Contents

Chapter 1

Differential Equations Basics

1.1 Modeling with Differential Equations: Boyce and DiPrima (1.1)

Generally speaking, **differential equations** are equations involving an unknown function and its derivatives. Differential equations are extremely useful tools in modeling real-world systems. Assuming these systems vary continuously in time, differential equations give us the power to predict the future course of such a system given its present state. If the system also depends on space, you can expect a differential equation to govern this behavior as well.

Definition: A **differential equation** is an equation that contains an unknown function and one or more of its derivatives. The **order** of a differential equation is the order of the highest derivative that occurs in the equation.

Definition: A function is called a **solution** of a differential equation if the equation is satisfied when the function and its derivatives are substituted into the equation.

In many ways differential equations are analogous to algebraic equations. In an algebraic equation like $2x^2 + x - 1 = 0$, we have an unknown x and we would like to know all explicit x which satisfy this equation. For differential equations, however, the unknown is a *function* and the equation involves the unknown and its derivatives.

Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things happen. For example, if we have a particle moving under the influence of a force field $\boldsymbol{F}(\boldsymbol{x})$, then Newton's second law tells us that

$$\frac{d^2\boldsymbol{x}}{dt^2} = \frac{1}{m}\boldsymbol{F}(\boldsymbol{x}(t))$$

where $\boldsymbol{x}$ represents the position vector of the particle in 3-dimensional space as a function of time.

Solving this differential equation would allow us to find the position of the particle at any later time assuming we know its position and velocity at a given initial time.

Another example comes from predator-prey models. For instance, consider populations of tundra wolves, given by $W(t)$, and caribou, given by $C(t)$, in northern Canada. The interaction between the two populations is modeled by

$$\frac{dC}{dt} = aC - bCW \ , \ \frac{dW}{dt} = -cW + dCW$$

where a, b, c, d are positive constants.

Differential equations form a broad sub-discipline of mathematics with a long and interesting history. In this course we will develop tools which will help us solve a large class of differential equations relevant for biology, sociology, economics, physics and engineering.

Before we launch into a discussion of solutions of differential equations and their origins, we need to establish some definitions and names. Just like the word "animal" describes a vast number of organisms on the planet, it is not terribly helpful if you want to be more descriptive (zebras, elephants, fish and birds are all animals, but they are all very different). In short, we need a way of **classifying** differential equations.

1.2 Classification of Differential Equations: BD (1.3)

As a great teacher once told me, it is important to organize your ignorance. In this section we introduce some terminology that will provide a framework for understanding the various types and features of differential equations.

Ordinary versus Partial Differential Equations. One of the most obvious classifications is based on whether the unknown function depends on a single independent variable or several. In the first case, only ordinary derivatives appear in the differential equation, and it is said to be an **ordinary differential equation** or ODE for short. In the second case, the derivatives are partial derivatives, and the equation is called a **partial differential equation** or PDE for short.

All of the examples we have considered so far are ODEs. An example of a PDE would be the one-dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

where $u(x,t)$ represents the disturbance of the wave medium from equilibrium. Here both position and time are variables.

Arguably partial differential equations provide more realistic models of the world. We need to understand ODEs before we can tackle PDEs which is a vastly more complicated subject. You have to learn how to walk before you can run!

Systems of Differential Equations. Another classification of differential equations depends on the number of unknown functions that are involved. If there is a single function to be determined, then one equation is sufficient. However, if there are two or more unknown functions (like in the Caribou/Tundra Wolf model), then a system of differential equations is required.

Order. As we will see, the order of a differential equation greatly impacts how we approach its analysis. The techniques used to solve first order equations are distinct from those used to approach a second order equation, etc. To keep things simple, we always assume in this course that it is possible to write a differential equation of order n in the form

$$y^{(n)} = f(t, y, y', \ldots, y^{(n-1)})$$

Linear versus Nonlinear. A crucial classification of differential equations is whether they are linear or nonlinear. The ordinary differential equation

$$F(t, y, y', \ldots) = 0$$

is said to be **linear** if F is a linear function of its variables. If this is not true of F, then the differential equation is said to be **nonlinear**.

An example of a nonlinear differential equation is the mathematical model of pendulum motion:

$$\theta''(t) = -\frac{g}{l} \sin \theta(t)$$

Here a bob attached to a thin light rod of length l makes an angle $\theta(t)$ with respect to the vertical. The dynamics of θ are governed by Newton's second law.

We will only skim the surface of solving nonlinear ODEs in this course. While many natural phenomena can be understood using linear differential equations, the vast majority of interesting systems are nonlinear. There is a whole branch of math devoted to understanding these nonlinear equations called Dynamical Systems Theory. You probably know it better as Chaos Theory.

1.3 Modeling with Differential Equations Continued: BD (1.1)

Now let's return to section 1.1 and look at how some differential equations arise when we model real-world phenomena. Constructing **mathematical models** is an art form that requires some practice to develop. I will try to give you a sense of how the process works by illustrating the construction of a few models taken from a broad range of disciplines.

In broad strokes, the three main steps in constructing a mathematical model are:

1. Identify important quantities.
2. Identify relevant principles.
3. Put it all together.

I would argue that there is a fourth step that we won't be able to implement in the short amount of time allotted to our course: this step would be to compare to experiment and refine as needed. The pattern of science is to propose a model based on observation or intuition, solve that model and then compare with experiments. If the model disagrees with experiment, scientists revise the theory to bring it in line with the observable world.

Model from Psychology: Learning Curves

Psychologists interested in learning theory study **learning curves**. A learning curve is the graph of a function $P(t)$, the performance of someone learning a skill as a function of the training time t. The derivative $P'(t)$ represents the rate at which performance improves. The performance function $P(t)$ represents the results of a hypothetical standardized test which is theoretically administered to the student throughout each moment of the learning process. Obviously, this is an approximation: testing only occurs at discrete time intervals (in fact, we only have five exams total!). But the approximation is a very good one for understanding general learning behavior. It is particularly helpful in the arena of so-called Machine Learning related to Artificial Intelligence.

If M is the maximum level of performance of which the learner is capable (think 100 percent on the test), I will explain why the differential equation

$$\frac{dP}{dt} = k\,(M - P)$$

where k is a constant is a reasonable model for learning. The underlying assumption of the model is that once a student performs the maximum M, the student has nothing left to learn and so the rate of change of performance should be 0; *i.e.* the student has mastered the material. The equation given above is the simplest differential equation expressing this assumption. We'll learn how to solve this equation soon, but qualitatively speaking, assuming the student starts with 0 prior knowledge, the graph looks similar to a square root function:

concave down, approaches $P = M$ asymptotically as $t \to \infty$. Assuming no prior knowledge means that $P(0) = 0$; *i.e.* we are specifying the **initial condition** of the model.

Model from Sociology/Biology: The Logistic Equation.

The first time you were exposed to the idea of a differential equation was back in precalculus. After your precalculus teacher introduced the number e, you most likely learned that one of its applications is in the modeling of simple bacteria growth.

Assume that $P(t)$ represents the population of bacteria growing in a culture with unlimited space and unlimited resources (thus identifying the population of bacteria as the important quantity). Given the way bacteria reproduce, it stands to reason that the instantaneous rate of change of the population should be proportional to the current population. In terms of an equation,

$$\frac{dP}{dt} = kP(t)$$

where k is a positive constant determined by observation. It turns out that the exponential function is the only non-trivial function satisfying this equation and so bacterial growth is exponential in time.

Of course, it is unrealistic to expect that the bacteria would continue to grow unimpeded forever: they would quickly either run out of space or food or both. So the petrie dish has a **carrying capacity**. The simplest realistic way to model such restricted population growth is with the **logistic differential equation**:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right).$$

(Please refer to section 2.5 for more details and models of population growth.)

Let's see how to construct this model. First, if P is relatively small, then $1 - \frac{P}{M} \approx 1$ and the differential equation becomes $\frac{dP}{dt} \approx kP$. Thus, when the population is small, the growth is exponential since the carrying capacity of the environment is not felt by the population yet.

Second, if $P < M$, then $\frac{dP}{dt} > 0$ meaning the population is growing; if $P > M$, then

$\frac{dP}{dt} < 0$ meaning the population is declining. This is the important feature we need to model a realistic host environment: when the population becomes too large, the species starts to die off to compensate for the finite resources of the environment.

Third, there are special solutions where nothing happens: $P = 0$ (a population cannot spontaneously appear out of nowhere) and $P = M$. These two constant solutions are referred to as **equilibrium solutions** since the populations do not change with time.

The logistic equation is the simplest model capturing all of these intuitive requirements. It is of course not unique and only applies in very simple situations. However, once again (for emphasis), if your model fails to capture what you see in experiment, you go back to the drawing board and revise! To err is human and understanding nature takes a great deal of trial and error.

Model from Physics: Harmonic Oscillator

Another example of a model system described by a differential equation is the model for the motion of a spring. The resulting solution (simple harmonic motion) is a topic often used in precalculus to motivate the introduction of sinusoidal functions.

Suppose we have a block attached to a spring which moves in one dimension without frictional damping. A combination of Newton's second law and Hooke's law gives us the differential equation

$$x''(t) = -\frac{k}{m}x(t)$$

where m is the mass, k is the spring constant and $x(t)$ represents the displacement from the equilibrium position as a function of time t. This model of spring motion comes from empirical observation. We will see that the most general solution to this equation is a linear combination of sine and cosine functions. It is an example of a **second order differential equation**:

Model from Physics: Free-falling Object with Drag

Suppose now we wish to model a small object of mass m falling from a great height. Let's model this situation with a differential equation. Specifically, I will be more careful to elaborate on the steps involved in constructing this particular model to give you a better sense of the overall process in the sciences.

Step 1: Identify what's Important. To say the world is complicated is an understatement. When we make a model to understand some system (which is part of our complicated and chaotic world), we must identify what is important and must be kept and what is negligible and can be forgotten.

In the example we are considering the object will have some irregular shape. You could choose to throw a sandwich, a ball, your textbook, etc. from the great height. Even if you chose an object with a simple shape, at the atomic level everything is rough and complex (except crystals but let's not worry about them). If we worry about the detailed shape of the object, we would be stuck in mathematical hell for a very long time without much progress.

So we step back and ask: is the shape of the object important for our purposes? If we want a general idea of how an object behaves as it falls from a great height, the answer is no. Notice though that we are trading precision for tractability. You shouldn't feel bad about doing this because, first it allows us to make progress and, second you would be amazed at how much we have progressed in physics by not sweating the details.

To make a long story short, we can safely assume that we can neglect the shape of our object. We continue with the process of identifying what is important and what is not. The position of the object above the ground is potentially important and so is the velocity.

When you identify an important quantity you want to give it a name: let's label the height about the ground h and the velocity v. You also want to make sure you set a specific system of units. Being consistent with your units is of course logically important but it also helps you identify mistakes or potential simplifications through **dimensional analysis**. We'll have more to say about this later. For now, let's fix our units for h to be meters in which case it is natural to select meters per second for v.

Generally your choice of units should reflect the relative size of the important quantities.

For instance, since we have an intuitive notion that the total time it takes an object to fall will be measured on the order of seconds or minutes, it wouldn't make sense to measure time in units like years or eons. Also, our height is great, but not that great: the earth's atmosphere only extends about 50 miles about the surface of the Earth. So meters or kilometers would be acceptable units but lights years or megaparsecs would be a bit extreme.

When our object falls, it falls through the earth's atmosphere. The atmosphere is turbulent, chaotic, unpredictable in many ways (think of how accurate weather forecasts are!). Should we worry about these things? Well, if we did, we would be stuck in mathematical hell for a while with little to show for our efforts. So let's forget about the complexity of the atmosphere. Also, gravity tends to die off as we move away from a gravitating body like the Earth. Should we take this into account? Well, not really, since our height is great, but not that great: we aren't launching anything into space in this experiment.

Hopefully you get a sense of how this process works: don't sweat the details and try to identify a small set of variables which are important. If you end up making too many simplifying assumptions, your model won't reflect your experiments. This just means you have to go back and find what you left out and repeat the model-building process. It happens to the best of us!

Step 2: Identify the Physical Principles. Steps 1 and 2 are not usually cleanly divided: one influences the other. But for the most part the division of labor holds. In this step you need to identify what laws and principles of the universe come into play. Science is an iterative process and even on the cutting edge of research you very rarely have to start from scratch.

In the context of our current problem, we would identify Newton's second law as being important since it relates the total force on an object to its acceleration. Based on what we concluded from the first step, the only important forces acting on our object are the forces exerted by the atmosphere and the earth via gravity.

Looking up such forces in a standard physics book we see that the force of the atmosphere is most easily modeled by $-\gamma v$ where γ is a constant and the minus sign indicates that the force works **against** motion. Moreover, if we assume that our object always remains close to the earth, the gravitational force on our object is constant: mg.

Step 3: Put It All Together. In this step, we synthesize our variables and our principles into one or several differential equations. In this example our model will consist of (initially) two differential equations:

$$m\frac{d^2h}{dt^2} = -\gamma v + mg \ , \ \frac{dh}{dt} = v$$

The first equation comes from Newton's second law and the second just comes from the definition of velocity. This is an example of a **system** of differential equations which are **coupled**; i.e. the two equations involve both variables.

Fortunately, we can decouple them in a natural way. Warning: this will generally not be possible and we'll see later in the course how to handle the more general situation. We can write down a single equation simply by plugging the second into the first:

$$m\frac{dv}{dt} = -\gamma v + mg$$

Once we know $v(t)$ we can integrate it to get $h(t)$ if we so desire.

We'll return to these ideas throughout the course whenever we need to build a mathematical model. Let's resume our general discussion of differential equations.

1.4 Concluding Remarks: BD (1.4)

To conclude this section, I will list three important questions which will concern us throughout the course:

1. Does a given differential equation actually have a solution? This is known as the existence problem or existence question. We will develop some theorems which will tell us under what conditions we can expect a differential equation to have a solution. Clearly, if our model doesn't have a solution our model is no good!

2. If our differential equation has at least one solution, how many distinct solutions are

there? This is known as the uniqueness problem. This question is relevant to realistic models because we expect our model to make specific and unambiguous predictions about the future.

3. If we know our differential equation has a solution, how do we find it? This is the $ 64,000 question. The bulk of our time will be devoted to developing analytical techniques for finding solutions to ODEs. Worst-case scenario: if you know a solution exists but can't find it using what you learn from this course, you can always find a numerical approximation using a computer and some numerical algorithm like Euler's method.

Chapter 2

First Order Differential Equations

2.1 Introduction

These lectures discuss differential equations of the form $y' = f(t, y)$; *i.e.* single, first order differential equations. We have looked at a few models giving rise to first-order differential equations already: the logistic equation, the learning curve equation and the free-fall equation. It would be nice to have an explicit formula for solutions of such differential equations. For these models integration techniques learned in Calculus II allow us to find such formulae since the underlying differential equations are **separable** (see below). However, for general $f(t, y)$, it will not be possible to find such formulae and we'll have to resort to numerical schemes like Euler's method. (See lecture notes on Numerical Methods.)

2.2 Separable Equations: BD (2.2)

Definition: A **separable equation** is a first-order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x times a function of y:

$$\frac{dy}{dx} = \frac{g(x)}{f(y)}$$

In differential form, this implies that

$$g(x)dx = f(y)dy$$

and so we may integrate both sides to obtain a formula for y in terms of x or vice versa:

$$\int g(x)dx = \int f(y)dy. \tag{2.2.1}$$

It is typically the case that the relationship will be implicit. In other words, it may not be possible to find a formula for y given only in terms of x. However, the resulting integration is still very useful and there are numerical schemes like Newton's method that can help us approximately invert the implicit relationship should it arise. Let's solve the logistic, learning curve and free-fall models of the previous lecture since they are all (as I hope you noticed) separable.

Recall the model for an object falling from a great height:

$$m\frac{dv}{dt} = -\gamma v + mg \tag{2.2.2}$$

This a separable differential equation since we can rewrite this equation in differential form as

$$\frac{dv}{-(\gamma/m)\,v + g} = dt$$

Integrating both sides of this equation, we find

$$-\frac{m}{\gamma}\ln|g - (\gamma/m)\,v| = t + C,$$

which gives an implicit relationship between v and t. I'll take the opportunity here to point out a strange but useful notation in differential equations: any modification to C (the constant of integration) during the solution process is lumped into C's definition. Here's an example of what I mean by this statement. If we exponentiate both sides of the above expression after dividing both sides by $\frac{m}{\gamma}$, it would be fine to write the equation as

$$|g - (\gamma/m)\,v| = Ce^{-\frac{\gamma}{m}t}$$

where the "C" in this line is not technically the same as the "C" above! You should think of "C" as a placeholder for something that is constant and will be fixed later by initial conditions.

If we start from rest, then $v(0) = 0$ and so

$$|g - (\gamma/m)\, v| = g e^{-\frac{\gamma}{m}t}.$$

At this point, we need to determine how to solve this absolute value equation for $v(t)$. Since the falling object should be accelerating (the velocity increases as the object falls) to terminal velocity, the only reasonable answer is

$$(\gamma/m)\, v - g = g e^{\frac{\gamma}{m}t}$$

or

$$v(t) = \frac{mg}{\gamma}\left(1 - e^{-\frac{\gamma}{m}t}\right).$$

Notice that after a very long time has passed the velocity is essential $\frac{mg}{\gamma}$, the so-called terminal velocity.

Another example from the previous lecture which is separable is the (normalized) logistic equation

$$\frac{dP}{dt} = P(1-P)$$

By normalized, I mean we take $k = 1$ and $M = 1$ just to keep the math as simple as possible. We employ the partial fraction decomposition method to obtain an exact solution to this equation. Specifically, we write the differential equation as

$$\frac{dP}{P(1-P)} = dt$$

and use the method of partial fractions to write

$$\frac{1}{P(1-P)} = \frac{1}{(1-P)} + \frac{1}{P}. \tag{2.2.3}$$

Using the partial fraction decomposition we may integrate both sides of the differential equation to find

$$\ln\left|\frac{P}{1-P}\right| = t + C.$$

To go further with this problem, we need to specify initial conditions. If, for instance, $P(0) = 0.5$, then we know $C = 1$ and furthermore

$$P(t) = \frac{1}{1 + e^{-t}}$$

is the correct solution. Notice that the model predicts that the population will grow from 0.5 to 1 after a very long time has passed. This makes sense because the system tends to equilibrium for long times.

An important fact to note (which I'll come back to later when we discuss the existence and uniqueness theorem) is that in both the logistic model and the free-fall model, there are equilibrium solutions (the terminal velocity in the case of free fall and the carrying capacity in the case of the logistic model) as well as solutions that vary in time. It is easy to check that the time-varying solutions never reach or cross the equilibrium solutions. In general, for well-behaved differential equations, solutions corresponding to distinct initial conditions can never cross. More on this later.

Example: The Fitzhugh-Nagumo neuron model is also separable. Use this observation to find explicit solutions for

$$\frac{dv}{dt} = -v\left[v^2 - (1+a)v + a\right]$$

with $a = \frac{1}{2}$ and compare with the conclusions we reached using the direction field analysis technique.

Partial ANS: Use partial fractions to write $\frac{1}{v(v^2-3/2v+1/2)} = \frac{2}{v} + \frac{-2v+3}{v^2-\frac{3}{2}v+\frac{1}{2}}$ and then integrate using the separable-equation technique to find

$$\frac{v^2\left(1-v\right)^2}{(1-2v)^4} = Ce^{-t}$$

Now certain choices have to be made. If we are at the equilibrium values $v = 1$ or $v = 0$, then $C = 0$. If $C \neq 0$, then we have to make a choice for the relative sign of v. For instance, one possibility is that we are above $v = 1$ which means $v > 1$ for all time. Then we can solve this explicitly for v using the quadratic formula.

Equations of the form $y'(t) = f(y(t))$ like the examples considered so far in this section are known as **autonomous equations**. Autonomous equations are separable and can be integrated to yield implicit relationships between y and t. Autonomous equations related to population growth are discussed in detail in the section 2.5 of the text. The detail there is far beyond what we need in this course, so we won't spend too much time on that section. However, the necessary terminology from that section (**equilibria**, **stability**, **logistic equation**, **carrying capacity**,etc.) has been seeded throughout these lecture notes so there is no great loss in passing over 2.5.

Example: In an elementary chemical reaction, single molecules of two reactants A and B form a molecule of the product C. The law of mass action states that the rate of reaction is proportional to the product of the concentrations $[A]$ and $[B]$. The reasoning behind the law goes something like this: in order to form the molecule C, it is necessary for one molecule of A and one molecule of B to come very close to each other in order to react. These molecules move through solvent (like water) in a container that is very, very large compared to the sizes of the individual molecules. If there are only a few molecules, it will take a very long time for a pair to meet up if they are moving through the solvent at random. However, if the concentration of either molecule is high, then there is a greater chance of a collision and so the rate of production of C is much higher. In short, the law of mass action for the reaction $A + B \to C$ gives us the differential equation

$$\frac{d[C]}{dt} = k[A][B].$$

Thus if the initial concentrations are a and b respectively and $[C] \equiv x(t)$, we have

$$\frac{dx}{dt} = k(a-x)(b-x).$$

If $x(0) = 0$ and $a \neq b$, we can solve this differential equation using the method of partial fractions:

$$\int \frac{dx}{(a-x)(b-x)} = \int k dt = kt + C.$$

Using partial fractions, we find that

$$\frac{1}{(a-x)(b-x)} = \frac{1}{b-a}\left(\frac{1}{a-x} - \frac{1}{b-x}\right)$$

and so

$$\int \frac{dx}{(a-x)(b-x)} = \frac{1}{b-a}\left[-\ln|a-x| + \ln|b-x|\right] = kt + C.$$

This equation is equivalent (after some basic algebra) to

$$Ce^{k(b-a)t} = \left|\frac{b-x}{a-x}\right|.$$

Using the initial condition $x(0) = 0$, we find that $C = \ln\left(\frac{b}{a}\right)$. With various assumptions about a and b, we can go further and solve explicitly for $x(t)$. However, this calculation is adequate.

An interesting feature about this problem is that it is possible to obtain the same solution through a different integration route. When you work with enough differential equations you'll come to see that there are often multiple approaches leading to the same solution. Discerning which is the best or easiest route takes time to develop and only comes about through practice.

For this example from chemistry, note that we can write the differential equation as

$$\int \frac{dx}{ab - (a+b)x + x^2} = \int k dt = kt + C$$

if we FOIL the denominator of the lefthand integral. We complete the square in the denominator to obtain

$$ab - (a+b)x + x^2 = (x - (a+b)/2)^2 + ab - (a+b)^2/4.$$

Note that the constant $ab - (a+b)^2/4 = \left(2ab - a^2 - b^2\right)/4 = -(a-b)^2/4$ is always nonpositive. Let's call this constant $-J^2$ just to tidy up the notation.

We now have the equation

$$\int \frac{dx}{(x-(a+b)/2)^2 - J^2} = \int k dt = kt + C$$

to solve. To compute the integral as it is written, let us define $u = x - (a+b)/2$ so that, in terms of the new variable u, we have

$$\int \frac{du}{u^2 - J^2} = \int k dt = kt + C.$$

The lefthand integral can be performed if we make a **trigonometric substitution** $u = J\cos(\theta)$, for instance.

$$\int \frac{du}{u^2 - J^2} = \int \frac{-J\sin(\theta)}{J^2\cos^2(\theta) - J^2} d\theta = \frac{1}{J}\int \csc(\theta) d\theta = kt + C.$$

From an integral table you can determine that

$$\int \csc(\theta) d\theta = \ln|\csc(\theta) - \cot(\theta)|$$

and so

$$\frac{1}{J}\ln|\csc(\theta) - \cot(\theta)| = kt + C.$$

If you back-substitute in order to write this expression in terms of $x(t)$, you'll find precisely the same answer we found when using partial fractions. Specifically,

$$|\csc(\theta) - \cot(\theta)| = \sqrt{\frac{J-u}{J+u}}$$

and so you find

$$Ce^{k(b-a)t} = \left|\frac{b-x}{a-x}\right|$$

as before.

This is the way it should be: different routes should produce equivalent expressions. Notice, though, how much effort went into solving the problem using a trig substitution compared to the effort expended using partial fractions. Seeing the best path forward in differential equations is a skill that only comes with practice and experience!

To conclude this example, it is much more realistic to assume that the forward reaction $A + B \to C$ takes place in solution along with the reverse reaction $C \to A + B$: in solution, you have A and B reacting to make a C and alongside this process you have the reverse process where a molecule of C spontaneously crumbles into a molecule of A and a molecule of B. It is not hard to see, based on the law of mass action, that the differential equation for $x(t)$ becomes

$$\frac{dx}{dt} = k_f(a - x)(b - x) - k_r x$$

where k_f is the rate constant for the forward reaction and k_r is the rate constant for the reverse reaction. This differential equation may be solved either using a trigonometric substitution or partial fractions. This equation represents the simple idea that the total rate of change is the difference between the rate of creation (or input) and the rate of destruction (or removal). This idea applies directly to so-called mixing problems.

So far, our examples have been exclusively autonomous. It is possible to find examples of differential equations that are not autonomous in physics. For instance, consider the **escape velocity problem**:

Example: According to Newton's Law of Universal Gravitation, the gravitational force on an object of mass m that has been projected vertically upward from the earth's surface is

$$F = \frac{mgR^2}{(x + R)^2}$$

where $x(t)$ is the object's distance above the surface at time t, R is the earth's radius, and g is the acceleration due to gravity. Also, by Newton's Second Law, $F = ma = m\frac{dv}{dt}$ and so

$$\frac{dv}{dt} = -\frac{gR^2}{(x + R)^2}$$

Suppose a rocket is fired vertically upward with an initial velocity v_0. Let h be the maximum height above the surface reached by the object. Show that

$$v_0^2 = \frac{2gRh}{R + h} \tag{2.2.4}$$

and interpret this equation when $h \to \infty$.

A particular family of problems leading to separable first order equations with a broad range of applications is the family of **mixing problems:**

Mixing Problems. A typical mixing problem involves a tank of fixed capacity filled with a thoroughly mixed solution of some substance, such as salt. A solution of a given concentration enters the tank with a fixed rate and the mixture, thoroughly stirred, leaves at a fixed rate, which may differ from the entering rate. If $y(t)$ denotes the amount of substance in the tank at time t, then $y'(t)$ is the rate at which the substance is being added minus the rate at which it is being removed. The mathematical description of this situation often leads to a first-order separable differential equation. We can use the same type of reasoning to model a variety of phenomena: chemical reactions, discharge of pollutants into a lake, injection of a drug into a bloodstream.

Example: A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L per minute. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?

Partial ANS:

$$\frac{dy}{dt} = \text{rate in} - \text{rate out} = \left(0.03\frac{\text{kg}}{\text{L}}\right)\left(25\frac{\text{L}}{\text{min}}\right) - \left(\frac{y(t)}{5000}\frac{\text{kg}}{\text{L}}\right)\left(25\frac{\text{L}}{\text{min}}\right)$$

It is a simple matter to alter the above mixing problem format to arrive at differential equations we cannot yet solve since they are not separable:

Example: A tank with a capacity of 400 L is full of a mixture of water and chlorine with a concentration of 0.05 g of chlorine per liter. In order to reduce the concentration of chlorine, fresh water is pumped into the tank at a rate of 4 liters per second. The mixture is kept stirred and pumped out a rate of 10 liters per second. Find the amount of chlorine in the tank as a function of time.

Example: A tank contains 100 gallons of water and 50 oz of salt. Water containing a salt concentration of $\frac{1}{4}\left(1+\frac{1}{2}\sin(t)\right)$ oz/gal flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the same rate.

1. Find the amount of salt in the tank at any time.
2. The long-time behavior of the solution is an oscillation about a certain constant level. What is the level? What is the amplitude of oscillation?

To solve these problems we now turn to section 2.1 and look into the method of the integrating factor.

(Extra examples if there is time/need:)

Example: A sphere with radius 1 m has temperature 15°C. It lies inside a concentric sphere of radius 2 m with temperature 25°C. The temperature $T(r)$ at a distance r from the common center of the spheres satisfies

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0.$$

Find $T(r)$.

Example: Find the solution of $y'(x) = xe^y$ that satisfies $y(0) = 0$.

Example: Find the solution of $x \ln x = y\left(1+\sqrt{3+y^2}\right) y'$ that satisfies $y(0) = 0$.

2.3 Linear Equations with Variable Coefficients: BD (2.1)

Definition: A first-order **linear** differential equation is one that can be put into the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions on a given interval. This type of equation occurs frequently in the sciences.

The key to solving such equations is to recognize that the righthand side is almost (but not quite) similar in form to a combination of functions which appears in the product rule. Recall:

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

Just for fun, if we let $g(x) = \mu(x)$ (some unspecified function) and $f(x) = y(x)$, then the product rule gives us

$$\frac{d}{dx}\left[\mu(x)y(x)\right] = \mu'(x)y(x) + \mu(x)\frac{dy}{dx}$$

If we let $\mu'(x) = P(x)$ then we have the right combination of y and P but not the right term involving $\frac{dy}{dx}$. So we can't use the product rule directly here.

Hope is not lost if we multiply both sides of our ODE by the mystery function $\mu(x)$:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x)$$

Now, if we choose $\mu(x)$ such that $\mu'(x) = \mu(x)P(x)$, then we have

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)\frac{dy}{dx} + \mu'(x)y = \mu(x)Q(x) = \frac{d}{dx}\left[\mu(x)y(x)\right]$$

The equations

$$\mu'(x) = \mu(x)P(x)\ ,\ \ \mu(x)Q(x) = \frac{d}{dx}\left[\mu(x)y(x)\right]$$

can be solved given what we already know!

The function $\mu(x)$ is called the **integrating factor**.

Example: Suppose we have a voltage source tied to a resistor and an inductor in a simple electrical circuit. The equation governing the electrical current in this circuit is given by

$$L\frac{dI}{dt} + RI = E(t) \tag{2.3.1}$$

where L is the inductance in henries (H), R is the resistance in ohms (Ω), $I(t)$ is the current in amperes (A) and $E(t)$ is the voltage source in volts (V). Suppose further that

$E(t) = E_0 \sin(\omega t)$; *i.e.* we have alternating voltage. Using the integrating factor we may solve this equation explicitly for $I(t)$.

Partial ANS: remember that integration by parts leads to the identity

$$\int e^{-at} \sin(bt) dt = -\frac{b\cos(bt) + a\sin(bt)}{a^2 + b^2} e^{-at} + C,$$

which can be used to compute the solution using the integrating factor method.

Example: Return to mixing problem examples.

Example: Solve the initial-value problem $x^2 y' + 2xy = \ln x$, $y(1) = 2$.

Example: Solve the initial-value problem $xy' = y + x^2 \sin(x)$, $y(\pi) = 0$.

Example: Solve the initial-value problem $ty' - y = t^2 e^{-t}$, $y(1) = 0$.

2.4 Exact Solutions and Integrating Factors: BD (2.6)

In the previous section we saw that it is possible to solve a certain class of linear first order differential equations using the integrating factor method. In this section we generalize the method to a broader class of first order equations.

First consider the differential equation $2x + y^2 + 2xyy' = 0$. This equation is neither separable nor linear. However, we notice something cute if we write this equation using differentials:

$$\left(2x + y^2\right) dx + (2xy)\, dy = 0 = \frac{\partial}{\partial x}\left(x^2 + xy^2\right) dx + \frac{\partial}{\partial y}\left(x^2 + xy^2\right) dy$$

If we define the function $\psi(x, y) = x^2 + xy$, then the chain rule in two variables allows us to rewrite the differential equation as

$$d\psi(x, y) = 0 \tag{2.4.1}$$

Of course, if the differential of ψ vanishes, then ψ must be constant. In other words, we've reduced a nonlinear first order differential equation to an algebraic equation! Specifically

$$c = \psi(x, y) = x^2 + xy^2$$

which defines y implicitly in terms of x.

Let's see how far we can go with this idea. If a first order differential equation of the form

$$M(x, y) + N(x, y)y' = 0$$

(where M, N, N_x, M_y are continuous on some rectangle R) has $M_y = N_x$ at each point of R, then there exists a function ψ such that $d\psi = 0$ is equivalent to the original ODE.

One proof of this statement relies on ideas from Calculus III: if we think of M and N defining a two dimensional vector field, we are asserting that if this vector field is conservative, then a potential function exists.

Example: Determine whether or not $(2x + 3) + (2y - 2)y' = 0$ is an exact differential equation. If it is exact, find the solution.

Example: Determine whether or not $(3x^2 - 2xy + 2)dx + (6y^2 - x^2 + 3)dy = 0$ is an exact differential equation. If it is exact, find the solution.

Example: For what values of the constants a, b, c is the differential equation $\frac{dy}{dx} = -\frac{ax+by}{bx+cy}$ exact? Find the solutions when the equation is exact.

Sadly, not every first order ODE will be exact: consider $x^2y^3 + x(1 + y^2)y' = 0$. However, it is sometimes possible to convert a differential equation that is not exact into an exact equation by multiplying the equation by a suitable integrating factor.

Let's start with the general case $M(x, y)dx + N(x, y)dy = 0$ and then we'll look at some examples. Let's multiply both sides by the unknown integrating factor $\mu(x,y)$:

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

This equation will be exact if and only if

$$\frac{\partial}{\partial y}\left[\mu M\right] = \frac{\partial}{\partial x}\left[\mu N\right]$$

First, it is not necessarily the case that a unique integrating factor will work here: as long as there is one, that's all we need. Second, solving this partial differential equation is as hard or harder than solving the original ODE. So we can't expect this method to always work.

Your best chance of finding an integrating factor occurs when μ is a function of a single variable, either x or y:

1. If $\frac{M_y - N_x}{N}$ is a function of x only, then there is an integrating factor μ that also depends only on x.
2. If $\frac{M_y - N_x}{M}$ is a function of y only, then there is an integrating factor μ that also depends only on y.

Example: Let's go back and look at the motivating example $x^2y^3 + x(1+y^2)y' = 0$. This equation is not exact, but becomes exact when multiplied by the integrating factor $\mu(x,y) = \frac{1}{xy^3}$. (Also, you can solve the problem with the integrating factor $\mu(x,y) = \frac{1}{xy(2x+y)}$ which shows that integrating factors need not be unique!)

Example: Show $ydx + (2x - ye^y)dy = 0$ is not exact, but becomes exact when multiplied by the integrating factor $\mu(x,y) = y$.

Example: Find an integrating factor and solve the equation $y' = e^{2x} + y - 1$.

Example: Find an integrating factor and solve the equation $dx + \left(\frac{x}{y} - \sin y\right) dy = 0$.

The concepts in section 2.6 are more difficult to motivate since they arise in highly specialized areas. I'll attempt now to explain the importance of integrating factor methods and potentials.

First, consider the motion of a pendulum under the influence of gravity. The angle made by the pendulum with the vertical $\theta(t)$ satisfies the nonlinear differential equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin(\theta(t)).$$

This equation cannot be solved without the concept of an elliptic integral, but we can say quite a bit about solutions using the integrating factor method. The idea is to change points of view and consider the angular velocity $\dot{\theta} = \frac{d\theta}{dt}$ as a function of θ instead of time. Using chain rule, we find

$$\frac{d^2\theta}{dt^2} = \dot{\theta}\frac{d}{d\theta}\dot{\theta} = -\frac{l}{g}\sin(\theta).$$

Using the integrating factor method, it is possible to show that solutions satisfy $\psi(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 - \frac{l}{g}\cos(\theta) = \text{constant}$. This equation is the sum of two parts: the first part corresponds to the kinetic energy and the second corresponds to the gravitational energy. So this equation basically tells us that solutions conserve total energy. Identifying conservation laws (energy, momenta, angular momenta, etc.) are indispensable tools in understanding very complicated physics problems.

Another way in which integrating factor methods are important is in illuminating physical quantities representing **inexact** differentials. Let's start with some basic concepts: we all know that if x represents some quantity, then dx represents a very small amount of that quantity. For instance, $dm = \rho(x)dx$ would represent an infinitesimal amount of mass sitting at position x where the body has mass density $\rho(x)$. If you want to find the total, then you need to integrate:

$$M_{tot} = \int dm = \int \rho(x)dx.$$

There are many physical quantities which can be represented infinitesimally and whose total can be obtained by integration, but this integration process depends on the way integration is performed.

To be concrete, suppose you walk from point A to point B. We all know this takes some amount of energy to perform: you won't go very far if you are starving. The energy required to move from point A to B depends on *how* you move from point A to B. If A and B are across the street, only a relatively small amount of energy is required. However, if you walk around town first and then cross the street, more energy is needed even though the result is the same.

Since the energy (or **work**) required to move between two points depends on how we move, we represent work by an **inexact** differential $đW$. Another inexact differential is heat transferred from the environment into a body $đQ$. Now it is very interesting to point out that the first law of thermodynamics guarantees that the total energy (which can be represented by an exact differential) is the sum of the heat transferred and the work done, both of which are inexact differentials:

$$dE = đW + đQ.$$

A scientist or engineer trying to understand heat transfer will inevitably run into this equation. Such a person would have to perform measurements and it is a general fact of life that quantities represented by exact differentials are measured with relative ease because there is a function associated with them. In this example, there is a function of system parameters giving us the total energy E, but there is no corresponding function for the work or the heat transferred.

So it would be very challenging to measure, say, the heat transferred. However, we can ask the question: is it possible to find a function related to the heat transferred that *is* relatively easy to measure? The answer is yes and the the function is obtained through the integrating factor method.

To see how this works, consider a system made up of an ideal gas. The pressure, temperature and volume of the gas are all related through the ideal gas law

$$PV = kT$$

where k is a constant. For an ideal gas it is a fact that the energy depends only on the temperature, so $dE = c(T)dT$ where $c(T)$ represents the specific heat capacity of the gas.

Rearranging the first law, we have the equation

$$c(T)dT - PdV = \text{đ}Q.$$

Using the ideal gas law we can write this expression as

$$c(T)dT - k\frac{T}{V}dV = \text{đ}Q.$$

It is not hard to show that the left hand side cannot be written as the differential of a potential. However, using the integrating factor method, we find that if we multiply both sides by $1/T$, then we can find a potential! In fact, $\text{đ}Q/T = dS$ where S is the entropy, the measure of disorder of the gas. The entropy is central to thermodynamics and is the corner stone of much modern physics.

These notes are only meant to motivate the integrating factor method and to give you a sense of where they arise. If you cannot follow the physics, don't worry. It's not essential for the rest of our course.

2.5 Existence Theorems: BD (2.4)

So far we've considered first order differential equations which can be solved either because they are separable or linear with nicely behaved coefficient functions or possess an integrating factor making them exact. Our examples mostly come from models of physical or natural phenomena so we of course expect to find solutions. We charged into each example not asking the more basic question: does this equation even have a solution?

Of course this is an important question to answer. If your differential equation has no solution why waste time trying to find one? Moreover if your equation is supposed to model reality and yet it has no solution, something went wrong with your model-building process.

Based on our experience with first order linear equations, the following theorem shouldn't come as a surprise to you:

Theorem: If the functions p and g are continuous on an open interval $I = (\alpha, \beta)$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each $t \in I$, and that also satisfies the initial condition $y(t_0) = y_0$.

The proof of this theorem essentially involves using the integrating factor to explicitly write $y(t)$ in terms of y_0, p, g. The important thing to carry away from this theorem is that given mild conditions on p and g, a **unique** solution always **exists**.

So you can look at your coefficient functions p and g and, using this theorem, immediately assess whether it is worth the time and effort to look for a solution to the differential equation: the theorem will tell you if a solution exists or not.

Moreover, according to the theorem, you don't need to waste time looking for alternative solutions. This is comforting since our equations are ostensibly supposed to model real-world situations.

A similar theorem exists for the much more complicated case of nonlinear first order differential equations:

Theorem: Let the functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $(\alpha, \beta) \times (\gamma, \delta)$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in (α, β), there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y) \ , \ y(t_0) = y_0$$

So existence and uniqueness are tied to the continuity of f and its first partial with respect to y. The proof of this theorem is covered in the text if you are interested in the details. We'll primarily be interested in the application of the theorem, not the underlying theory. However, it is such an important theorem that I want to at least sketch the details for you.

Sketch of Proof. To prove the general theorem, it is enough to prove the following:

Auxiliary Theorem: If f and f_y are continuous in a rectangle $R: \; |t| \le a \; , \; |y| \le b$, then there is some interval $|t| \le h \le a$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem $y' = f(t, y) \; , \; y(0) = 0$.

If there is such a function, it must satisfy the **integral equation**

$$\phi(t) = \int_0^t f[s, \phi(s)] \; ds$$

This comes from integrating the equation $y' = f(t, y)$ and using the FTC (fundamental theorem of calculus). To get to $\phi(t)$ we generate a sequence of functions $\phi_j(t)$ through the **Picard iteration method**.

The idea is to define $\phi_0(t) = 0$ and let $\phi_1(t) = \int_0^t f[s, \phi_0(s)] ds$. Both functions satisfies the initial value condition, but neither necessarily solves $y' = f(t, y)$. We generate more functions recursively

$$\phi_{j+1}(t) = \int_0^t f[s, \phi_j(s)] ds$$

To prove the auxiliary theorem, we need to answer the following questions:

1. Do all of the members of the sequence exist?
2. Does the sequence converge?
3. If the sequence converges, does the limiting function satisfy the original integral equation?
4. Are there other solutions?

The detailed answer to these questions can be found in the text. Suffice it to say that the sequence does converge to a unique, nicely-behaved function ϕ which satisfies

$$\phi(t) = \int_0^t f[s, \phi(s)] \; ds$$

Note: The overall process is very similar to the way we compute $\sqrt{2}$ using an infinite sequence: define $a_0 > 0$ and look at the value the sequence $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ converges (which is necessarily $\sqrt{2}$).

As a sanity check, consider the problem $y'(t) = -y(t) - 1$ which has solution $y(t) = e^{-t} - 1$ for $y(0) = 0$. Show that the Picard iteration method produces the correct answer.

Note: The theorem for *linear* differential equations is a special case of the more general one for $y' = f(t, y)$. (Prove if there is time!)

Note: It is possible to weaken the conditions of the theorem: we can ensure the existence of a solution just based on the continuity of f but we cannot guarantee uniqueness without more assumptions.

For instance, consider the initial value problem $y' = y^{\frac{1}{3}}$, $y(0) = 0$. It is possible to show that $y(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$, $y(t) = -\left(\frac{2}{3}t\right)^{\frac{3}{2}}$ and $y = 0$ all satisfy the IVP. In fact, there are an infinite number of continuous, differentiable solutions to this IVP. This does not contradict our theorem because $\frac{\partial f}{\partial y}$ fails to be continuous.

Interval of Definition. The solution of the linear equation $y' + p(t)y = g(t)$ subject to the initial condition $y(t_0) = y_0$, exists throughout any interval about t_0 in which the functions p and g are continuous.

However, for the general case $y' = f(t, y)$ the interval in which a solution exists may have no simple relationship to the function f. For example, consider the ODE $y' = y^2$, $y(0) = 1$. It is not hard to show that a solution to this problem is $y(t) = \frac{1}{1-t}$. Clearly, the solution becomes unbounded as we approach 1. Therefore, the solution only exists on the interval $(-\infty, 1)$.

Nothing about the original IVP problem statement indicated that 1 is in any way special! The moral of the story is that solutions to nonlinear differential equations will have an **interval of definition** which may be difficult or impossible to determine from the initial condition and f.

In the same example, if we let $y(0) = y_0$ be general, the solution will have the form

$y(t) = \frac{y_0}{1-y_0 t}$. This shows that singularities in solutions to nonlinear first order ODEs may depend on the initial conditions as well as the differential equation itself.

Another way in which linear equations differ from nonlinear equations is in the specification of general solutions. When solving a linear first order ODE, we inevitably encounter an unknown constant. This constant can be fixed to match a specific initial condition we have in mind or it can be allowed to roam free and give us all possible solutions to the ODE. (Think of the constant of integration in Calculus I/II!)

Families of solutions to nonlinear first order ODEs on the other hand are usually not that easy to characterize.

Example: Consider the IVP

$$y' = \frac{-t + (t^2 + 4y)^{\frac{1}{2}}}{2}\ ,\ y(2) = -1$$

1. Verify that both $y_1(t) = 1 - t$ and $y_2(t) = -\frac{t^2}{4}$ are solutions. Where are the solutions valid?
2. Explain why the existence of two solutions does not contradict our theorems.
3. Show that $y = ct + c^2$, where c is an arbitrary constant, satisfies the differential equation (not the IVP of course) for $t \geq -2c$. If $c = -1$, we obtain y_1. Show that there is no choice of c that gives us y_2.

Because "additional solutions" to nonlinear equations are difficult to characterize, we only use the term **general solution** when discussing linear equations.

You may wonder about situations where the coefficient functions are discontinuous. This happens frequently in physics and engineering. In the most common scenario, the coefficient functions are piecewise-smooth. To solve such a problem, you would solve the differential equation on each sub-domain where the coefficients are smooth and then match across the endpoints:

Example: Solve the IVP $y' + 2y = g(t)$, $y(0) = 0$ where $g(t) = 1$, $t \in [0, 1]$ and $g(t) = 0$, $t > 1$.

Example: Solve the IVP $y' + 2p(t)y = 0$, $y(0) = 1$ where $p(t) = 2$, $t \in [0, 1]$ and $p(t) = 1$, $t > 1$.

One final remark before I finish this section with examples: when solving first order equations, it will be common to arrive at an **implicit** formula for the solution. In other words, it may not be possible to solve for y as an explicit function of the independent variable. If this happens, don't panic! It's perfectly natural and you can use a computer to numerically solve for y if your problem demands it.

Example: Determine (without solving the problem) an interval in which the solution of $(t - 3)y' + \ln(t)\ y = 2t$, $y(1) = 2$ is certain to exist.

Example: Determine (without solving the problem) an interval in which the solution of $(4 - t^2)y' + 2t\ y = 3t^2$, $y(-3) = 1$ is certain to exist.

Example: State the region in the ty-plane where a unique solution with a given initial point in this region is guaranteed to exist for the differential equation $y' = \frac{t-y}{2t+5y}$.

Example: State the region in the ty-plane where a unique solution with a given initial point in this region is guaranteed to exist for the differential equation $y' = \frac{1+t^2}{3y-y^2}$.
Example: Solve the IVP $y' = \frac{t^2}{y(1+t^3)}$, $y(0) = y_0$ and determine how the interval in which the solutions exist depends on y_0.

2.6 First Order Difference Equations: BD (2.9)

While a continuous model leading to a differential equation is reasonable and attractive for many problems, there are some cases in which a discrete model may be more natural:

1. Compound interest models

2. Some population growth models

3. Some probability models (especially those with discrete random variables).

More importantly, discrete models are often used to understand nonlinear systems which generally possess a characteristic called **chaos**. In layman's terms, chaos is extreme sensitivity to initial conditions. This extreme sensitivity leads to unpredictability even if full solutions to the model equations are known. Examples of chaotic systems are earth's atmosphere and the general three body problem.

Discrete models based on the original continuous model capture much of the chaotic behavior but are in general more easily analyzed.

Definition: A **first order difference equation** is defined recursively as

$$y_{n+1} = f(n, y_n)$$

where $n = 0, 1, 2, \ldots$. You can think of n as representing discrete time.

Much of the terminology we presented for first order ODEs carries immediately over to the discrete case. We don't have much time to explore first order difference equations or their applications in understanding chaotic systems. To get a taste of how to deal with linear first order difference equations we'll look at a few examples involving **autonomous** difference equations; i.e. equations in which f depends only on y. These examples may be solved using an **iteration** process: since $y_{n+1} = f(y_n)$ it is easy to see that $y_n = f^n(y_0)$ in general where

$$f^n(x) = \underbrace{(f \circ f \cdots \circ f)}_{n \text{ times}}(x).$$

Example: Solve $y_{n+1} = -0.9y_n$ in terms of the initial condition y_0. Describe the behavior as $n \to \infty$.

Example: Solve $y_{n+1} = \sqrt{\frac{n+3}{n+1}}y_n$ in terms of the initial condition y_0. Describe the behavior as $n \to \infty$.

Example: Solve $y_{n+1} = -0.5y_n + 6$ in terms of the initial condition y_0. Describe the behavior as $n \to \infty$.

Chapter 3

Second Order Differential Equations

This chapter deals with second order linear differential equations. It employs much of the terminology we introduced in the linear systems lectures. Second order linear equations appear often in engineering and the physical sciences. One reason for this is that the ubiquitous Newton's second law is a relation between force, mass and acceleration: acceleration, of course, is the second derivative of position with respect to time.

Applying Newton's second law to the damped spring-mass system of the linear systems lectures, we find

$$mx'' + \gamma x + kx = 0$$

It turns out that a diverse array of phenomena obey equations of *similar form*. For instance, if we have a resistor, capacitor and an inductor all wired in *series* to a potential source $E(t)$ in a circuit, then

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

where Q is the charge on the capacitor.

It is interesting to note that when two systems obey differential equations of similar form, one can be used to simulate the other. Hence, it is possible to simulate the behavior of this circuit with a spring-mass system immersed in a viscous fluid or vice versa. A more sophisticated example involves fluids and black holes: certain fluid equations are mathematically similar to equations describing black holes near the event horizon. This has opened up a new field of black hole simulation called analogue gravity.

The equation governing the LRC circuit has a **forcing term** on the righthand side and hence cannot be **homogeneous**. A forcing term in the damped spring-mass system can help engineers understand a phenomenon known as **resonance** in which the natural oscillation of the spring responds to the oscillation frequency of a forcing term $F = F_0 \cos(\omega_0 t)$. Understanding resonance phenomena is important if you are building, say, a bridge which experiences cross winds. You wouldn't want the cross winds to set up a resonance which literally shakes your bridge to pieces! On the other hand, you can use resonance to your advantage in designing seismographs which are intended to detect weak periodic signals.

We will discuss some interesting applications of second order linear equations in this portion of the course. First, we need to set the stage for second order linear equations.

3.1 Homogeneous Equations with Constant Coefficients: BD (3.1/3.4)

The most general second order differential equation has the form

$$y'' = f(t, y', y')$$

in complete analogy to the first order case. We will focus primarily on **linear** equations of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

throughout these lectures. An **initial value problem** in this context involves *two* separate initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$. In general an n-th order equation demands n initial conditions.

Think about solving $\frac{d^2x}{dt^2} = 0$: one integration gives us $x'(t) = c_1$ and another gives us $x(t) = c_1 t + c_2$. Moral: second order implies two integrations which implies two constants of integration.

Definition: A second order linear equation is said to be **homogeneous** if the term $g(t)$ is

0 for all t. Otherwise, the equation is called **nonhomogeneous**.

To allay any fears about existence or uniqueness of solutions, we state without proof:

Theorem: Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t)\ ,\ y(t_0) = y_0\ ,\ y'(t_0) = y_0'$$

where p, q, and g are continuous on an open interval I. Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I.

To keep the mathematics as simple as possible we consider first the case of homogeneous second order equations with constant coefficients:

$$ay'' + by' + cy = 0$$

where $a \neq 0$.

Let's do something drastic and just *guess* a solution of the form e^{rt} and see where it leads us. If we substitute this function into the second order ODE and cancel the common exponential factors, we have the **characteristic equation**:

$$ar^2 + br + c = 0$$

This is nothing but a quadratic which we know how to solve! Let's assume for the moment that the roots of this polynomial are distinct: we either have two distinct real roots or a conjugate pair of complex roots. Call these roots r_1 and r_2. The roots lead to two possible solutions

$$e^{r_1 t}\ ,\ e^{r_2 t}$$

Since the two roots are distinct, these functions are linearly independent (i.e. the only constants c_j such that $c_1 e^{r_1 t} + c_2 e^{r_2 t} = 0$ are $c_j = 0$). Therefore, $e^{r_1 t}, e^{r_2 t}$ form a set of **fundamental solutions** and we can write *any* solution as the linear combination

$$\phi(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$

which is called the **general solution**. The constants c_1 and c_2 are determined through the initial conditions.

Example: Find the general solution of $y'' + 2y' - 3y = 0$.

Example: Find the general solution of $6y'' - y' - y = 0$.

Example: Find the solution to the IVP $y'' + 4y' + 3y = 0$, $y(0) = 1$, $y'(0) = 1$.

Example: Find the solution to the IVP $2y'' + y' - 4y = 0$, $y(0) = 0$, $y'(0) = 1$.

Complex values can be handled via Euler's formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Example: Find the general solution of $y'' - 2y' + 2y = 0$.

Example: Find the general solution of $y'' - 2y' + 6y = 0$.

Example: Find the solution to the IVP $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = 1$.

Example: Find the solution to the IVP $y'' + 2y' + 2y = 0$, $y(\pi/4) = 2$, $y'(\pi/4) = -2$.

3.2 Fundamental Solutions: BD (3.2)

Returning to the general second order linear equation $y'' + p(t)y' + q(t)y = g(t)$, it is not hard to show that if we have two solutions to this equation y_1 and y_2 then any linear combination $c_1y_1 + c_2y_2$ will also be a solution. This is known as the **principle of superposition**.

We saw this in the case of constant coefficients. Before we can make claims about general solutions, we need to introduce an invaluable tool in ODE theory: the **Wronskian**.

Definition of the Wronskian. Given two differentiable functions y_1, y_2 the **Wronskian** is defined by

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

We can use the Wronskian to phrase the following theorem:

Theorem: Suppose that y_1 and y_2 are two solutions of

$$y'' + p(t)y' + q(t)y = 0$$

with

$$W(y_1, y_2)(t_0) \neq 0.$$

Then there are constants c_1 and c_2 such that $c_1y_1 + c_2y_2$ is a solution to

$$y'' + p(t)y' + q(t)y = 0 \ , \ y(t_0) = y_0 \ , \ y'(t_0) = y_0'.$$

The implication of this theorem is that if there is a point t_0 where the Wronskian of y_1 and y_2 is nonzero, then these functions form a **fundamental set of solutions**.

Another way of stating the theorem: To find the general solution, and therefore all solutions, we need to find two solutions whose Wronskian is nonzero.

The reasoning behind this theorem is easy to understand: if y_1 and y_2 are to be fundamental solutions, then it must be possible to solve

$$c_1y_1(t_0) + c_2y_2(t_0) = y_0 \ , \ c_1y_1'(t_0) + c_2y_2'(t_0) = y_0'$$

In matrix form,

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

This matrix equation will always have a unique solution if and only if the coefficient matrix is invertible; i.e. the Wronskian does not vanish.

We can summarize this section as follows. To find the general solution of the differential equation

$$y'' + p(t)y' + q(t)y = 0 \ , \ \alpha < t < \beta$$

we must first find two functions y_1 and y_2 that satisfy the differential equation in $\alpha < t < \beta$. Then we must make sure that there is a point in the interval where the Wronskian is nonzero. Under these circumstances y_1 and y_2 form a fundamental set of solutions.

Example: Find the Wronskian of $e^{2t}, e^{-3t/2}$.

Example: Find the Wronskian of $e^t \sin(t)$, $e^t \cos(t)$.

Example: Verify that $y_1 = \cos(2t)$, $y_2 = \sin(2t)$ are solutions to $y'' + 4y = 0$. Do they constitute a fundamental set of solutions?

Example: Verify that $y_1 = e^t$, $y_2 = te^t$ are solutions to $y'' - 2y' + y = 0$. Do they constitute a fundamental set of solutions?

3.3 Linear Independence and the Wronskian: BD (3.3)

Two functions f and g are said to be **linearly dependent** on an interval I if there exist two constants k_1 and k_2, not both zero, such that $k_1 f(t) + k_2 g(t) = 0$. Otherwise, they are **linearly independent** on I.

Linear dependence is tied to the Wronskian of f and g in the following way:

Theorem: If f and g are differentiable functions on an open interval I and if $W(f,g)(t_0) \neq 0$ for some point t_0 in I, then f and g are linearly independent on I. Moreover, if f and g are linearly dependent on I, then $W(f,g)(t) = 0$ for every t in I.

Example: Determine if $f(t) = t^2 + 5t$, $g(t) = t^2 - 5t$ are linearly independent using the Wronskian.

Example: Determine if $f(t) = t$, $g(t) = t^{-1}$ are linearly independent using the Wronskian.

Example: The Wronskian of two functions is $t \sin^2(t)$. Are the functions linearly dependent or independent?

It may surprise you to know that there is actually a simple formula for the Wronskian of two solutions of a second order linear homogeneous equation:

Abel's Theorem: If y_1 and y_2 are solutions of the differential equation

$$y'' + p(t)y' + q(t)y = 0\ ,\ \ \alpha < t < \beta$$

where p and q are continuous, then

$$W(y_1, y_2)(t) = c\exp\left\{-\int p(t)dt\right\}$$

where c is a certain constant that depends on y_1 and y_2 but not on t. Further, W is either 0 for all $t \in I$ or else is never zero.

Example: Find the Wronskian of $t^2y'' - t(t+2)y' + (t+2)y = 0$ without solving the equation.

Example: Find the Wronskian of $(1-t^2)y'' - 2ty' + \alpha(\alpha+1)y = 0$ without solving the equation.

Let me end this section with two observations. First, notice how many linear algebra terms appeared in this section. The solution set of the differential equation considered here acts very much like a **vector space**. In fact, using the principle of superposition you can show that the set of solutions is necessarily a vector subspace of the space of all continuous functions on I. This suggests a deep connection between differential equation theory and linear algebra. We will only skim the surface of this connection.

Second, despite its theoretical character, the Wronskian will be useful to us later when we need to calculate a second, linearly independent solution given an initial solution. In other words, Abel's formula will allow us to generate a second solution assuming we can find a single solution.

3.4 Repeated Roots of the Characteristic Equation: BD (3.5)

If our characteristic equation returns repeated roots, then we cannot find two linearly independent solutions using this equation. It is possible to construct the second solution given an initial solution through two possible routes.

The Way of the Wronskian. We know how to compute the Wronskian of two solutions to the differential equation $ay'' + by' + c = 0$ using Abel's Theorem:

$$W(y_1, y_2, t) = C \exp\left\{-\int \frac{b}{a} dt\right\} = Ce^{-\frac{b}{a}t}$$

Let's assume for simplicity that $C = 1$ without loss of generality.

By definition of the Wronskian, we have $y_1 y_2' - y_2 y_1' = e^{-\frac{b}{a}t}$. Since we know y_1 from the characteristic equation approach, this gives us a *first order* equation for y_2! We can use this to solve for y_2.

Reduction of Order. The second approach also yields a first order equation which determines the second independent solution. The idea is to suppose $y_2 = v(t)y_1$ where $v(t)$ is an unknown function. Using the original differential equation, we arrive at an equation for $v(t)$:

$$y_1 v'' + (2y_1' + py_1)\, v' = 0$$

If we let $V(t) = v'(t)$, we have a first order equation in V which we can solve through standard methods.

However you approach the problem of repeated roots, we have the following summary:

Summary of Second Order Homogeneous Equations with Constant Coefficients.

1. If the roots of the characteristic equation are real but unequal, then $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.
2. If the roots are complex conjugates, then $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ and there is an additional

required step in which the complex numbers are eliminated via Euler's formula.

3. If the roots are repeated,

$$y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$$

Example: Solve $9y'' + 6y' + y = 0$.

Example: Solve $y'' - 2y' + y = 0$.

Example: Solve $9y'' - 12y' + 4y = 0\ ,\ y(0) = 2\ ,\ y'(0) = -1$.

Example: Solve $t^2 y'' - 4ty' + 6y = 0\ ,\ t > 0$ using the method of reduction of order if $y_1(t) = t^2$.

Example: Solve $t^2 y'' + 2ty' - 2y = 0\ ,\ t > 0$ using the method of reduction of order if $y_1(t) = t$.

3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients: BD (3.6)

With our understanding of constant coefficient homogeneous equations, we turn to the more complicated situation of nonhomogeneity. Consider the differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

with corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

In this most general setting we notice a division of labor is possible. Suppose we know how to find a fundamental set of solutions for the homogeneous equation. Call the solutions y_1 and y_2 like usual.

It turns out that we can always write *any* solution to $y'' + p(t)y' + q(t)y = g(t)$ as the sum of two pieces:

$$\phi(t) = \underbrace{c_1 y_1 + c_2 y_2}_{\text{homogeneous part}} + \underbrace{Y(t)}_{\text{particular part}}$$

The **particular part** or **particular solution** $Y(t)$ is any solution to $y''+p(t)y'+q(t)y = g(t)$. The point of making this decomposition is to see that we are halfway to finding a full general solution to the nonhomogeneous equation by first finding a fundamental set of solutions to the corresponding homogeneous equation (which should be by far easier). Also, the c's are fixed by imposing the initial conditions as usual.

The hard part is usually finding the particular part of the general solution. In this chapter, we study two popular methods for finding particular solutions: the method of undetermined coefficients and the method of variation of parameters. In this section we discuss **the method of undetermined coefficients**.

You might as well call it the method of best guessing because that's essentially all that goes into the process. The method of undetermined coefficients works when your function $g(t)$ is made up of elementary functions like power functions, trig functions, etc. and your coefficient functions p and q are constant. Despite the limitations, the method works for a fairly large number of equations relevant to mathematical modeling.

With that said, if the method of undetermined coefficients doesn't work for your particular problem, then you should consider using variation of parameters which we'll cover next.

The gist of method is to make a well-educated guess about the functional form of a particular solution to the nonhomogeneous differential equation based on the functional form of $g(t)$. For example, consider

$$y'' - 3y' - 4y = -8e^t \cos(2t)$$

We know that derivatives turn exponential functions into exponential functions and same goes for trig functions. So we hazard a guess about the particular solution $Y(t)$:

$$Ae^t \cos(2t) + Be^t \sin(2t) \tag{3.5.1}$$

I include a sine term here even though one doesn't appear in $g(t)$ because the first derivative of sine/cosine is related to cosine/sine so I expect sine to turn up when I start taking derivatives.

If I plug this guess into the differential equation I find that the guess works so long as $A = \frac{10}{13}$ and $B = \frac{2}{13}$. Hence the coefficients A and B which were initially undetermined have now been determined.

There are two aspects of undetermined coefficients which make the method difficult to understand/work with. First, the algebra gets very tedious very fast. Simple errors tend to propagate when that happens leading to false conclusions. Second, and more importantly, our initial guesses (though most would agree to be intuitive) don't always work out.

Consider $y'' + 4y = 3\cos(2t)$. Based on the last example I wouldn't disagree with you if you wanted to try a guess of the form $Y(t) = A\cos(2t) + B\sin(2t)$. But if you plug this into the equation you find there is no choice of A and B that works!

The issue here is that our guess has precisely the same form as general solution to the corresponding homogeneous equation. So no wonder we get 0!

What confuses many students is what comes next: most authors will write something like "Well, that didn't work, so let's tack on a power of t to our original guess because it's the next easiest thing to try". It does work (show this), but there is very little intuition about why it works.

Moreover, in some cases, even this won't work and you have to tack on yet another power of t in a haphazard way to find a particular solution! At least for second order equations you never have to add more than 2 powers of t.

Here's a general recipe for implementing the method of undetermined coefficients: if we are to solve the differential equation $ay'' + by' + c = g(t)$, we must

1. Find the general solution of the corresponding homogeneous equation.

2. Make sure that $g(t)$ involves nothing more than exponential functions, sines, cosines,

polynomials or sums or products of such functions. If this is not the case, use variation of parameters (next section).

3. If $g(t) = g_1(t) + \cdots + g_n(t)$, then form n subproblems, each of which contains only one of the terms g_1 , $\ldots g_n$. Since the differential equation is linear, we are free to add up the particular solution for each subproblem to find the overall particular solution.

4. For the i-th subproblem assume a particular solution $Y_i(t)$ of the form

   (a) $t^s\left(A_0t^n + \cdots + A_n\right)$, $s = 0, 1, \text{or } 2$ if $g_i = a_0t^n + \cdots + a_n$.

   (b) $t^s\left(A_0t^n + \cdots + A_n\right)e^{\alpha t}$, $s = 0, 1, \text{or } 2$ if $g_i = \left(a_0t^n + \cdots + a_n\right)e^{\alpha t}$.

   (c) $t^s\left[\left(A_0t^n + \cdots + A_n\right)e^{\alpha t}\cos(\beta t) + \left(B_0t^n + \cdots + B_n\right)e^{\alpha t}\sin(\beta t)\right]$, $s = 0, 1, \text{or } 2$ if $g_i = \left(a_0t^n + \cdots + a_n\right)e^{\alpha t}\left\{\sin(\beta t) \text{ or } \cos(\beta t)\right\}$.

5. Find a particular solution Y_i for each of the subproblems. The sum $Y_1 + \cdots + Y_n$ is a particular solution of the nonhomogeneous equation.

6. Form the sum of the particular solution and the general solution to the homogeneous problem.

7. Use the initial conditions to determine the values of the arbitrary constants remaining in the solution.

If you follow this algorithm correctly you can implement the method of undetermined coefficients for any problem where it is relevant. If the algorithm fails to produce a solution, you need to try something else like the variation of parameters method.

Example: Find the general solution of $y'' - 2y' - 3y = 3e^{2t}$.

Example: Find the general solution of $y'' - 2y' - 3y = -3te^{-t}$.

Example: Find the general solution of $y'' + 9y = t^2e^{3t} + 6$.

Example: Find the general solution of $y'' + y = 3\sin(2t) + t\cos(2t)$.

Example: Find the solution of $y'' - 2y' - 3y = 3te^{2t}$, $y(0) = 1$, $y'(0) = 0$.

There is a slick trick which provides an alternative way of computing solutions to $ay'' + by' + cy = g(t)$. Define D to be the derivative operator with respect to t. We can show that

$$ay'' + by' + cy = (D - r_1)\,(D - r_2)\,y = g(t)$$

where r_1 and r_2 are the zeros of the characteristic equation. If we define $u = (D - r_2)\,y$, then we can solve

$$(D - r_1)\,u = g(t)\ ,\ u = (D - r_2)\,y$$

to find the general solution to the nonhomogenous problem.

3.6 Variation of Parameters: BD (3.7)

The advantage of using **variation of parameters** is that it is completely general: it applies to any equation of the form $y'' + p(t)y' + q(t)y = g(t)$. One of the major disadvantages is that we must have the general solution to the corresponding homogeneous problem (with $g(t) = 0$) in order to apply the method. This is a big assumption since even the homogenous case with variable coefficients is difficult to solve. Also, the application of the method involves computation of messy integrals.

It's primarily used as an alternative to the method of undetermined coefficients and as a tool to prove theorems in differential equations theory.

The method works as follows: assume we know the general solution to the corresponding homogeneous problem $c_1y_1(t) + c_2y_2(t)$. As a guess for the nonhomogeneous problem, we try a function similar to this one only with c_1 and c_2 replaced with unknown parameter functions $u_1(t)$ and $u_2(t)$:

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

Plugging into the nonhomogeneous ODE gives us the system of equaitons

$$u_1'(t) = \frac{-y_2(t)g(t)}{W(y_1, y_2, t)} \; , \; u_1'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2, t)}$$

These can be immediately integrated (in principle) to obtain the following theorem:

Theorem: If the functions p, q and g are continuous on an open interval I, and if the functions y_1 and y_2 are linearly independent solutions of the homogeneous equation corresponding to

$$y'' + p(t)y' + q(t)y = g(t),$$

then a particular solution is

$$Y(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W(y_1, y_2, t)} dt + y_2(t) \int \frac{y_1(t)g(t)}{W(y_1, y_2, t)} dt,$$

and the general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t).$$

Of course you see the two major obstacles in using this theorem: first, you need to know the homogeneous solutions y_1 and y_2 and, second, you need to be able to evaluate the integrals appearing in the expression for $Y(t)$.

Example: Find the general solution to $y'' + y = \tan t \; , \; 0 < t < \pi/2$.

Example: Find the general solution to $t^2 y'' - 2y = 3t^2 - 1 \; , \; 0 < t \; , \; y_1(t) = t^2 \; , \; y_2(t) = t^{-1}$.

Application: To finish off this chapter, we will examine one of the most important second order linear differential equations in physics and engineering:

$$ay'' + by' + cy = F_0 \cos(\omega t)$$

1. First, we'll look at the case where $F_0 = 0$ (i.e, no forcing). We'll investigate the different possible solution behavior (i.e. overdamped motion when the frictional drag is very high, critically damped and underdamped oscillations).

2. Second, after we have a good grasp on the unforced case, we'll turn the forcing on and turn the damping off.

3. When the frequency of the forcing function is different from the natural frequency of the system, we encounter the **beat** phenomenon.

4. When the frequency of the forcing function is the same as the natural frequency of the system, we encounter the **resonance** phenomenon.

5. With the damping turned back on, we'll explore the **forced response** or **steady-state solution** a long time after the forcing is turned on.

Understanding these points will shed light on vast array of physics phenomena from why bridges can collapse due to cross-winds to the reason why water droplets produce rainbows.

Chapter 4

Higher Order Differential Equations

This chapter deals with higher order linear differential equations. The good news: virtually everything from the second order case carries over easily to the higher order case. The bad news: computations are more intense as compared to the second order case.

We will be concerned with the general linear equation

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} \cdots + p_n(t)y = g(t)$$

As you would expect, we have the following theorem regarding **fundamental sets of solutions**:

Theorem: If the functions $p_1, \ldots, p_n$ are continuous on the open interval I, if the functions $y_1, \ldots, y_n$ are solutions to the homogeneous problem and if the **Wronskian**

$$W(y_1, \ldots, y_n) = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

is nonzero for at least one point in I, then every solution of the homogeneous equation can be expressed as a linear combination of the solutions $y_1, \ldots, y_n$.

We call a linear combination of such $y_1, \ldots, y_n$ a **general solution**.

Functions $f_1, \ldots, f_n$ are said to be **linearly dependent** on an interval I if there exist constants $k_1, \ldots, k_n$, not all zero, such that $\sum_j k_j f_j(t) = 0$. Otherwise, they are **linearly independent** on I.

Linear dependence is tied to the Wronskian as in the case of second order differential equations.

Just like the second order case, it turns out that we can always write *any* solution to $\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} \cdots + p_n(t)y = g(t)$ as the sum of two pieces:

$$\phi(t) = \underbrace{\sum_j c_j y_j}_{\text{homogeneous part}} + \underbrace{Y(t)}_{\text{particular part}}$$

The **particular part** or **particular solution** $Y(t)$ is any solution to $\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} \cdots + p_n(t)y = g(t)$.

Our progression mirrors the second order case exactly: we'll first discuss homogeneous equations with constant coefficients, then we will tackle the nonhomogeneous (constant coefficient) case using the method of undetermined coefficients. We'll finish with the variation of parameters technique which is generally applicable.

4.1 Homogeneous Equations with Constant Coefficients: BD (4.2)

Let's start by *guessing* a solution of the form e^{rt} for the equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$$

and see where it leads us. If we substitute this function into the ODE and cancel the common exponential factors, we have the **characteristic equation**:

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_n = 0$$

This is nothing but a degree n polynomial. Beyond a certain degree there are no nice formulas for roots of degree n polynomials like the quadratic case. However, we still have the following possibilities:

1. For real and unequal roots $r_1, \ldots, r_n$ the general solution has the form
$$c_1 e^{r_1 t} + \cdots + c_n e^{r_n t}.$$
2. If we have purely imaginary roots, but none of the roots are repeated, the general solution has the form
$$c_1 \sin(\mathrm{Im} r_1 \ t) + \cdots + c_n \sin(\mathrm{Im} r_n \ t) + d_1 \cos(\mathrm{Im} r_1 \ t) + \cdots + d_n \cos(\mathrm{Im} r_n \ t)$$
where we simplify complex exponentials via Euler's formula as usual.
3. If the root r_1 is repeated, say, with multiplicity s, then
$$e^{r_1 t} \ , \ t e^{r_1 t} \ , \ \ldots \ , \ t^{s-1} e^{r_1 t}$$
must be included in the general solution.

Example: Find the general solution of $y''' - y'' - y' + y = 0$.

Example: Find the general solution of $y^{(4)} + y = 0$.

Example: Find the general solution $y^{(8)} + 8y^{(4)} + 16y = 0$.

Example: Find the solution to the IVP $y^{(4)} - 4y''' + 4y'' = 0$, $y(1) = -1$, $y'(1) = 2$, $y''(1) = 0$, $y'''(1) = 0$.

Before we press on to the method of undetermined coefficients, let me mention a generalization of Abel's theorem:

Abel's Theorem: If y_1 , ... , y_n are solutions of the differential equation

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} \cdots + p_n(t)y = 0 \ , \ \alpha < t < \beta$$

where p_1 , ... , p_n are continuous, then

$$W(y_1, \ldots, y_n)(t) = c \exp\left\{-\int p_1(t)dt\right\}$$

where c is a certain constant that depends on y_1 , ... , y_n but not on t. Further, W is either 0 for all $t \in I$ or else is never zero.

4.2 Method of Undetermined Coefficients: BD (4.3)

Here's a general recipe for implementing the method of undetermined coefficients for the higher order case. Note that all considerations carry over from the second order case. The primary difference is that the terms proposed for the nonhomogeneous part of the solution may need to be multiplied by higher powers of t up to power n. If we are to solve the differential equation $a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = g(t)$, we must

1. Find the general solution of the corresponding homogeneous equation.

2. Make sure that $g(t)$ involves nothing more than exponential functions, sines, cosines, polynomials or sums or products of such functions. If this is not the case, use variation of parameters (next section).

3. If $g(t) = g_1(t) + \cdots + g_m(t)$, then form m subproblems, each of which contains only one of the terms g_1 , $\ldots g_m$. Since the differential equation is linear, we are free to add up the particular solution for each subproblem to find the overall particular solution.

4. For the i-th subproblem assume a particular solution $Y_i(t)$ of the form

   (a) $t^s \left(A_0 t^m + \cdots + A_m\right)$, $s = 0, 1, \ldots, n$ if $g_i = \gamma_0 t^m + \cdots + \gamma_m$.

   (b) $t^s \left(A_0 t^m + \cdots + A_m\right) e^{\alpha t}$, $s = 0, 1, \ldots, n$ if $g_i = \left(\gamma_0 t^m + \cdots + \gamma_m\right) e^{\alpha t}$.

   (c) $t^s \left[\left(A_0 t^m + \cdots + A_m\right) e^{\alpha t} \cos(\beta t) + \left(B_0 t^m + \cdots + B_m\right) e^{\alpha t} \sin(\beta t)\right]$, $s = 0, 1, \ldots, n$ if $g_i = \left(\gamma_0 t^m + \cdots + \gamma_m\right) e^{\alpha t}\left\{\sin(\beta t) \text{ or } \cos(\beta t)\right\}$.

5. Find a particular solution Y_i for each of the subproblems. The sum $Y_1 + \cdots + Y_m$ is a particular solution of the nonhomogeneous equation.

6. Form the sum of the particular solution and the general solution to the homogeneous problem.

7. Use the initial conditions to determine the values of the arbitrary constants remaining in the solution.

If you follow this algorithm correctly you can implement the method of undetermined coefficients for any problem where it is relevant. If the algorithm fails to produce a solution, you need to try something else like the variation of parameters method.

4.3 Variation of Parameters: BD (4.4)

The method works as follows: assume we know the general solution to the corresponding homogeneous problem $c_1 y_1(t) + \cdots + c_n y_n(t)$. As a guess for the nonhomogeneous problem, we try a function similar to this one only with the coefficients replaced with unknown parameter functions $u_1(t)$, ... , $u_n(t)$:

$$y(t) = u_1(t)y_1(t) + \cdots + u_n(t)y_n(t)$$

Unlike the second order case, there isn't a very tidy formula for the parameter functions $u_j(t)$. However, it is still theoretically possible to write down a set of first order differential equations for the parameter functions $u_j(t)$ which can (sometimes) be solved via integration.

Chapter 5

Laplace Transform

The Laplace Transform is an integral technique that allows us to change a differential equation into an algebraic equation. The Laplace Transform is defined as follows

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$$

whenever the integral converges. The conditions for convergence of the integral are mild since the exponential decays very rapidly for positive s. Specifically, if f is piecewise continuous on the interval $[0, A]$ for any positive A and if $|f(t)| \leq Ke^{at}$ for $t \geq M$ for some set of positive constants a, K, and M, then the Laplace transform of f exists for $s > a$.

Example: Suppose $f(t) = 1$ for $t \geq 0$. Then $\mathcal{L}\{1\} = \frac{1}{s}$ for $s > 0$.

Example: Suppose $f(t) = e^{at}$ for $t \geq 0$. Then $\mathcal{L}\{f\} = \frac{1}{s-a}$ for $s > a$.

Example: Suppose $f(t) = \delta(t - a)$, the **Dirac Delta Function**. Then $\mathcal{L}\{f\} = e^{-sa}$ for $a > 0$.

There are a few properties of the Laplace transform that make it a useful tool for understanding some differential equations. First off, it's **linear**: $\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$ for constants a and b assuming that the Laplace transforms of the individual functions exist. Also,if f is continuous and f' is piecewise continuous on any interval $0 \leq t \leq A$ and suppose further that $|f(t)| \leq Ke^{at}$ for $t \geq M$ for positive constants K, a and M, then $\mathcal{L}\{f'\}$ exists and is equal to

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \tag{5.0.1}$$

for $s > a$.

The proof is a simple application of the integration by parts technique. This property shows that the Laplace transform in a sense turns derivatives (or primes) into powers of s. This conclusion is best illustrated by computing the Laplace transform $f^{(n)}(t)$ (assuming the relevant integrals converge):

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \ldots sf^{(n-2)}(0) - f^{(n-1)}(0). \tag{5.0.2}$$

In short, the Laplace transform can turn some differential equations into algebraic equations, which are easier to solve. The problem is finding the inverse Laplace transform of the result. There is an integral formula allowing us to invert the Laplace transform, but it's use requires knowledge of complex analysis. Instead, we usually have to use the table of elementary Laplace transforms available on page 321 of the 10th edition of the text.

Example: Solve the differential equation $y'' + 5y' + 6y = 0$ with initial conditions $y(0) = 1, y'(0) = 0$ using the Laplace Transform.

Example: Solve the differential equation $y'' + 2y' + y = 0$ with initial conditions $y(0) = 0, y'(0) = -5$ using the Laplace Transform. (Be aware of the multiple root!)

Example: Solve the differential equation $y'' - 2y' + 2y = \cos(t)$ with initial conditions $y(0) = 1, y'(0) = 0$ using the Laplace Transform.

Example: Solve the differential equation $y'' - 2y' + 2y = e^{-t}$ with initial conditions $y(0) = 0, y'(0) = 1$ using the Laplace Transform.

Though we won't have time to delve into the Laplace Transform in this course, it is a remarkable tool that finds uses in many areas of physics and engineering, particularly when we need to solve certain partial differential equations. Since this area is beyond the scope of our course, let me conclude this discussion of the Laplace transform by convincing you of its utility when we must solve a linear differential equation with constant coefficients and a driving term that is **discontinuous**. Such problems arise in electrical engineering, for instance, when we want to model the current flow through a circuit that is suddenly switched on. In physics, the Dirac Delta function above is used to model impulses; *i.e.* sudden jolts or kicks to the system.

Example: Solve the differential equation $y'' - 2y' + 2y = H(t-1)$ with initial conditions $y(0) = 0, y'(0) = 1$ using the Laplace Transform.

Example: Solve the differential equation $y'' - 2y' + 2y = \delta(t-1)$ with initial conditions $y(0) = 0, y'(0) = 1$ using the Laplace Transform.

Chapter 6

Series Solutions

In this portion of the course we will look at how to use infinite series to solve the differential equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

As we know, solving the homogeneous case is crucial to understanding how to solve the nonhomogeneous equation. Furthermore, the techniques we will develop here are applicable to higher order equations but we retain relative simplicity just by focusing on the second degree case.

Initially, we will assume P, Q and R are polynomials, but series solutions may also be constructed when these coefficient functions are **analytic**. By analytic, I mean that the function equals its Taylor series expansion at least locally (within some interval).

Infinite series are used to find solutions of a broad class of differential equations which appear in the natural sciences. Here are just a few applications to motivate our discussion:

1. **Bessel's equation**

   $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right)y = 0$$

   arising in the context of electromagnetic wave propagation in cylindrical waveguides, pressure amplitudes of inviscid rotational flows and vibrational modes of a drum,

2. **Hermite's equation**

   $$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0$$

arising in probability theory and quantum mechanics, and

3. the **Airy equation**
$$y'' \pm k^2 xy = 0$$
used to understand semiconductor devices and the properties of rainbows.

I will assume you are familiar with the standard treatment of power series presented in all Calculus II courses. If you need to refresh your memory, take a look at section 5.1 of the text.

6.1 Series Solutions Near an Ordinary Point: BD (5.2/5.3)

Starting with the equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

we will focus on the case where the coefficient functions are polynomials without common factors. If a common factor exists, just cancel it off.

Now for some terminology: a point x_0 is called a **regular point** if $P(x_0) \neq 0$. If that's the case, then we can rewrite the equation as

$$\frac{d^2y}{dx^2} + \frac{Q(x)}{P(x)}\frac{dy}{dx} + \frac{R(x)}{P(x)}y = 0$$

near the point x_0 since we know that the rational coefficient functions are necessarily continuous in an interval around x_0. Let's call this interval I. Then we are guaranteed a unique solution to the initial value problem $y(x_0) = y_0$ and $y'(x_0) = y_0'$ in this interval.

If on the other hand $P(x_0) = 0$, then one of the functions Q and R is non-zero at x_0 since we canceled off common factors. That means at least one of the rational functions Q/P, R/P blows up at x_0. In this case we call x_0 a **singular point**. We'll deal with singular points later.

As an example I will analyze the "standard" Airy equation (which has relevance for understanding semiconductor technology) near the point $x_0 = 0$.

The standard Airy equation is given by

$$y'' - xy = 0$$

and cannot be solved using the analytic techniques we have covered so far. But let's press on with a guess: all of the coefficient functions are polynomials in x. Let's try a polynomial solution of the form $ax + b = y_{\text{guess}}$. Plugging this into the equation, we find $-x\,(ax + b) = 0$ which can only be true if both coefficients are 0. Since we aren't interested in the trivial solution, this guess didn't work out.

But let's be persistent and try a higher order polynomial: $ax^2 + bx + c = y_{\text{guess}}$. Now we can choose $2a = c$, $b = 0$ (which is promising!) but we are still left with a cubic term that doesn't vanish. If you try a cubic guess, you'll be left with a quartic term that doesn't vanish. See the pattern?

So this reasoning leads us to the following question: can we try a guess solution which is of **infinite** degree? In other words, can we try

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

and see if we can come up with coefficients which make this sum a solution of the Airy equation?

It is possible to show that

$$a_{3n+2} = 0\ ,\ a_{3n} = \frac{a_0}{3n\,(3n-1)\,(3n-3)(3n-4)\dots(6\times 5)(3\times 2)}$$

and

$$a_{3n+1} = \frac{a_1}{3n\,(3n+1)\,(3n-2)(3n-3)\dots(7\times 6)(4\times 3)}$$

allow us to solve the Airy equation. We can arrive at these formulae by plugging the series expansion into the differential equation, finding a recursive definition of the coefficients and then solving the recursive relationship.

This solution looks unappealing, but it is actually quite useful. For one, we don't necessarily need many terms of the series if all we are interested in is approximating the answer. This is in fact necessary if, for instances, you are trying to graph the solution. Also, infinite series are made up of power functions which are very easy to manipulate. In fact, we can move derivatives and integrals past the summation symbol as long as we are within the radius of convergence of our series.

In general, we have

$$\begin{aligned} y = a_0 &\left[1 + \frac{x^3}{2\times 3} + \frac{x^6}{2\times 3\times 5\times 6} + \dots \frac{x^{3n}}{2\times 3\cdots\times(3n-1)(3n)} + \dots\right] \\ + a_1 &\left[x + \frac{x^4}{3\times 4} + \frac{x^7}{3\times 4\times 6\times 7} + \dots \frac{x^{3n+1}}{3\times 4\cdots\times(3n)(3n+1)} + \dots\right] \end{aligned}$$

We can show that the radii of convergence of the two series appearing above are both infinite and so the series converges for all real x.

However, we should note that approximating the solution to Airy's equation by truncating the infinite series will only work reasonably near $x = 0$ since that is where we chose to expand the solution.

This phenomenon is general and we can illustrate it with a simple example.

Example: Consider the differential equation $y' - y = 0$. We know that the solution is obviously $y(x) = Ce^x$ for some choice of C depending on the initial condition. For sake of argument, suppose we didn't know this and just applied the series method for finding the solution.

It is not too hard to show that we would find (upon expanding near $x = 0$) that

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

This is nothing but the Taylor series expansion of the exponential function near $x = 0$. In the figure below, I plot the exact solution (e^x) along with the first few partial sums of the Taylor series: Notice how the approximate curves are very close to e^x near 0, but fail to

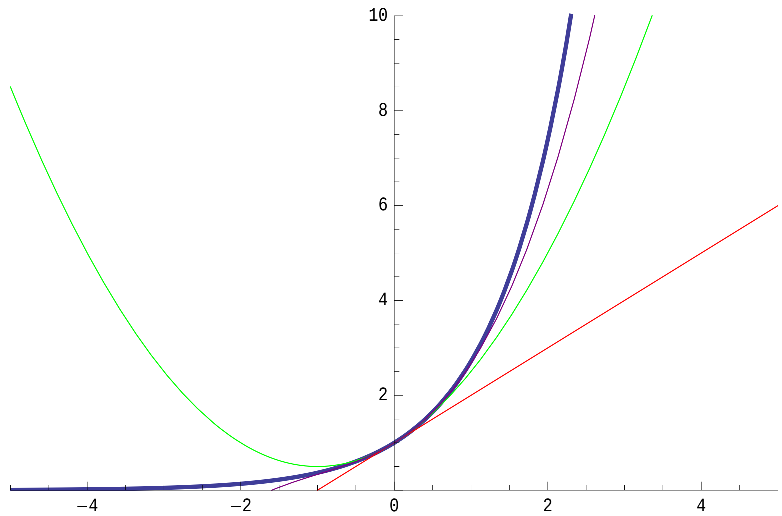


Figure 6.1: Figure: Plot of e^x (in blue) along with $1 + x$ (red), $1 + x + x^2/2$ (green) and $1 + x + x^2/2 + x^3/6$ (purple).

approximate the exponential as we move away from $x = 0$. Notice also that the more terms we include, the better the approximation, at least near $x = 0$.

A relatively simple way to determine the interval in which a given partial sum is a reasonably accurate approximation is to compare graphs of that partial sum and the next one. When the two differ appreciably, we can be confident that the original partial sum is no longer accurate. Referring back to figure 1, the red line corresponds to the partial sum $1+x$. The next partial sum would be $1 + x + x^2/2$ which is plotted in green. We notice an appreciable gap between the two graphs near $x = \pm 0.5$. This then is the limit of the approximation's validity.

The partial sum $1 + x + x^2/2$ (in green) and the next partial sum $1 + x + x^2/2 + x^3/6$

(in purple) are noticeably different outside of ± 1. So $1 + x + x^2/2$ is no longer a good approximation outside of $(-1, 1)$.

If we want a good approximation for the solution of $y' - y$ near say $x = 1$ using series, then the best plan of action would be to expand near $x = 1$. In this case, the series solution would have the form

$$y(x) = \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n.$$

For completeness, I plot the first few partial sums below

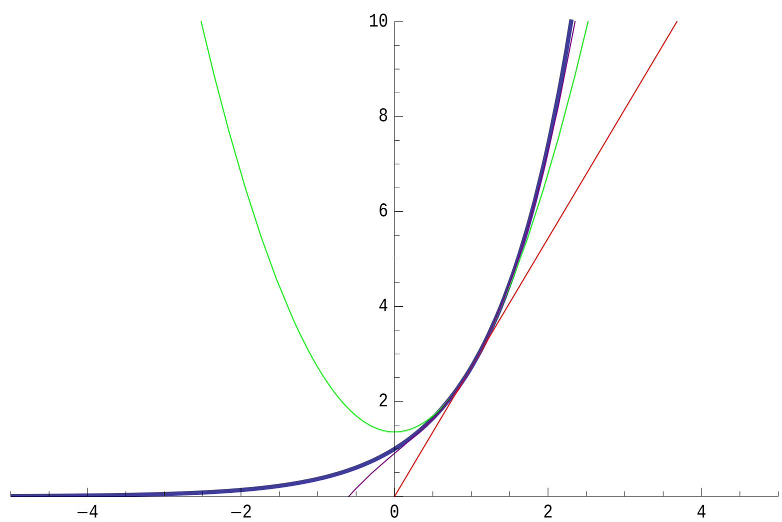


Figure 6.2: Figure: Plot of e^x (in blue) along with ex (red), $e(x + (x-1)^2/2)$ (green) and $e(x + (x-1)^2/2 + (x-1)^3/6)$ (purple).

Example: Find the power series solution of the equation $y'' - xy' - y = 0$ near $x_0 = 1$. Find the first four terms in each of two solutions y_1 and y_2. Compute the Wronskian to show that y_1 and y_2 form a fundamental set of solutions.

Partial ANS: The recursive definition of the coefficients is given by $(n+2)a_{n+2} - a_{n+1} - a_n = 0$. We also have

$$y_1(x) = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \ldots$$

and

$$y_2(x) = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \ldots$$

Example: Find the power series solution of the equation $2y'' + (x+1)y' + 3y = 0$ near $x_0 = 2$. Find the first four terms in each of two solutions y_1 and y_2. Compute the Wronskian to show that y_1 and y_2 form a fundamental set of solutions.

Partial ANS: The recursive definition of the coefficients is given by $2(n+2)(n+1)a_{n+2} + 3(n+1)a_{n+1} + (n+3)a_n = 0$. We also have

$$y_1(x) = 1 - \frac{3}{4}(x-2)^2 + \frac{3}{8}(x-2)^3 + \frac{1}{64}(x-2)^4 + \dots$$

and

$$y_2(x) = (x-2) - \frac{3}{4}(x-2)^2 + \frac{1}{24}(x-2)^3 + \frac{9}{64}(x-2)^4 + \dots$$

Example: The Hermite Equation The equation $y'' - 2xy' + \lambda y = 0$ where λ is a constant, is known as the Hermite equation.

1. Find the first four terms in each of two solutions about $x = 0$ and show that they form a fundamental set of solutions.

2. Observe that if λ is a nonnegative even integer, then one or the other series solutions terminates and becomes a polynomial.

3. The Hermite polynomial $H_n(x)$ is defined as the polynomial solution of the Hermite equation with $\lambda = 2n$ for which the coefficient of x^n is 2^n. Find $H_0(x)$, $H_1(x)$ and $H_2(x)$.

We have assumed in our discussion that P, Q and R are polynomials without common factors. This situation is not general enough. We want to be able to use the series method on a broader class of problems.

This leads us to a natural and general definition of an **ordinary point**: an ordinary point x_0 of the differential equation if the functions $p = Q/P$ and $q = R/P$ are analytic at x_0. A function is analytic at a point if it equals its Taylor series expansion at that point. Otherwise, x_0 is called a **singular point**.

With this definition of an ordinary point we have the following theorem:

Theorem 5.3.1: If x_0 is an ordinary point of the differential equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0,$$

then the general solution is

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 y_1(x) + a_1 y_2(x),$$

where a_0 and a_1 are arbitrary, and y_1 and y_2 are two power series solutions that are analytical at x_0. These solutions form a fundamental set. Further, the radius of convergence for each of the series solutions is at least as large as the minimum of the radii of convergence of the series for p and q.

Example: Can we determine a series solution about $x = 0$ for the differential equation $y'' + \cos(x)y' + (1 + x^3)y = 0$, and if so, what is the radius of convergence?

Example: Determine a lower bound for the radius of convergence of series solutions about $x_0 = 4, 0$ for the differential equation $(x^2 - 2x - 3)y'' + xy' + 4y = 0$.

Example: Determine a lower bound for the radius of convergence of series solutions about $x_0 = -1/2, 0$ for the differential equation $(1 + x^2)y'' + 2xy' + 4x^2y = 0$.

Partial ANS: In the first case, the series converges at least for $|x + 1/2| < \sqrt{5}/2$. In the second case, $|x| < 1$. What's interesting about this problem is that Theorem 5.3.1 only guarantees the existence of a series solution in $|x| < 1$ for the second case. On the other hand $1 + x^2 \neq 0$ for all real x so we know that a unique solution should exist on the entire real line. Basically, the resolution of the paradox is that there may be a solution on the real line which does not have a power series about $x = 0$ that converges everywhere.

Example: The Chebyshev Equation The Chebyshev differential equation is $(1-x^2)y'' - xy' + \alpha^2 y = 0$, where $\alpha = 0$.

1. Determine two solutions in powers of x for $|x| < 1$.
2. Show that if α is a nonnegative integer n, then there is a polynomial solution of degree n.

6.2 Regular Singular Points: BD (5.4/5.5)

We now turn to the problem of finding solutions to differential equations near so-called singular points. These are points where $P(x_0) = 0$ but at least one of $Q(x_0)$ and $R(x_0)$ is nonzero.

It turns out that the most interesting behavior of a differential equation occurs near its singular points, so we simply cannot ignore them. However, their analysis is difficult since we cannot immediately apply the series method of the previous few sections.

To simplify the analysis as much as possible, we assume that our singular points are **regular**:

$$(x - x_0)\,\frac{Q(x)}{P(x)}\ ,\ (x - x_0)^2\,\frac{R(x)}{P(x)}$$

both have convergent Taylor series about x_0. Otherwise the singular point is called **irregular**. Irregular singular points are very complicated and lie outside the scope of this course.

Example: Find all singular points of $x^2y'' + xy' + (x^2 - \nu^2)\,y = 0$ and classify them.

Example: Find all singular points of $(x + 2)^2(x - 1)y'' + 3(x - 1)y' - 2(x + 2)y = 0$ and classify them.

Example: Find all singular points of $xy'' + y' + (\cot x)\,y = 0$ and classify them.

The general approach to regular singular points is outlined in Theorem 5.6.1. I'll state it for completeness shortly. Basically, the procedure is to assume a series solution of the form $y = x^r \sum_{n=0}^{\infty} a_n x^n$ where $a_0 \neq 0$. We assume for simplicity that $x = 0$ corresponds to the singular point of interest. If not, we can always translate the singular point from $x_0 \neq 0$ to 0 without affecting the problem. We also assume our attention is focused on $x > 0$; for

$x < 0$ we simply redefine the variable to be $\xi = -x > 0$ and proceed as I'll describe.

Since we assume that $x = 0$ is a regular singular point, the functions $xp(x) = xQ(x)/P(x)$ and $x^2R(x)/P(x) = x^2q(x)$ have finite limits as $x \to 0^+$ and are analytic at $x = 0$.

This means that we can write $xp(x) = \sum_{n=0}^{\infty} p_n x^n$, $x^2q(x) = \sum_{n=0}^{\infty} q_n x^n$, on some interval $|x| < \rho$ about the origin, where $\rho > 0$.

We write

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

as

$$x^2\frac{d^2y}{dx^2} + x[xp(x)]\frac{dy}{dx} + [x^2q(x)]y = 0.$$

This is not necessary when P, Q and R are polynomials, however.

Notice that p_0 and q_0 are necessarily finite by assumption.

Substituting the series expansion $y = x^r \sum_{n=0}^{\infty} a_n x^n$ into this differential equation yields an equation for r and a relation for the coefficients a_n. The equation for r is called the **indicial equation** and the values of r are called the **exponents at the singularity**.

This process is known as the **Method of Frobenius**. In all cases we consider in this class, it is possible to find at least one solution of the form $y = x^r \sum_{n=0}^{\infty} a_n x^n$. It is possible (as we will see) that method only yields on series solution. In order to find another to complete a fundamental set, we could use the method of reduction of order or some other process we've see in the course.

It is probably most beneficial to look at a few examples dealing with this method to understand its finer points.

Example: Euler Equations One of the most basic family of equations with regular singular points is the family of Euler equations:

$$x^2y'' + \alpha xy' + \beta y = 0$$

where α and β are real constants.

According to the Frobenius method, we can identify $xp(x) = \alpha$ and $x^2q(x) = \beta$. Let's attempt to find solutions of the form $y = x^r \sum_{n=0}^{\infty} a_n x^n$. For simplicity, assume that $x > 0$.

Plugging this into the differential equation, we find that all of the coefficients $a_n, n > 0$ are 0. We also find the **indicial equation**

$$r(r-1) + \alpha r + \beta$$

which is a quadratic in the **exponents at the singularity**.

As we know from algebra, there will be three possibilities for the exponents: they will either be distinct and real, equal or complex conjugates.

Real, Distinct Roots: It is not hard to show that $y = c_1 x^{\frac{1}{2}} + c_2 x^{-1}$ for $x > 0$ in this case.

Equal Roots: We can show that $y_1 = x^{r_1}$ and $y_2 = x^{r_1} \ln x$ is another obtained through the method of reduction of order. Again, $x > 0$.

Complex Roots: The most general solution in this case is $y = c_1 x^{\lambda} \cos(\mu \ln x) + c_2 x^{\lambda} \sin(\mu \ln(x))$ where the complex roots are $r_1 = \lambda + i\mu = r_2^*$.

For $x < 0$, we define $\xi = -x > 0$ and repeat the above exercise.

Example: For the equation $2xy'' + y' + xy = 0$, show that there is a regular singular point at $x = 0$. Determine the indicial equation, the recurrence relation, and the roots of the indicial equation. Find the series solution $(x > 0)$ corresponding to the larger root.

Example: For the equation $xy'' + y = 0$, show that there is a regular singular point at $x = 0$. Determine the indicial equation, the recurrence relation, and the roots of the indicial equation. Find the series solution ($x > 0$) corresponding to the larger root.

Example: For the equation $x^2y'' + xy' + (x - 2)y = 0$, show that there is a regular singular point at $x = 0$. Determine the indicial equation, the recurrence relation, and the roots of the indicial equation. Find the series solution ($x > 0$) corresponding to the larger root.

Example: The Laguerre differential equation is $xy'' + (1 - x)y' + \lambda y = 0$. Show that $x = 0$ is a regular singular point. Determine the indicial equation, its roots, and the recurrence relation. Find one solution for $x > 0$. Show that if $\lambda = m$ is a positive integer, this solution reduces to a polynomial.

Example: The Bessel Equation of Order Zero is $x^2y'' + xy' + x^2y = 0$. Construct

$$J_0(x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m!^2}$$

as a solution for $x > 0$. This is known as **the Bessel function of the first kind of order zero**.

Construct the second solution

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{x}{2}\right) J_0(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m} m!^2} x^{2m}\right]$$

where $H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$ and $\gamma = \lim_{n\to\infty}(H_n - \ln n)$ is the **Euler-Mascheroni constant**. This is known as the **Bessel function of the second kind of order zero**.

For completeness, I include here Theorem 5.6.1 summarizing the method of Frobenius:

Theorem 5.6.1 Consider the differential equation

$$x^2\frac{d^2y}{dx^2} + x[xp(x)]\frac{dy}{dx} + [x^2q(x)]y = 0.$$

where $x = 0$ is a regular singular point. Then $xp(x)$ and $x^2q(x)$ are analytic at $x = 0$ with convergent power series expansions

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n \ , \ x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n$$

for $|x| < \rho$, where $\rho > 0$ is the minimum of the radii of convergence of the power series for $xp(x)$ and $x^2q(x)$. Let r_1 and r_2 be the roots of the indicial equation

$$F(r) = r(r-1) + p_0 r + q_0 = 0$$

with $r_1 \geq r_2$ if r_1 and r_2 are real. Then in either the interval $-\rho < x < 0$ or the interval $0 < x < \rho$, there exists a solution of the form

$$y_1(x) = |x|^{r_1} \left[1 + \sum_{n=1}^{\infty} a_n(r_1) x^n \right]$$

where the $a_n(r_1)$ are given by the recurrence relation with $a_0 = 1$ and $r = r_1$.

If $r_1 - r_2$ is not zero or a positive integer, then in either the interval $-\rho < x < 0$ or the interval $0 < x < \rho$, there exists a second solution of the form

$$y_2(x) = |x|^{r_2} \left[1 + \sum_{n=1}^{\infty} a_n(r_2) x^n \right]$$

The power series converge at least for $|x| < \rho$.

If $r_1 = r_2$, then the second solution has the form

$$y_2(x) = y_1(x) \ln|x| + |x|^{r_1} \sum_{n=1}^{\infty} b_n(r_1) x^n$$

If $r_1 - r_2 = N$, a positive integer, then the second solution has the form

$$y_2(x) = a y_1(x) \ln|x| + |x|^{r_2} \sum_{n=1}^{\infty} c_n(r_2) x^n$$

Each of the series converges at least for $|x| < \rho$ and defines a function analytic in some neighborhood of 0. In all three cases, the two solutions y_1 and y_2 form a fundamental set of solutions of the given differential equation.

Chapter 7

First Order Linear Systems of Differential Equations

There are a few really good reasons for studying first order (linear) systems:

1. Many realistic models of natural phenomena are described by **systems** of simultaneous ordinary differential equations. Examples include (but are not limited to) spring-mass systems consisting of two or more masses, parallel LRC circuits and predator-prey systems.

2. General n-th order differential equations can be converted into a system of first order differential equations anyway so there isn't much to lose pedagogically speaking and much to gain in making this detour.

3. Understanding nonlinear systems near equilibria requires the use of first order linear systems of equations.

Let's set the stage for the general theory of systems of first order differential equations.

7.1 Introduction: BD (7.1)

When we began our journey I introduced a very simple predator-prey model:

$$\frac{dC}{dt} = aC - bCW \ , \ \frac{dW}{dt} = -cW + dCW$$

where a, b, c, d are positive constants.

This system of equations quantifies the interaction between a population of caribou (the prey) and a population of tundra wolves (the predators). This is an example of the most general first order system

$$\begin{aligned} x_1'(t) &= F_1(t, x_1, x_2, \ldots, x_n) \\ x_2'(t) &= F_2(t, x_1, x_2, \ldots, x_n) \\ &\vdots \\ x_n'(t) &= F_n(t, x_1, x_2, \ldots, x_n) \end{aligned}$$

Here, we label all of our dependent variables by $x_j(t)$ and we take our independent variable to be t. For example, in the predator-prey model, we could choose $C(t) = x_1(t)$, $W(t) = x_2(t)$ in which case $F_1(t, x_1, x_2) = ax_1 - bx_1x_2$ and $F_2(t, x_1, x_2) = -cx_2 + dx_1x_2$.

Using x_j to denote our dependent variable set is standard and is quite economical when you are dealing with large systems with many components (think of complex chemical reactions or sectors of a realistic economy).

Definition: If each of the functions $F_1, \ldots, F_n$ is a linear function of the dependent variables, then the system of equations is said to be **linear**.

Clearly, the predator-prey model is nonlinear. A linear example arises from studying parallel LRC circuits:

Example: Suppose we have a capacitor, a resistor and an inductor all wired in parallel to a voltage source. Using physics (specifically Kirchhoff's laws), we find that the current through the inductor and the voltage drop across the capacitor obey

$$\frac{dI}{dt} = \frac{V}{L}\ , \ \frac{dV}{dt} = -\frac{I}{C} - \frac{V}{RC}$$

The equations here describe a linear system of first order differential equations. Furthermore, this system is **homogeneous** since there are no functions of time being added to the terms involving I and V.

Our primary focus will be on linear systems of first order differential equations. Understanding the linear case is crucial to understanding how nonlinear systems behave near equilibria as we'll see.

As further motivation, consider the single second order differential equation

$$y'' + 2y' - y = 0$$

At this stage we don't know how to solve such a differential equation. As I mentioned in the preface it is always possible to convert an n-th order differential equation into a system of first order differential equations:

The trick is to define $x_1(t) = y(t)$, $x_2(t) = y'(t)$, $\dots$, $x_{n-1}(t) = y^{(n-1)}(t)$. That way $\frac{dx_{n-1}}{dt} = y^{(n)}$ which is determined by all of the other x_j through the original differential equation and $\frac{dx_j}{dt} = x_{j+1}$ for $j < n-1$.

Going back to our original example, if $x_1(t) = y(t)$ and $x_2(t) = y'(t)$, then

$$x_1'(t) = x_2(t) \ , \ x_2'(t) = -2x_2 + x_1$$

which is a linear first order system.

Sections 7.2 and 7.3 of the text review the necessary linear algebra terms. I will proceed assuming you have seen this terminology before. If you need a refresher, take a few minutes to read those sections of the text.

7.2 Basic Theory of Systems of First Order Linear Equations: BD (7.4)

Explicitly, the equations that concern us in this section are of the form

$$\begin{aligned}
x_1'(t) &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + g_1(t) \\
x_2'(t) &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + g_2(t) \\
&\vdots \\
x_n'(t) &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + g_n(t)
\end{aligned}$$

To discuss this system most effectively, we write it in matrix form:

$$\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x} + \boldsymbol{g}(t).$$

Not only does this clean up the notation but it also suggests strong parallels between such systems and single first order differential equations.

Definition: A vector $\boldsymbol{x} = \boldsymbol{\phi}(t)$ is said to be a **solution** if its components satisfy the original system of equations.

Throughout this section we assume that $\boldsymbol{P}$ and $\boldsymbol{g}$ are continuous on some interval $I = (\alpha, \beta)$: this is sufficient to guarantee the existence of solutions on I.

To simplify our lives, we will at first only consider systems of the form $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$; i.e. $\boldsymbol{g}(t) = 0$ and the system is said to be **homogeneous**.

Since our equations are linear if we have two solutions $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$, the **linear combination**

$$c_1\boldsymbol{x}^{(1)} + c_2\boldsymbol{x}^{(2)}$$

is also a solution for any choice of constant values c_1 and c_2.

In fact, we have the following theorem:

Theorem: If the vector functions $\boldsymbol{x}^{(1)}\ ,\ \ldots\ ,\ \boldsymbol{x}^{(n)}$ are **linearly independent** solutions of the system $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$ for each point of I, then each solution $\boldsymbol{x} = \boldsymbol{\phi}(t)$ of the system can be written as a linear combination

$$\boldsymbol{\phi} = c_1\boldsymbol{x}^{(1)} + \cdots + c_n\boldsymbol{x}^{(n)}$$

in exactly one way.

If we think of the **weights** $c_1 \ldots c_n$ as being arbitrary, the linear combination

$$c_1\boldsymbol{x}^{(1)} + \cdots + c_n\boldsymbol{x}^{(n)}$$

is called the **general solution** and the set of **fundamental solutions** $\boldsymbol{x}^{(1)}\ ,\ \ldots\ ,\ \boldsymbol{x}^{(n)}$ form a **basis** for our solution space.

You may wonder how to test $\boldsymbol{x}^{(1)}\ ,\ \ldots\ ,\ \boldsymbol{x}^{(n)}$ for linear independence. There is a theorem which guarantees that if $\boldsymbol{x}^{(1)}\ ,\ \ldots\ ,\ \boldsymbol{x}^{(n)}$ forms a solution set and is linearly independent at some t_0 (i.e. $\boldsymbol{x}^{(1)}(t_0)\ ,\ \ldots\ ,\ \boldsymbol{x}^{(n)}(t_0)$ is a collection of linearly independent vectors) then $\boldsymbol{x}^{(1)}\ ,\ \ldots\ ,\ \boldsymbol{x}^{(n)}$ is linearly independent for *all* $t \in I$.

We conclude this section with a theorem which guarantees the existence of at least one fundamental set:

Theorem: Let $\boldsymbol{e}^{(1)}\ ,\ \ldots\ ,\ \boldsymbol{e}^{(n)}$ denote the n-dimensional **standard basis** and further let $\boldsymbol{x}^{(1)}\ ,\ \ldots\ ,\ \boldsymbol{x}^{(n)}$ be the solutions to the system $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$ with initial conditions

$$\boldsymbol{x}^{(1)}(t_0) = \boldsymbol{e}^{(1)}\ ,\ \ldots\ ,\ \boldsymbol{x}^{(n)}(t_0) = \boldsymbol{e}^{(n)},$$

respectively, where t_0 is any point in I. Then $\boldsymbol{x}^{(1)}\ ,\ \ldots\ ,\ \boldsymbol{x}^{(n)}$ form a fundamental set of solutions.

7.3 Homogeneous Linear Systems with Constant Coefficients: BD (7.5/7.6/7.7)

We will concentrate most of our attention on systems of homogeneous linear equations with constant coefficients; that is

$$\boldsymbol{x}' = A\boldsymbol{x}$$

where A is a constant $n \times n$ real matrix.

The case $n = 2$ is particularly important and lends itself to visualization in the x_1x_2-plane, called the **phase plane**. By evaluating $A\boldsymbol{x}$ at a large number of points and plotting the resulting vectors one obtains a direction field of tangent vectors to solutions of the system of differential equations. A qualitative understanding of the behavior of the solutions can usually be gained from a direction field.

Definition: A plot that shows a representative sample of trajectories for a given system is called a **phase portrait**.

If A happens to be diagonalizable, then it is possible to decouple the system of equations by writing $A = VDV^{-1}$. In fact, it is possible to show that the solution will be $\boldsymbol{x}(t) = e^{tA}\boldsymbol{x}(0)$ as long as A is diagonalizable:

$$\frac{d}{dt}e^{tA} = \frac{d}{dt}\left(V \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & \\ 0 & e^{\lambda_2 t} & 0 & \dots \\ \vdots & & & \end{bmatrix} V^{-1} \right) = V \begin{bmatrix} \lambda_1 e^{\lambda_1 t} & 0 & \dots & \\ 0 & \lambda_2 e^{\lambda_2 t} & 0 & \dots \\ \vdots & & & \end{bmatrix} V^{-1} = Ae^{tA}$$

This is in complete analogy to solving the equation $x'(t) = ax(t)$ which we all know has the exponential function as the unique nontrivial solution.

It should hopefully be clear from this calculation that the nature of the eigenvalues and the corresponding eigenvectors determine the nature of the general solution of the system $\boldsymbol{x}' = A\boldsymbol{x}$. If we assume that A is a real-valued matrix, there are three possibilities for the eigenvalues of A:

1. All eigenvalues are real and different from each other.
2. Some eigenvalues occur in complex conjugate pairs.
3. Some eigenvalues are repeated.

In the second and third cases we must be careful to make sure our matrix A is in fact diagonalizable. Diagonalizability is guaranteed if we have n linearly independent eigenvectors. This condition may not be met if we have repeated eigenvalues. To keep the discussion as non-technical as possible, I will postpone discussing the case of repeated eigenvalues

Note: Hermitian and symmetric matrices are *always* diagonalizable and arise often in practical applications. So we are not losing much generality by focusing on the diagonalizable A case.

Example: Consider the system defined by $A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$. Find its general solution and sketch a phase portrait.

ANS: For any initial conditions, the trajectories spiral out, away from the origin. The origin (which is always an equilibrium point for $\boldsymbol{x}' = A\boldsymbol{x}$) is known as a **spiral point**. Since

$$e^{tA} = \begin{bmatrix} e^{2t}\left(\cos(t) + \sin(t)\right) & e^{2t}\sin(t) \\ -2e^{2t}\sin(t) & e^{2t}\left(\cos(t) - \sin(t)\right) \end{bmatrix},$$

we can obtain the solution for any initial condition $\boldsymbol{x}(0)$ via matrix multiplication.

Example: Consider the system defined by $A = \begin{bmatrix} -7 & 10 \\ -4 & 5 \end{bmatrix}$. Find its general solution and sketch a phase portrait.

ANS: For any initial conditions, the trajectories asymptotically approach the origin. The origin is known as an **spiral point**.

$$e^{tA} = \begin{bmatrix} e^{-t}\left(\cos(2t) - 3\sin(2t)\right) & 5e^{-t}\sin(2t) \\ -2e^{-t}\sin(2t) & e^{-t}\left(\cos(2t) + 3\sin(2t)\right) \end{bmatrix}$$

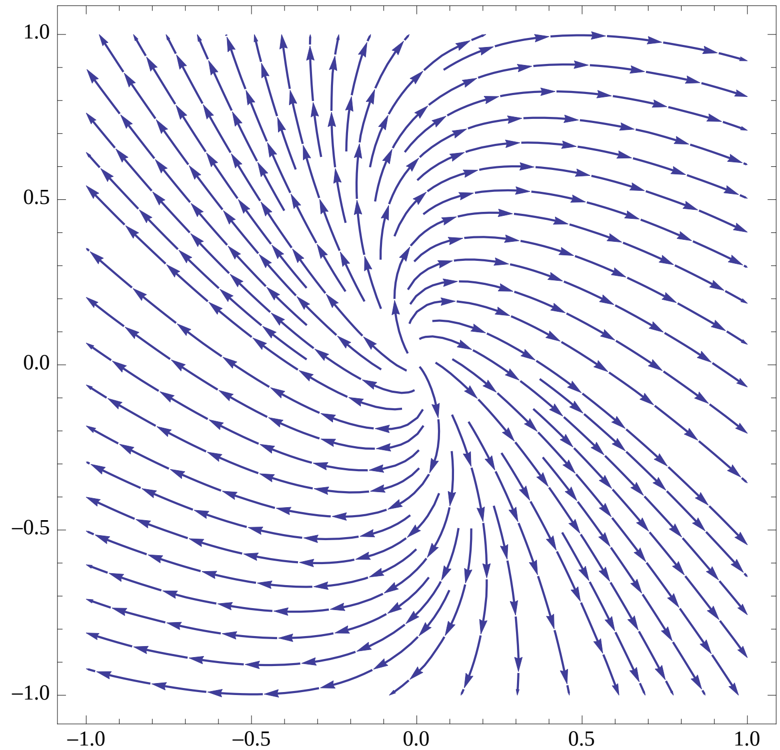


Figure 7.1: Phase portrait in phase space.

Example: Consider the system defined by $A = \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}$. Find its general solution and sketch a phase portrait.

ANS: Some trajectories approach the origin at first and then change direction and move away. The origin in this case is known as a **saddle point**.

$$e^{tA} = \begin{bmatrix} \frac{1}{7}e^{-t}\left(2+5e^{7t}\right) & -\frac{5}{7}e^{-t}\left(-1+e^{7t}\right) \\ -\frac{2}{7}e^{-t}\left(-1+e^{7t}\right) & \frac{1}{7}e^{-t}\left(5+2e^{7t}\right) \end{bmatrix}$$

For second order systems with real coefficients we have:

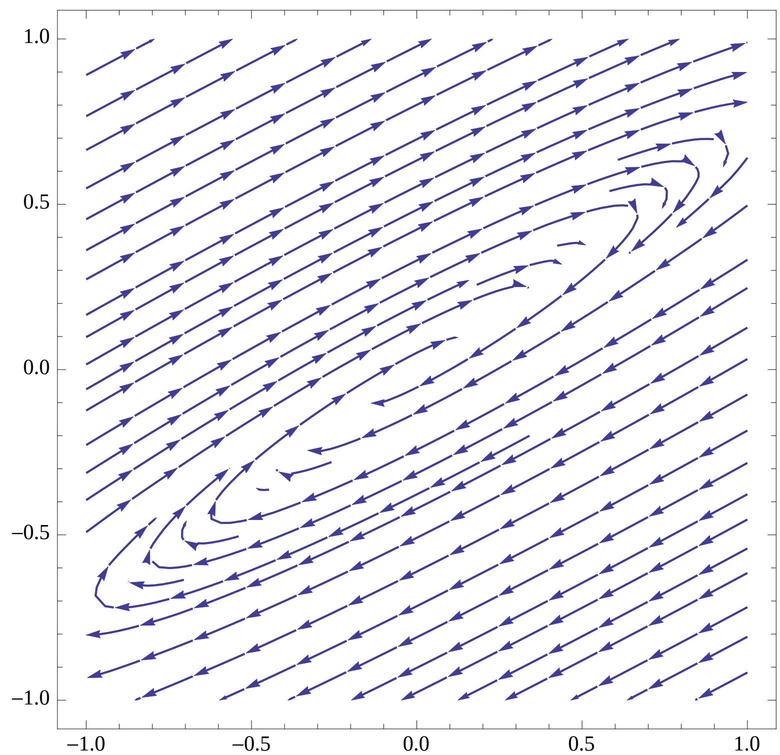


Figure 7.2: Phase portrait in phase space.

1. Eigenvalues of A have opposite signs: $\mathbf{0}$ is a saddle point.
2. Eigenvalues of A have same sign but are unequal: $\mathbf{0}$ is a node.
3. Eigenvalues of A are complex with nonzero real part: $\mathbf{0}$ is a spiral point.

In physics, we can use linear systems of equations to solve order-n linear differential equations relevant to physical systems.

Example: Suppose a block is attached to a spring which moves in a viscous fluid while attached to a spring. The friction of the fluid acts to slow the block down, while the spring supplies a linear restoring force.

$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt}$$

Set up an equivalent linear system of differential equations and solve. Determine the resul-

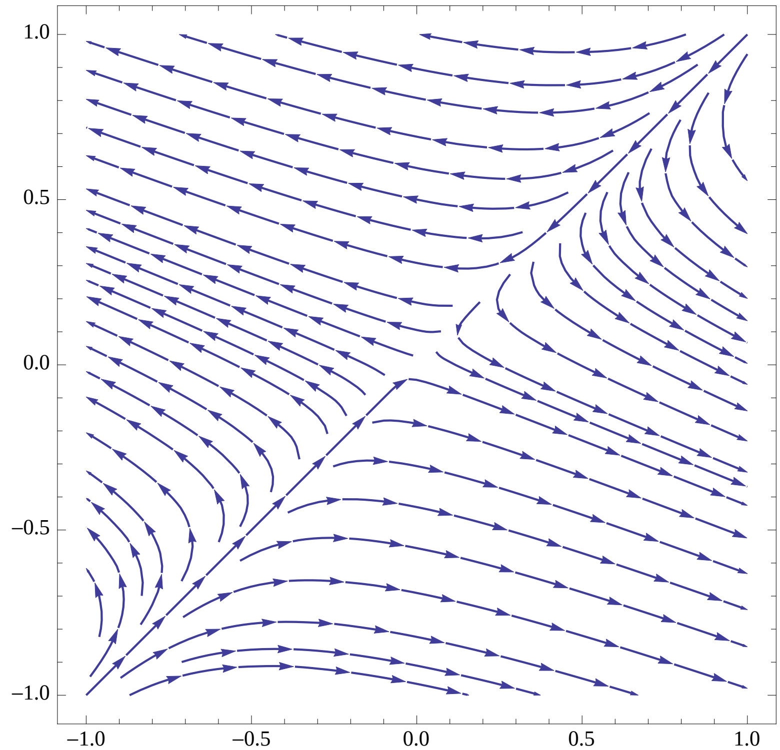


Figure 7.3: Phase portrait in phase space.

tant trajectories assuming $\frac{k}{m} = 2$, $\frac{\gamma}{m} = 1$.

Example: Find the position of a charged particle moving in a constant magnetic field pointing in the x_3 -direction as a function of time if the particle is initially traveling with velocity v_0 along the x_1-direction.

HINT: the force is given by $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$.

While linear systems appear in many contexts, nonlinear equations are far more common. Linear systems of equations can be used to understand the behavior of a nonlinear system near a so-called equilibrium point. We conclude this section with an example illustrating the idea:

Example: For the tundra wolf-caribou predator-prey model, analyze the behavior of the system close to each equilibrium point.

7.4 Fundamental Matrices: BD (7.7)

The matrix $\boldsymbol{\Psi}(t)$ whose columns form a fundamental set of solutions for the equation $\boldsymbol{x}' = \boldsymbol{P}(t)\boldsymbol{x}$ is called a **fundamental matrix** for the system.

The symbol $\boldsymbol{\Phi}(t)$ is reserved for the fundamental matrix satisfying $\boldsymbol{\Phi}(t_0) = \boldsymbol{I}$. When computable, the fundamental matrix $\boldsymbol{\Phi}(t)$ can be used to solve the system with initial condition $\boldsymbol{x}^0$:

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}^0.$$

In fact, the matrix is most useful when you have to solve the same system multiple times with different initial conditions.

We have already encountered the fundamental matrix concept: for $\boldsymbol{P}(t) = A$, the fundamental matrix is none other than e^{tA}!

Example: For the system

$$\boldsymbol{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{bmatrix} \boldsymbol{x}$$

compute the fundamental matrix $\boldsymbol{\Phi}(t)$.

7.5 Nonhomogeneous Linear Systems: BD (7.9)

Finally, let's return to **nonhomogeneous** system

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{g}(t)$$

where A is diagonalizable. We begin by writing $A = VDV^{-1}$ where D is a diagonal matrix consisting of the eigenvalues of A.

Whenever two matrices A and B are related via $A = PVP^{-1}$ we say the matrices are

similar and that P provides a **similarity transformation**. Another way of viewing this situation is that A and B are different **representations** of the same linear transformations in different **bases**. The basis transformation is then given by P.

So, if we transform the problem (originally stated in the standard basis) into the eigenvector basis using V, we have

$$\boldsymbol{u}' = D\boldsymbol{u} + \boldsymbol{G}(t)$$

where $\boldsymbol{u} = V^{-1}\boldsymbol{x}$ and $\boldsymbol{G}(t) = V^{-1}\boldsymbol{g}(t)$.

Now the u equations are completely decoupled and we can solve them through standard techniques. Once we have $\boldsymbol{u}$, we can find $\boldsymbol{x}$ via $V\boldsymbol{u} = \boldsymbol{x}$.

Example: Find the general solution to

$$\boldsymbol{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} e^t \\ t \end{bmatrix}$$

Chapter 8

Numerical and Qualitative Studies of Differential Equations

8.1 Direction Fields and Euler's Method: BD (1.1/ 2.7)

The vast majority of differential equations encountered at the cutting edge of research or in the business world do not have known explicit solutions. This seems to steal some of the thunder from this introductory course on differential equations. Aren't we here to learn analytic methods for finding explicit solutions for ODEs? This is true and the techniques you learned are important, but life is imperfect and I can't teach you a single recipe that will solve all of your ODE problems.

On the other hand, I view this situation as rather liberating since the only way to study such situations (instead of invoking some theoretically beautiful but obscure analytic trick to find an explicit solution) is to use numerical algorithms which are often much more straightforward to implement. Also, the computer does the bulk of the dirty work! We only have time to cover the basics of numerical analysis, so let's get started.

First, it is possible to visualize general solutions to some differential equations using something called a **direction field**. In general, suppose we have a first-order differential equation of the form

$$y' = F(x, y)$$

where F is some expression in x and y and is called the **rate function**. Recall what y' means: it represents the slope of the tangent line to a solution of the differential equation. Specifically, even if we don't know an explicit relationship between x and y which satisfies the ODE, we can at least say that at the point (x, y) the solution must have slope $F(x, y)$.

If we draw short line segments with slope $F(x, y)$ at several points (x, y), the result is called the **direction field** (or **slope field**). These line segments indicate the direction in which a solution curve is heading, so the direction field helps us visualize the general shape of these curves.

Example: Let's look at a differential equation of the form $y' = F(x, y)$ where we already know a family of solution curves. Recall that we looked at the differential equation $y' = -y^2$ in the previous section and found that $y = \frac{1}{x+C}$ describes a family of solution curves.

If we use a computer to generate the direction field for an equation of this type, the computer essentially does the following: it lays a fine grid down on top of the xy-plane. At each grid point, the computer computes the slope of a short line segment based on the differential equation $y' = F(x, y)$. The computer then plots all of these segments together to produce a nice visualization of ODE.

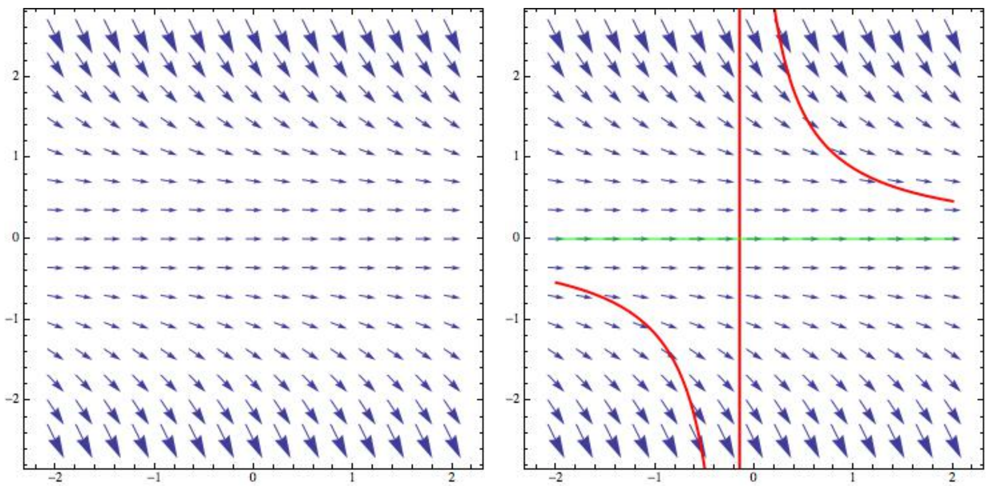


Figure 8.1: Direction field for $y' = -y^2$ plus plots of some solutions.

On top of this slope field I draw one solution from the family (in red) and the misfit solution $y = 0$ (in green). Notice how well the arrows in the direction field parallel the behavior of the curves describing the two solutions.

Example: Now let's look at an example where we don't have a family of explicit solutions. Recall the Fitzhugh-Nagumo model for the electrical impulse in a neuron:

$$\frac{dv}{dt} = -v\left[v^2 - (1+a)v + a\right].$$

For concreteness, let's choose $a = \frac{1}{2}$. The direction field is plotted below in Figure 2. In the first frame of Figure 2, I plot just the direction field. In the second frame, I plot the three equilibrium solutions $v = 0$, $v = \frac{1}{2}$, $v = 1$ in green, red and orange respectively. The equilibrium solutions we can find easily simply by setting $v'(t) = 0$. The direction field, however, gives us additional information. Let's look at the orange line: the arrows above are pointing downwards **towards** the orange line whereas the arrows below are pointing upwards **towards** the orange line.

This means that if you start your system with initial conditions slightly below or above the orange line, the system will eventually asymptote towards this value. Another way of looking at it is if you are in the $v = 1$ equilibrium and perturb the system just a little bit, that perturbation eventually fades away and arrive back into the $v = 1$ equilibrium.

For the red line, we have the opposite situation: arrows above point up and away and arrows below point down and away. Therefore, any initial condition chosen slightly above the red line will eventually asymptote to the orange line. Any initial condition chosen slightly below the red line will asymptote to the green line.

We say that $v = 1$ (orange) is a **stable** equilibrium value and $v = \frac{1}{2}$ is an **unstable** equilibrium value. You can think of stability in terms of ball on top of a hill or at the bottom of a valley. If you perfectly balance the ball at the top of the hill, it will remain motionless for all time. But if a slight breeze comes along, the ball will roll down the hill never to return. This is an unstable situation. However, if you situate a ball at the bottom of a valley, any slight kick to the ball will eventually bring it back to rest at the bottom of the valley. This is a stable situation.

In physics and finance knowing the equilibrium values of your model is usually insufficient: you also need to make some statement about the stability of the equilibrium points.

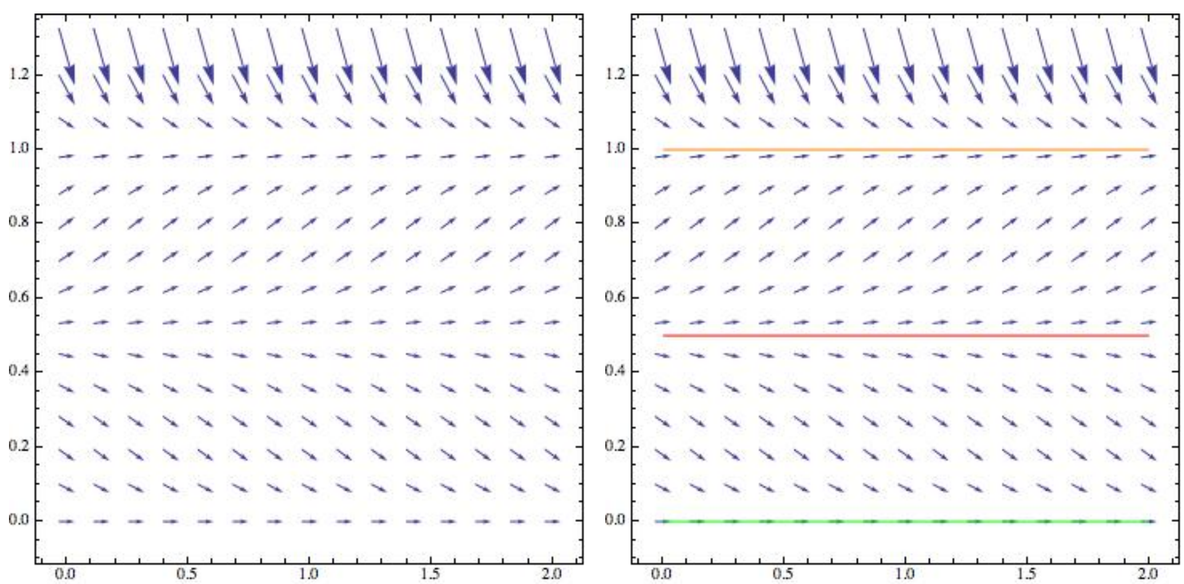


Figure 8.2: Direction field for neuron model plus plots of equilibrium solutions.

Example: A similar sort of behavior can be seen in the (normalized) logistic equation

$$\frac{dP}{dt} = P(1-P).$$

For convenience, we have taken all constants to be 1. Let's construct the direction field, analyze it as we did for the neuron model and see if we can compute exact solutions with which to compare.

We see once again that we have two distinct equilibria. The non-trivial one $P = 1$ is stable whereas $P = 0$ is unstable. In the long run solutions with non-trivial initial conditions will asymptote to $P = 1$. An exact non-trivial solution may be obtained by using the integration method known as partial fraction decomposition.

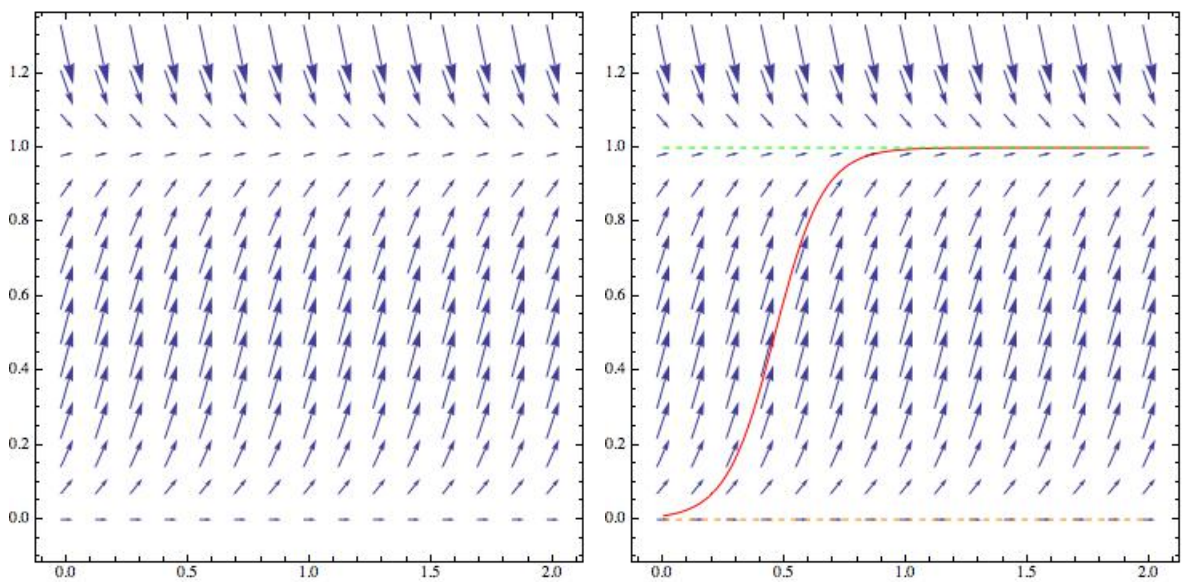


Figure 8.3: Direction field for the (normalized) logistic equation plus plots of equilibrium solutions and exact solution.

Before we move on and discuss Euler's Method, I wish to take a moment to discuss a method for visualizing the behavior of some *systems* of differential equations like the Caribou-Tundra Wolf model I used to initiate these lectures on differential equations. Specifically the equations

$$\frac{dC}{dt} = aC - bCW \ , \ \frac{dW}{dt} = -cW + dCW$$

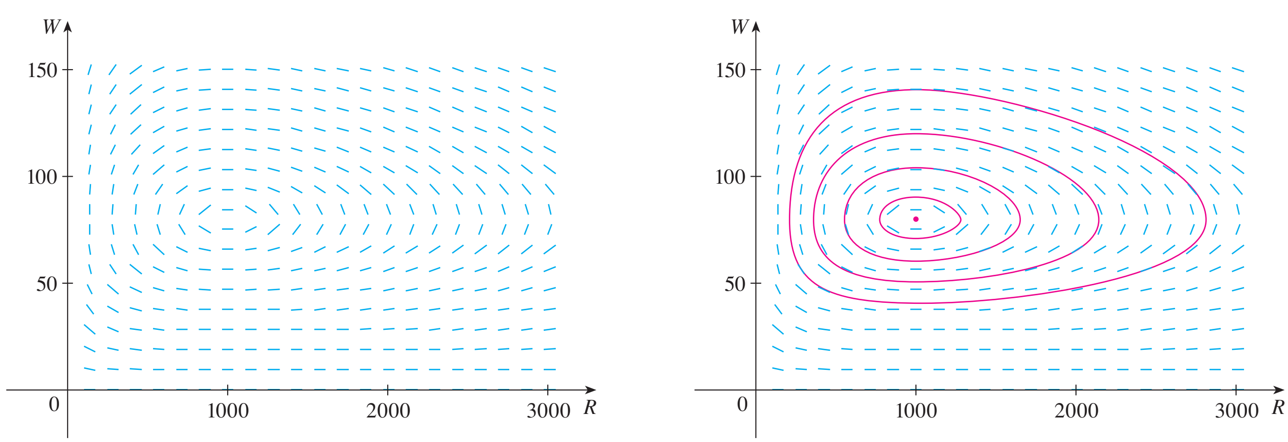
are known as **predator-prey equations**, or the **Lotka-Volterra equations**. Equations such as these arise in many different contexts in which you have two (or more) groups competing over limited resources. (Think of applications in economics, sociology and evolutionary biology!) Unfortunately, it is usually impossible to find explicit solutions since the equations are **coupled** (the derivative of one variable depends on both variables) and **non-linear** (products of variables occur). We can, however, use graphical methods to analyze the equations.

Let's choose concrete values for the constants: $a = 0.08$, $b = 0.001$, $c = 0.02$, $d = 0.00002$. We can easily solve these equations for **equilibria** by setting C' and W' equal to 0: $W = 80$, $C = 1000$. By using the chain rule, it is possible to construct the direction field for the system of differential equations in the so-called **phase plane** which is a fancy way of describing the CW-plane. Solution curves are referred to as **phase trajectories** and a **phase portrait** consists of equilibrium points and typical phase trajectories like in the figure below.

Using the chain rule, we have

$$\frac{dW}{dC} = \frac{\frac{dW}{dt}}{\frac{dC}{dt}} = \frac{-0.02\ W + 0.00002\ CW}{0.08\ C - 0.001\ CW}$$

which we use to construct the phase portrait.



Euler's Method: Studying direction fields gives us an idea of how to find solutions to differential equations of the form $y' = F(x, y)$: start at some point in the direction field (our initial condition) and then follow the line segments to trace out an approximate numerical solution.

In order for a computer to generate numerical approximations to your differential equation, you have to select a **step size**. You have control over the independent variable x but the computer cannot handle a continuum variable which is a mathematical idealization anyway. A computer has a finite memory and therefore can only represent a finite (though large!) number of real numbers as finite decimals. So it is necessary to take "discrete steps" in any program implementing the algorithm we sketched above.

With your suitably chosen step size h and your initial x-value x_0, you can generate a large (but finite) number of points along the x-axis defined recursively by $x_n = x_{n-1} + h$. Now we can formally state **Euler's Method** to find a numerical approximation to solutions of the differential equation $y' = F(x, y)$.

Euler's Method: Approximate values for the solution of the initial-value problem $y' = F(x, y)$, $y(x_0) = y_0$, with step size h, at $x_n = x_{n-1} + h$, are

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

The following examples illustrate the idea and were gratefully borrowed from Paul's Online Notes on Euler's Method:

Example: For the initial value problem $y' + 2y = 2 - e^{-4t}$, $y(0) = 1$ use Euler's Method with step size of $h = 0.1$ to find approximate values of the solution at $t = 0.1, 0.2, 0.3, 0.4, .5$. Compare them to the exact values of the solution at these points.

The exact answer is $y(t) = 1 + \frac{1}{2}e^{-4t} - \frac{1}{2}e^{-2t}$. A table of values is provided below:
Example: Repeat the previous example only this time give the approximations at $t = 1, 2, 3, 4, 5$. Use $h = 0.1, h = 0.05, h = 0.01, h = 0.005, h = 0.001$ for the approximations.
As you would expect decreasing the step size generally increases the accuracy of the approximation. A worst-case scenario uniform bound on the error of Euler's Method is given by

$$|e_n| \leq Mh^2/2$$

Time, t_n	Approximation	Exact	Error
$t_0 = 0$	$y_0 = 1$	$y(0) = 1$	0 %
$t_1 = 0.1$	$y_1 = 0.9$	$y(0.1) = 0.925794646$	2.79 %
$t_2 = 0.2$	$y_2 = 0.852967995$	$y(0.2) = 0.889504459$	4.11 %
$t_3 = 0.3$	$y_3 = 0.837441500$	$y(0.3) = 0.876191288$	4.42 %
$t_4 = 0.4$	$y_4 = 0.839833779$	$y(0.4) = 0.876283777$	4.16 %
$t_5 = 0.5$	$y_5 = 0.851677371$	$y(0.5) = 0.883727921$	3.63 %

Approximations

Time	Exact	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	0.9414902	0.9313244	0.9364698	0.9404994	0.9409957	0.9413914
$t = 2$	0.9910099	0.9913681	0.9911126	0.9910193	0.9910139	0.9910106
$t = 3$	0.9987637	0.9990501	0.9988982	0.9987890	0.9987763	0.9987662
$t = 4$	0.9998323	0.9998976	0.9998657	0.9998390	0.9998357	0.9998330
$t = 5$	0.9999773	0.9999890	0.9999837	0.9999786	0.9999780	0.9999774

Percentage Errors

Time	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	1.08 %	0.53 %	0.105 %	0.053 %	0.0105 %
$t = 2$	0.036 %	0.010 %	0.00094 %	0.00041 %	0.0000703 %
$t = 3$	0.029 %	0.013 %	0.0025 %	0.0013 %	0.00025 %
$t = 4$	0.0065 %	0.0033 %	0.00067 %	0.00034 %	0.000067 %
$t = 5$	0.0012 %	0.00064 %	0.00013 %	0.000068 %	0.000014 %

where e_n is the error of the n-th step and M is the maximum of $|y''(t)|$ on the domain of interest $[a, b]$ where y is a solution to the differential equation.

Basically, error goes down with decreasing step size as h^2. Similar error estimates were encountered in our discussion of various numerical schemes for integration.

Example: For the initial value problem $y' - y = -\frac{1}{2}e^{\frac{t}{2}}\sin(5t) + 5e^{\frac{t}{2}}\cos(5t)$, $y(0) = 0$ use Euler's Method to find the approximation to the solution at $t = 1, 2, 3, 4, 5$ using $h = 0.1$, 0.05, , 0.01 , 0.005 , 0.001.

The exact solution is $y(t) = e^{\frac{t}{2}} \sin(5t)$.

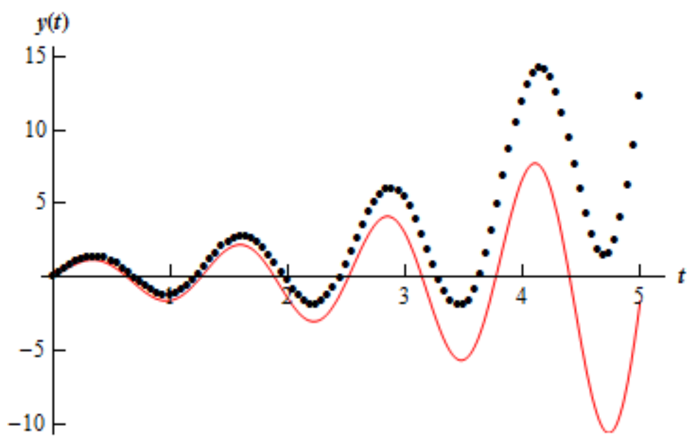
Approximations

Time	Exact	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	-1.58100	-0.97167	-1.26512	-1.51580	-1.54826	-1.57443
$t = 2$	-1.47880	0.65270	-0.34327	-2.18657	-1.35810	-1.45453
$t = 3$	2.91439	7.30209	5.34682	3.44488	3.18259	2.96851
$t = 4$	6.74580	15.56128	11.84839	7.89808	7.33093	6.86429
$t = 5$	-1.61237	21.95465	12.24018	1.56056	0.0018864	-1.28498

Percentage Errors

Time	$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.005$	$h = 0.001$
$t = 1$	38.54 %	19.98 %	4.12 %	2.07 %	0.42 %
$t = 2$	144.14 %	76.79 %	16.21 %	8.16 %	1.64 %
$t = 3$	150.55 %	83.46 %	18.20 %	9.20 %	1.86 %
$t = 4$	130.68 %	75.64 %	17.08 %	8.67 %	1.76 %
$t = 5$	1461.63 %	859.14 %	196.79 %	100.12 %	20.30 %

Here's a graph of the exact solution together with the numerical approximation.



Euler's method is the tip of the iceberg. As the last example shows, Euler's method is imperfect even for basic IVPs. There is a whole industry out there for improving numerical algorithms for solving differential equations. A few lines of computer code, written in a high-level programming language and executed (often within a few seconds) on a relatively inexpensive computer, suffice to solve numerically a wide range of differential equations. As we saw in the previous examples the usual output is a table of numbers, listing selected values of the independent variable and the corresponding values of the dependent variable. A good book on the topic is Numerical Analysis by Burden and Faires.