$

Ans: $\dfrac{dy}{dx} = \dfrac{[3x^2 - 4e^{3x\cos^2(4x+1)} + \sin^3(5x+1)][2\cos(x^2-4x) + 2x(-\sin(x^2-4x))(2x-4) + e^{3}}{(3x^2 - 4e^{3x\cos^2(4x+1)} + \sin^3(5x+1))^2}$

$\dfrac{[2x\cos(x^2-4x) + e^{3x^2-1}][6x - 4e^{3x\cos^2(4x+1)}(3\cos^2(4x+1) + 3x(2\cos(4x+1))(-\sin(4x+1))(4)}{(3x^2 - 4e^{3x\cos^2(4x+1)} + \sin^3(5x+1))^2}$

mm. $y = \arccos x$

Ans: $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$

oo. $y = \ln x$

Ans: $\dfrac{dy}{dx} = \dfrac{1}{x}$

pp. $y = \ln(x^2 + 4x - 1)$

Ans: $\dfrac{dy}{dx} = \dfrac{2x+4}{x^2+4x-1}$

qq. $y = \arcsin(x^2 - 5x + 1)$

Ans: $\dfrac{dy}{dx} = \dfrac{2x-5}{\sqrt{1-(x^2-5x+1)^2}}$

rr. $y = \ln\dfrac{1}{x}$

Ans: $\dfrac{dy}{dx} = -\dfrac{1}{x}$

ss. $y = \ln(\ln x)$

Ans: $\dfrac{dy}{dx} = \dfrac{1}{x\ln x}$

tt. $y = \dfrac{1}{\ln x}$

Ans: $\dfrac{dy}{dx} = -\dfrac{1}{x(\ln x)^2}$

uu. $y = \arctan^2(\ln 3x)$

Ans: $\dfrac{dy}{dx} = 2\arctan(\ln(3x))\dfrac{1}{x[1+(\ln(3x))^2]}$

vv. $y = \dfrac{\ln(3x)}{\sin(\ln x^2)}$

Ans: $\dfrac{dy}{dx} = \dfrac{\sin(\ln x^2)\frac{1}{x} - \ln(3x)(\cos(\ln(x^2))\frac{2}{x})}{(\sin(\ln(x^2)))^2}$

ww. $y = \dfrac{3e^{4x}\sin^{-1}(3x^2+1)}{\ln x \sin 5x + 3x^2\cos^2 x}$

Ans: $\dfrac{dy}{dx} = \dfrac{(\ln x\sin(5x) + 3x^2\cos^2 x)[12e^{4x}(\sin^{-1}(3x^2+1) + 3e^{4x}\left(\frac{6x}{\sqrt{1-(3x^2+1)^2}}\right)}{(\ln x\sin(5x)+3x^2\cos^2 x)^2}$

$$-\frac{(3e^{4x}\sin^{-1}(3x^2+1))\left[\frac{1}{x}\sin(5x) + 5\ln x\cos(5x) + 3(2x\cos^2 x - 2x^2\cos\right.}{(\ln x\sin(5x)+3x^2\cos^2 x)^2}$$

2. Find the equation of the line tangent to the given function $f$ at the given point $x = a$:

a. $f(x) = 2$, $a = 3$

Ans: $y = 0$

b. $f(x) = 5x - 1$, $a = -2$

Ans: $y = 5x - 1$

c. $f(x) = x^2 - 2x + 4$, $a = 1$

Ans: $y = 3$

d. $f(x) = \sin x$, $a = \dfrac{\pi}{6}$

Ans: $y - \dfrac{1}{2} = \dfrac{\sqrt{3}}{2}\left(x - \dfrac{\pi}{6}\right)$

e. $f(x) = \ln x$, $a = 3$

Ans: $y - \ln 3 = \dfrac{1}{3}(x-3)$

f. $f(x) = e^{3x}$, $a = -2$

Ans: $y - \dfrac{1}{e^6} = \dfrac{3}{e^6}(x+2)$

g. $f(x) = xe^x - \ln 4x$, $a = 1$

Ans: $y - (e - \ln 4) = (2e-1)(x-1)$

3. a. Define a function that has a left and right handed limit at a point $a$, but $f$ does not have a limit at $a$.

b. Define a function that has a limit at a point $b$ but is undefined at $b$.

c. Define a function that has a limit at point $c$, is defined at $c$, but is discontinuous at $c$.

d. Define a function that is continuous at point $d$ but is not differentiable at $d$.

e. Define a function that is first differentiable at point $p$ but is not second differentiable at point $p$.

f. Draw the graph of a function which has **all** of the properties in a, b, c, d, e, above.

Ans: There are more than one correct answer. The function defined below satisfies all conditions a through e:

$$f(x) = \begin{cases} -3 & \text{if} \quad x \leq -2 \\ -2 & \text{if} \quad -2 < x < -1 \\ \frac{\sin x}{x} & \text{if} \quad -1 \leq x < 1 \\ \frac{x^2-4}{x-2} & \text{if} \quad 1 \leq x < 3, x \neq 2 \\ 5 & \text{if} \quad x = 2 \\ \sqrt[3]{x-4} & \text{if} \quad 3 \leq x \leq 5 \\ \sqrt[3]{(x-6)^4} & \text{if} \quad 5 < x \end{cases}$$

$\lim_{x \to -2^-} f(x) = -3$

$\lim_{x \to -2^+} f(x) = -2$

$\lim_{x \to -2} f(x)$ Does not exist.

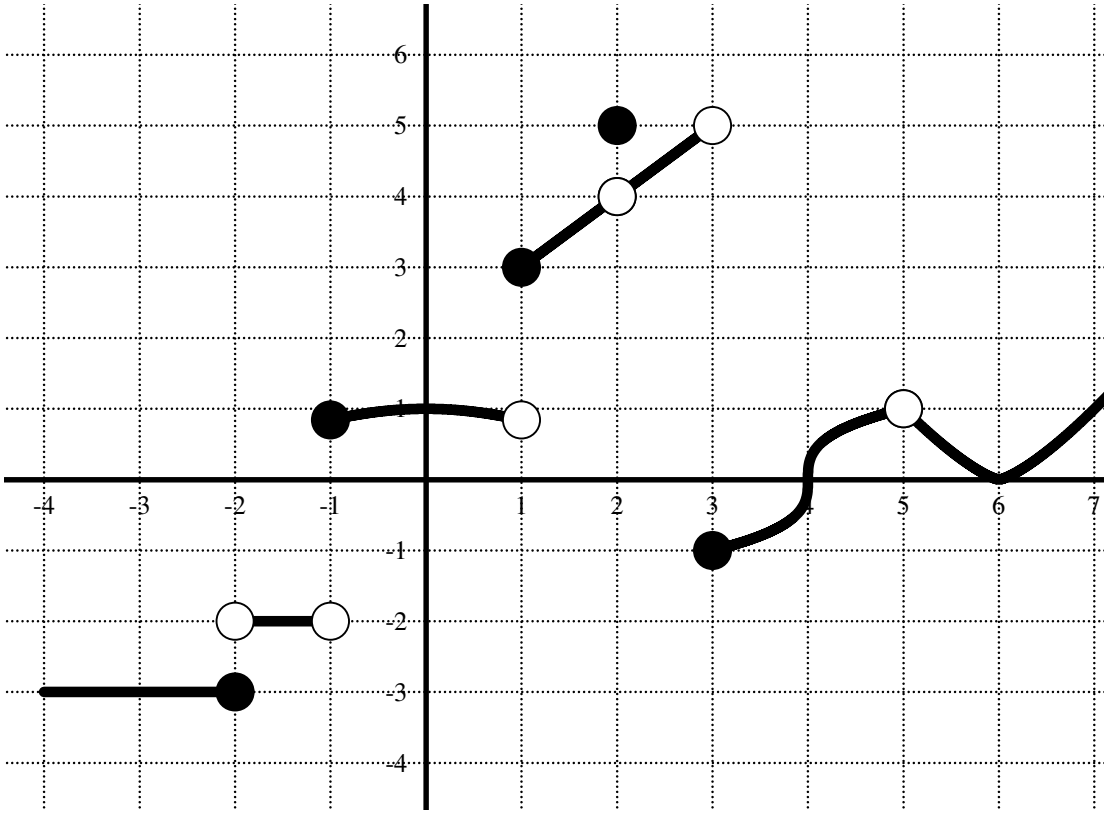
$\lim_{x \to 0} f(x) = 1$, $f(0)$ is undefined.

$\lim_{x \to 2} f(x) = 4$, $f(2) = 5$, $f$ is discontinuous at 5.

$f$ is continuous at $x = 4$ but not differentiable at 4.

$f$ is first differentiable at $x = 6$ but not second differentiable at 6.

The graph of $f$ is drawn below.




4. Think About:

a. Given the graph of the original function $f(x)$, how can you draw the graph of its derivative, $f'(x)$?

b. Given the graph of the derivative of a function, $f'(x)$, how can you draw the graph of the original function, $f(x)$?

5. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ by implicite differentiation:

a. $y \sin x = x \cos y$

Ans: $\dfrac{dy}{dx} = \dfrac{\cos y - y \cos x}{\sin x + x \sin y}$

b. $x^2 y + y^2 x = 5$

Ans: $\dfrac{dy}{dx} = \dfrac{-2xy - y^2}{x^2 + 2xy}$

c. $2y - 4x \sin x = e^y$

Ans: $\dfrac{dy}{dx} = \dfrac{4 \sin x + 4x \cos x}{2 - e^y}$

d. $\ln(xy) + \dfrac{x}{y} = 1$

Ans: $\dfrac{dy}{dx} = \dfrac{-y^2 - xy}{xy - x^2}$

e. $e^{xy} + \ln(x + y) = 2xy$

Ans: $\dfrac{dy}{dx} = \dfrac{2y(x+y) - ye^{xy}(x+y) - 1}{-2x(x+y) + xe^{xy}(x+y) + 1}$

6. Find $f''(x)$ and $f'''(x)$:

a. $f(x) = 4$

Ans: $f'(x) = 0$ $f''(x) = 0$ $f'''(x) = 0$

b. $f(x) = 3x^2$

Ans: $f'(x) = 6x$ $f''(x) = 6$ $f'''(x) = 0$

c. $f(x) = e^x$

Ans: $f'(x) = e^x$ $f''(x) = e^x$ $f'''(x) = e^x$

d. $f(x) = \sin x$

Ans: $f'(x) = \cos x$ $f''(x) = -\sin x$ $f'''(x) = -\cos x$

e. $f(x) = \ln x$

Ans: $f'(x) = \dfrac{1}{x}$ $f''(x) = -\dfrac{1}{x^2}$ $f'''(x) = \dfrac{2}{x^3}$

f. $f(x) = \arccos x$

Ans: $f'(x) = -\dfrac{1}{\sqrt{1-x^2}}$ $f''(x) = -\dfrac{x}{(1-x^2)^{3/2}}$ $f'''(x) = -(1-x^2)^{-3/2} - 3x^2(1-x^2)^{-5/2}$

g. $f(x) = \dfrac{x^2+1}{x-1}$

Ans: $f'(x) = \dfrac{x^2-2x-1}{(x-1)^2}$ $f''(x) = \dfrac{4}{(x-1)^3}$ $f'''(x) = -\dfrac{12}{(x-1)^4}$

h. $f(x) = e^{\sin x}$

Ans: $f'(x) = \cos x e^{\sin x}$ $f''(x) = e^{\sin x}(\cos^2 x - \sin x)$ $f'''(x) = e^{\sin x}(\cos^3 x - 3\sin x \cos x - \cos x)$

i. $f(x) = \sqrt{x}$

Ans: $f'(x) = \dfrac{1}{2\sqrt{x}}$ $f''(x) = -\dfrac{1}{4\sqrt{x^3}}$ $f'''(x) = \dfrac{3}{8\sqrt{x^5}}$

j. $f(x) = x^{5/4}$

Ans: $f'(x) = \dfrac{5}{4}x^{1/4}$ $f''(x) = \dfrac{5}{16}x^{-3/4}$ $f'''(x) = -\dfrac{15}{64}x^{-7/4}$

7. Find the linearization of the given function at the given point $a$:

a. $f(x) = 2x - 3$, $a = 2$

Ans: $L(x) = 2x - 3$, the linearization of a line is itself.

b. $f(x) = 2x - 3$, $a = 4$

Ans: $L(x) = 2x - 3$, the linearization of a line is itself.

c. $f(x) = 2x - 3$, $a = 10$

Ans: $L(x) = 2x - 3$, the linearization of a line is itself.

d. $f(x) = 4x^2 - x + 3$, $a = -1$

Ans: $L(x) = -9x - 1$

e. $f(x) = \sin x$, $a = 0$

Ans: $L(x) = x$

f. $f(x) = \cos x$, $a = \dfrac{\pi}{4}$

Ans: $L(x) = \dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}\left(x - \dfrac{\pi}{4}\right)$

g. $f(x) = e^x$, $a = 0$

Ans: $L(x) = x + 1$

h. $f(x) = e^x$, $a = 1$

Ans: $L(x) = e + e(x - 1)$

i. $f(x) = \ln x$, $a = 1$

Ans: $L(x) = x - 1$

j. $f(x) = \ln x$, $a = 2$

Ans: $L(x) = (\ln 2) + \dfrac{1}{2}(x - 2)$

k. $f(x) = \arctan x$, $a = 0$

Ans: $L(x) = x$