Math 3A Fall 2015 Extra Credit Problem Set

P.R. Zulkowski *

May 20, 2015

1 Introduction and Instructions

The purpose of this extra credit problem set is to compute the definite integral $\int_0^a x^m dx$ for $a > 0$ and $m \in \mathbb{N}$ using Riemann sums. You do NOT need to attempt all problems to receive partial extra credit. Full credit for attempting any given problem in this set will be given ONLY when your argument is written out neatly, legibly, logically and correctly. I reserve the right to award credit based on my assessment of how well you adhere to these criteria. I reserve the right to alter the content of the problem set below when necessary and at any time prior to the due date of **Wednesday, December 9th**. You may work on these problems in groups, but the work you submit must be your own.

You may find it useful to read Appendix E: Sigma notation in the text. Most of the problems below can be attempted after reading this appendix: you need not wait until we reach the section on integrals to begin this problem set!

*pzulkowski@peralta.edu

2 A Necessary Limit

In order to compute $\int_0^a x^m dx$ using Riemann sums, we first need to establish the following identity:

$$\lim_{N\to\infty} \frac{1}{N^{m+1}} \sum_{j=0}^{N-1} j^m = \frac{1}{m+1}. \tag{2.1}$$

We will work towards this identity in stages.

1. **(1 point)** First, let's assume that $m = 0$. Show that the identity Eq. (2.1) holds in this case.

2. **(2 points)** Assume that $m = 1$. Using the identity $\sum_{j=0}^{N-1} j = \frac{N(N-1)}{2}$, show that Eq. (2.1) holds in this case as well.

3. **(7 points)** For $m \geq 2$, we need to use the idea of a telescoping sum.

   (a) **(2 points)** Show that the sum $\sum_{j=0}^{N-1} \left\{(1+j)^3 - j^3\right\}$ equals N^3. Essentially, most of the terms in the sum cancel, leaving just N^3. The sum collapses like an old style telescope.

   (b) **(4 points)** Expand the cube in the above sum to show that $\sum_{j=0}^{N-1} j^2 = \frac{N(N-1)(2N-1)}{6}$.

   (c) **(1 point)** Use the formula from the previous problem to show that Eq. (2.1) holds for $m = 2$.

4. **(18 points)** At this point, it may be helpful to read up on the Principle of Induction, in particular, the Principle of Complete Induction. Wikipedia has nice article on this topic. You do not need a thorough understanding of induction to complete this assignment.

   The purpose of this problem is to prove Eq. (2.1) for $m \geq 3$.

(a) **(2 points)** Show that the sum $\sum_{j=0}^{N-1} \left\{(1+j)^{m+1} - j^{m+1}\right\}$ telescopes to N^{m+1}.

(b) **(8 points)** Expand $(1+j)^{m+1}$ using the Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \tag{2.2}$$

where

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \tag{2.3}$$

and $j! = j \times (j-1) \times (j-2) \ldots 2 \times 1$ is the factorial of j.

Show that the sum from the previous step reduces to

$$N^{m+1} = \binom{m+1}{m} \sum_{j=0}^{N-1} j^m + \sum_{k=0}^{m-1} \binom{m+1}{k} \sum_{j=0}^{N-1} j^k. \tag{2.4}$$

(c) **(8 points)** We now invoke the Principle of Complete Induction to prove the identity Eq. (2.1). Let's assume that $\lim_{N\to\infty} \frac{1}{N^{l+1}} \sum_{j=0}^{N-1} j^l = \frac{1}{l+1}$ holds true for all integers between 0 and $m-1$. It is certainly true for $l = 0, 1, 2$ as demonstrated above. With this inductive assumption, we need to show that $\lim_{N\to\infty} \frac{1}{N^{m+1}} \sum_{j=0}^{N-1} j^m = \frac{1}{m+1}$ and the proof will be complete by induction.

Use Eq. (2.4) to complete the inductive argument.

3 The Definite Integral

5. **(2 points)** To evaluate $\int_0^a x^m dx$, let us break the interval $[0, a]$ up into N equal pieces. Use the left endpoint Riemann sum definition of the definite integral to show that

$$\int_0^a x^m dx = \lim_{N\to\infty} \left(\frac{a^{m+1}}{N^{m+1}}\right) \sum_{j=0}^{N-1} j^m \tag{3.1}$$

and use Eq. (2.1) to prove that

$$\int_0^a x^m dx = \frac{a^{m+1}}{m+1}. \tag{3.2}$$

In this problem set, we evaluated a definite integral by computing the limit of a Riemann sum. However, in most applications this will not be possible. Many integrations you will see in calculus will depend in some way on the Fundamental Theorem of Calculus. You will learn many integration techniques in Math 3B. However, even these techniques will fall short in real world integrations. There, we often employ numerical methods which originate with the Riemann sum.