Contents

Chapter 1

Precalculus Review

1.1 Function Basics: Stewart Section 1.1

Calculus has been called the mathematics of taking limits. We'll soon get to limits and all they imply, but we need to have a firm understanding of the objects we are taking the limits *of*, namely **functions**. In section 1.1, Stewart has a nice discussion of the four ways to represent a function:

1. verbally (by a description in words)
2. numerically (by a table of values)
3. visually (by a picture or graph)
4. algebraically (by an explicit formula)

I encourage you to read this portion of the text on your own. However, while the first two options are useful in some applications, we will almost exclusively use the third (graphical representation) and fourth (explicit formula) options in this course.

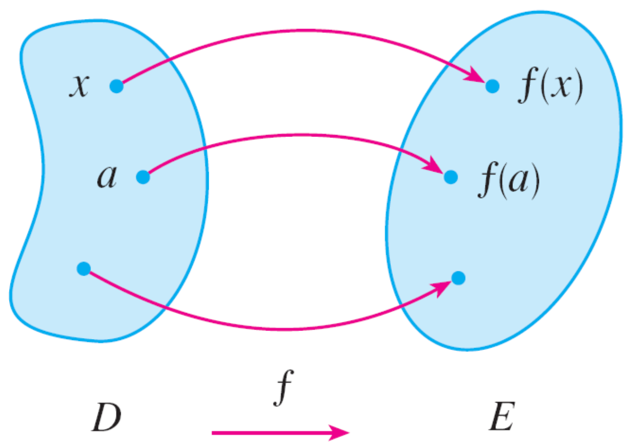
What is a function?

Definition: A **function** f is a rule that assigns to each element x in a set D (called **the domain**) exactly one element, called $f(x)$, in a set E (named **the codomain**).

In this course, we will always assume that D and E are sets of real numbers.

Definition: The number $f(x)$ is called the **value of f at x**.

Definition: The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.



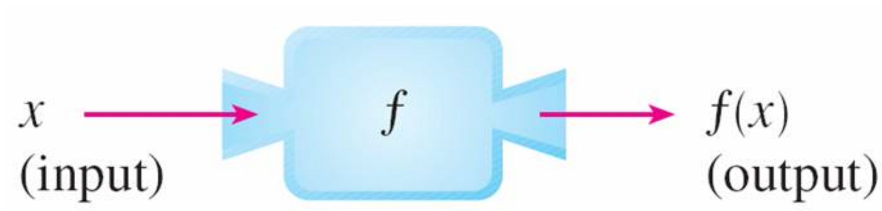
Definition: A symbol that represents an arbitrary number in the domain of a function f is called an **independent variable**.

Definition: A symbol that represents a number in the range of a function f is called a **dependent variable**.

These definitions arise because we may vary x as we please while $y = f(x)$ has no choice but to depend on x through f.

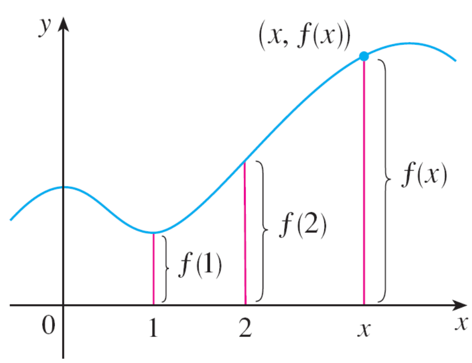
It is sometimes helpful to visualize f as a machine which takes as input x from the domain and produces exactly one output $y = f(x)$ in the range.

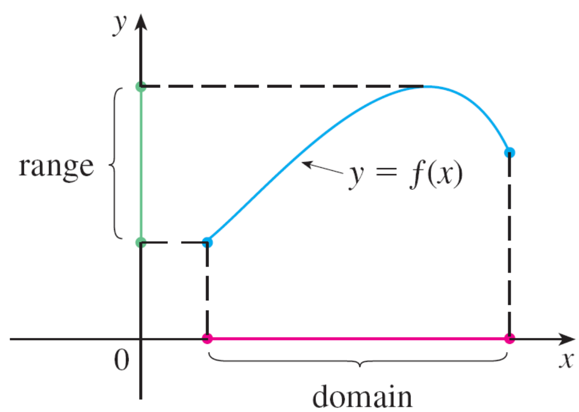
The most common and useful way to visualize a function is through its graph. If f is a function with domain D, then its **graph** is the set of ordered pairs $\{(x, f(x))|x \in D\}$.



In words, the graph consists of all ordered pairs (x, y) in the plane $\mathbb{R}^2$ such that x is in the domain of f and $y = f(x)$.

- Since the y-coordinate of any point (x, y) on the graph is $y = f(x)$, $f(x) =$ height of the graph above x.
- The graph of f also allows us to picture both domain and range.





Example: Sketch the graph of $f(x) = x^2$ and state the domain and range.

Example: Sketch the graph of $f(x) = 3x + 1$ and state the domain and range.

Remember, lines are normally specified in one of two ways: we can specify two points on the line or (equivalently) we can specify the slope and the y-intercept.

In the first case, we can plot the two points separately, then connect them with the unique line going through them. In the second case, we may obtain two points from the formula $y = mx + b$ and then plot the line as before.

The most convenient choice is usually to find the x and y intercepts. The y-intercept is easy: $(0, b)$. The x-intercept comes from setting $y = 0$ and solving for x: $(-\frac{b}{m}, 0)$.

In precalculus, we came across many functions with domains given by subsets of the real numbers. Given an algebraic formula for the function, we looked for "trouble spots" where the function became ill-defined. These trouble spots were excluded: every remaining real number necessarily belonged to the domain. Trouble can arise if we divide by 0, take the square root of a negative number, etc.

Example: Find the domain of the function $f(x) = 5 + \sqrt{3x - 2}$.

Example: Find the domain of the function $f(x) = \frac{x+4}{x^2-9}$

Example: Find the domain of the function $f(x) = \sqrt{3 - x} - \sqrt{2 + x}$

If a function is given by a formula and the domain is not stated explicitly, we assume the domain is the set of all numbers for which the formula makes sense and defines a real number.

Finding the range of a function is a little trickier. Remember, y is in the range of f if and only if there is at least one x satisfying $y = f(x)$. That is the definition of the range.

So, to find the range, we set $y = f(x)$ and try to solve for y if possible. We won't always be able to solve this equation, but it can tell us conditions on y when it is possible to solve $y = f(x)$. These conditions define the range.

Example: Find the range of the function $f(x) = 5 + \sqrt{3x - 2}$.

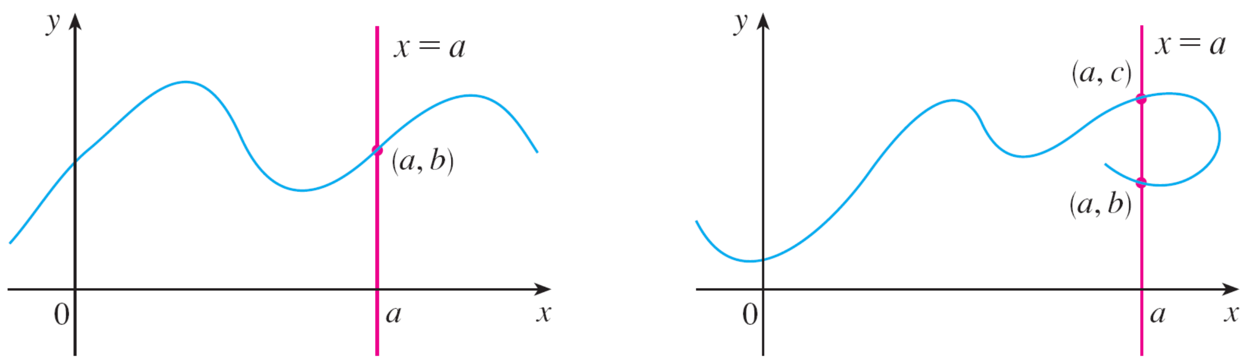
Example: Find the range of the function $f(x) = \frac{3x+5}{x-9}$

Not all curves in the plane represent functions. Remember the equation $x^2 + y^2 = 1$? It describes the unit circle which is a circle centered at the origin with radius 1. We'll have more to say about this circle when we come to the trig functions. This curve cannot represent the graph of a function, because over every x between -1 and 1 there are *two* y's. Remember, for every input we can only have one output!

How can we determine which curves in $\mathbb{R}^2$ represent functions? Remember the Vertical Line Test!

The Vertical Line Test: A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

The vertical line test works precisely because the graph of a function should have only



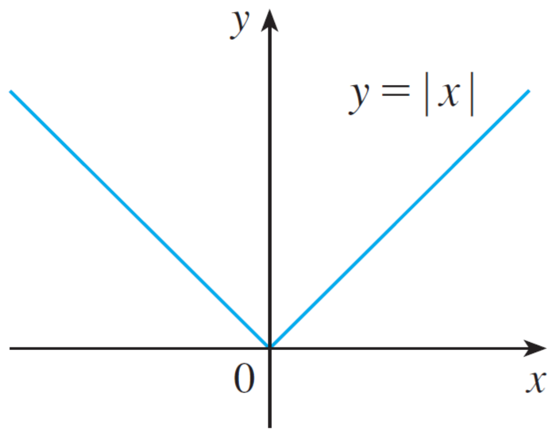
one value above any x in the domain. Any more, and there cannot be a unique output. Going back to the unit circle example, we could split the circle in half and take the top part. This portion of the circle passes the vertical line test and is indeed the graph of a function: $f(x) = \sqrt{1-x^2}$.

So far, the functions we've considered have graphs which can be drawn without lifting our pencils. These are described by a single formula for every x in the domain of f. A very important class of functions are defined by different formulas in different parts of their domains: these are the piecewise defined functions:

The most important example of a piecewise defined function is the absolute value function.

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

The graph of $y = |x|$ is sketched by remembering that $y = x$ when x is positive and



$y = -x$ when x is negative. So over the positive x axis, we draw the line $y = x$; over the negative x-axis, we draw the line $y = -x$.

To sketch the graph of any piecewise defined function, we sketch each formula separately for the specified domain, then put all of the pieces together:

Example: Sketch the graph of

$$f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

and evaluate $f(2), f(-1), f(0)$.

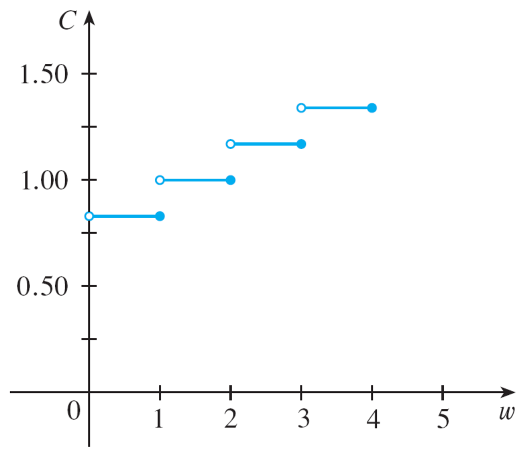
Example: Sketch the graph of

$$f(x) = \begin{cases} x+9 & \text{if } x < -3 \\ -2x & \text{if } |x| \leq 3 \\ -6 & \text{if } x > 3 \end{cases}$$

and evaluate $f(2), f(-1), f(0)$.

Another important class of piecewise functions are the **step functions**. These functions are piecewise constant. In other words, they take on different constant values over different parts of the real line:

We'll have more to say about step functions when we come to Chapter 2. Suffice it to



say that step functions are critical to understanding how to define Riemann integrals. Integrals and derivatives are the primary objects in calculus.

Some functions are special because of the way they behave when x is replaced with $-x$. These functions are said to be **symmetric**.

Definition: A function is said to be **even** if $f(-x) = f(x)$. The graph of such a function is symmetric about the y-axis.

Examples: $y = x^2, x^4, \ldots$ or simply $y = x^{2n}$. (Hence the definition of even)

Example: $y = \cos(x)$ (we haven't discussed trig functions yet, but we'll show that $\cos(x)$ is even.

Example: $y = |x|$.

Definition: A function is said to be **odd** if $f(-x) = -f(x)$. The graph of such a function is symmetric about the origin.

Examples: $y = x, x^3, \ldots$ or simply $y = x^{2n+1}$. (Hence the definition of odd)

Example: $y = \sin(x)$ (we haven't discussed trig functions yet, but we'll show that $\sin(x)$ is odd.

Not all functions will be even or odd (think of $y = x^2 + x$). But when the function has a symmetry, it can make computing the integral of f over a symmetric domain very easy indeed as we'll see.

When we discuss derivatives of functions, we will see that they give us information about when a function is **increasing** and **decreasing**.

Definition: A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \ , \ x_1, x_2 \in I \ , \ x_1 < x_2$$

It is called **decreasing** on an interval I if

$$f(x_1) > f(x_2) \ , \ x_1, x_2 \in I \ , \ x_1 < x_2$$

Example: Find the interval where $f(x) = x^2$ is increasing (decreasing).

Example: Find the interval where $f(x) = \frac{3x+|x|}{x}$ is increasing (decreasing).

Note: this example is meant to be tricky. First, we have to exclude $x = 0$ from the domain since it is a "trouble spot." Second, the graph is that of a step function. Using the strict definition of increasing/decreasing, this function is nowhere increasing/decreasing.

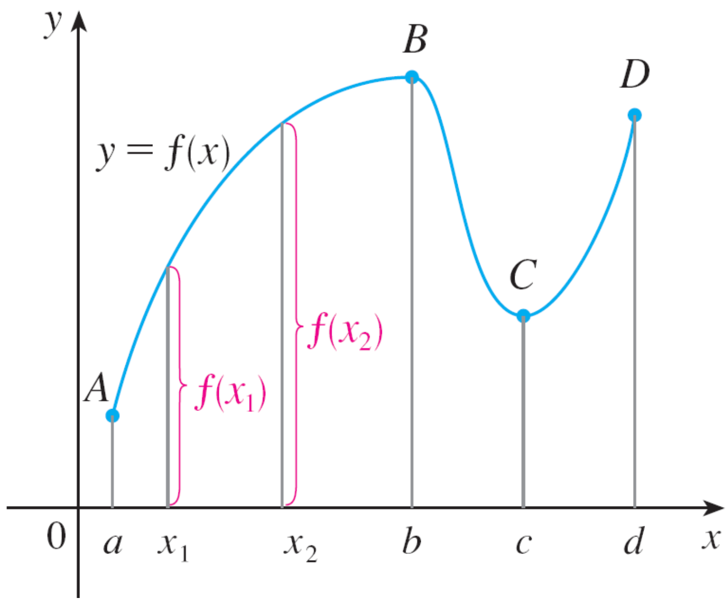


Figure 1.1: This graph represents a function that increases on (a, b), decreases on (b, c) and increases on (c, d).

1.2 Some Essential Functions: Stewart 1.2

This section has a very nice discussion of mathematical models and what they mean. I strongly encourage you to read this material on your own. What we need from section 1.2 is the Catalog of Essential Functions.

Linear functions are those functions of the form $f(x) = mx + b$ where $m, b \in \mathbb{R}$. They are known as linear functions since their graphs are lines in the plane $\mathbb{R}^2$. Here, m is the **slope** of the line and b is the **y intercept**. As we mentioned before, one way to plot the graph of $f(x) = mx + b$ is by using the x and y intercepts $(-\frac{b}{m}, 0)$, $(0, b)$. Plot these two points then connect with a line.

Linear functions are special cases of **polynomials**.

Definition: A function $P(x)$ is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, \ldots$ are called the **coefficients** of the polynomial.

The domain of any polynomial is $\mathbb{R}$. If the leading coefficient a_n is nonzero, then the **degree** of the polynomial is n.

Example: A polynomial of degree 1 is of the form $P(x) = mx + b$ and so is a linear function.

Example: A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic**. The graph of a quadratic is always a parabola obtained by shifting, stretching and possibly reflecting the graph of $y = x^2$.

The way to see this is to recall the very important technique of **completing the square**.

By completing the square, $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$.

Recall the **Transformations of Functions** from Precalculus: So, to graph $y = ax^2 + bx + c$,

Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Vertical and Horizontal Stretching and Reflecting Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

we must shift $y = x^2$ by $\frac{b}{2a}$ units right or left, scale by a, then shift up or down by $c - \frac{b^2}{4a}$.

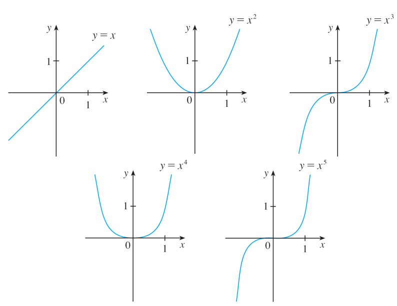
Example: Graph $y = x^2 + x + 1$.

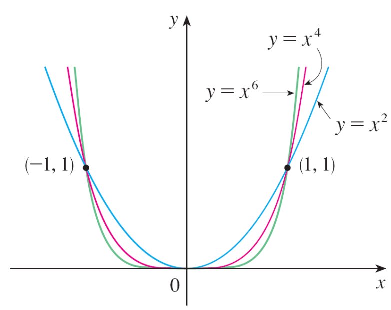
Polynomials by definition are made up of **power functions**; i.e. each **term** is a power function.

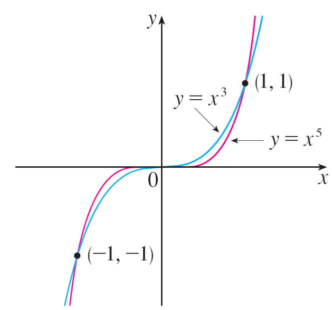
Definition: A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**.

Example: The terms of a polynomial are proportional to x^m where m is a nonnegative integer.

The general shape of the graph of $f(x) = x^m$ depends on whether m is odd or even. If m is odd, then the graph is similar to the graph of $y = x^3$; if m is even, the graph is similar to $y = x^2$.





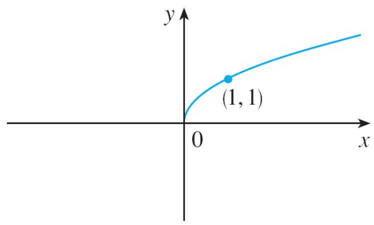


Example: When $a = \frac{1}{m}$ (where m is positive integer), the function $f(x) = x^a = x^{\frac{1}{m}}$ is called a **root function**. For $m = 2$, the function is the square root function $f(x) = x^{\frac{1}{2}} = \sqrt{x}$ which has domain $[0, \infty)$. When $m = 3$, we have the **cube root** function $f(x) = x^{\frac{1}{3}}$ whose domain is all $\mathbb{R}$.

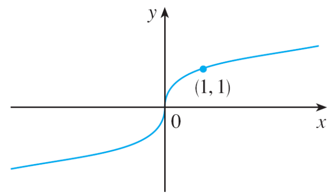
When m is odd, the graph of $f(x) = x^{\frac{1}{m}}$ is similar to the graph of $f(x) = x^{\frac{1}{3}}$; when m is even, the graph of $f(x) = x^{\frac{1}{m}}$ is similar to the graph of $f(x) = x^{\frac{1}{2}}$.

Example: When $a = -1$, $x^a = x^{-1} = \frac{1}{x}$ is called the **reciprocal function**.

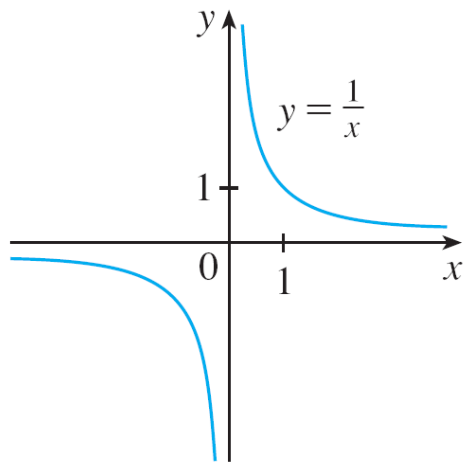
- Graph has symmetry through the origin since $f(x) = -f(-x)$.



$f(x) = \sqrt{x}$



$f(x) = \sqrt[3]{x}$



- Graph has vertical and horizontal **asymptotes**
- **Asymptotes** are lines such that the distance between the curve and the line approaches zero as they approach infinity. In other words, the curve gets closer and closer to the asymptote the farther out you go.

A **rational function** f is a ratio of two polynomials:

$$f(x) = P(x)/Q(x)$$

where P and Q are polynomials. The domain of f is the set of all x where $Q(x) \neq 0$.

A function f is called an **algebraic function** if it can be constructed using algebraic operations starting with polynomials.

Example: Any rational function is an algebraic function

Example: $f(x) = \sqrt{x^2 + 4}$

Example: $f(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}}$.

The remainder of section 1.2 briefly describes the trigonometric functions, the exponential function and the logarithmic function. These functions are so important that we'll spend a bit more time recalling their definitions and properties.

We'll start with the exponential.

1.3 Exponential Functions: Stewart Section 1.4

In general, an **exponential function** is a function of the form

$$f(x) = a^x$$

where $a > 0$ is a constant.

The x in a^x can be any real number so the domain of an exponential function is $\mathbb{R}$. Remember that the real numbers are made up of **rational** and **irrational** numbers.

If x is rational, there exist $p, q \in \mathbb{Z}$, $q \neq 0$ such that $x = \frac{p}{q}$. (We may as well take $q > 0$, so let's do so for simplicity). Let's recall now what $a^{\frac{p}{q}}$ means.

First, if x is a positive integer n, then $a^n = a \times a \times \cdots \times a$ by definition. If $x = 0$, then we define $a^0 = 1$ for consistency. If x is a negative integer $-n$, then we define

$$a^{-n} = \frac{1}{a^n}.$$

This takes care of the definition of a^x for all $x \in \mathbb{Z}$. So much for the integers, what about rationals like $\frac{p}{q}$ for $q > 0$? First, we note that we have the following **laws of exponents**:

Theorem: If a and b are positive real numbers and n and m are any integers, we have

$$\begin{aligned} a^{n+m} &= a^n a^m \\ (a^n)^m &= a^{nm} \\ (ab)^n &= a^n b^n \end{aligned} \tag{1.3.1}$$

These properties follow immediately from the definition of a^n for $n \in \mathbb{Z}$.

We want these properties to hold for all real-valued exponents, so we must define

$$a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p$$

From our discussion of power functions, we know that $a^{\frac{1}{q}}$ is defined to be the q-th root of a. Therefore, a^x is defined for $x \in \mathbb{Q}$.

The tricky part is defining a^x for x irrational. There is an amazing mathematical fact about irrational numbers: given any irrational number, we can always find a rational number arbitrarily close by.

For example, consider $\sqrt{2}$ which was the first irrational number to be discovered. If you plug in $\sqrt{2}$ into your calculator, you'll get something like

$$1.41421356\ldots$$

Like all irrational numbers, $\sqrt{2}$ has an infinite decimal expansion; i.e. the decimal representation of $\sqrt{2}$ does not terminate.

Your calculator is giving you a very good approximation to $\sqrt{2}$, not the actual number itself.

To show that there is always a rational number arbitrarily close to $\sqrt{2}$, we **truncate** the decimal. Specifically, we take the decimal expansion out to a specified place and then chop off the rest.

If we truncate at the first digit, we have 1 which is a rational number a distance of about $0.41421356\ldots$ units away from $\sqrt{2}$.

If we truncate at the first digit, we have $1.4 = \frac{14}{10}$ which is a rational number a distance of about $0.01421356\ldots$ units away from $\sqrt{2}$.

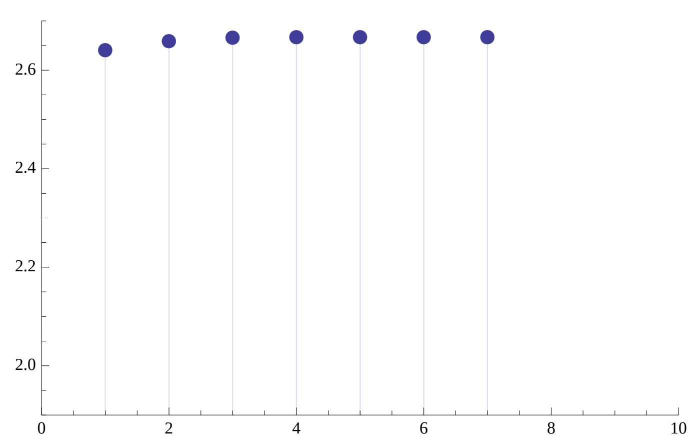
If we truncate at the third digit, we have $1.41 = \frac{141}{100}$ which is a rational number a distance of about $0.00421356\ldots$ units away from $\sqrt{2}$.

Hopefully, you see the pattern: if we truncate farther out, the rational number leftover is closer to the original irrational number. Why do we care? We care, because we know how to compute (in principle) $a^{\frac{p}{q}}$ and we can use this to define a^x where x is irrational in the following way:

Let $r_1, r_2, \ldots$ be the rational numbers you find after truncating the decimal expansion of x (irrational) at the first digit, second digit, and so on.

The numbers a^{r_1} , a^{r_2} , $\ldots$ are well-defined and very rapidly approach a number which we call a^x.

Let's illustrate this idea with $x = \sqrt{2}$ and $a = 2$. By plotting $2^{r_1}, 2^{r_2}, \ldots,$ we see that



the numbers rapidly level out near 2.65.

To get a better sense of what is happening, let us look at the numbers $2^{r_1}, \ldots$ themselves:

The numbers $2^{r_1} \ldots$ clearly **converge** to $2.66514414\ldots$. We are foreshadowing the concept

1	2
1.4	2.63902
1.41	2.65737
1.414	2.66475
1.4142	2.66512
1.41421	2.66514
1.414213	2.665143
1.4142135	2.66514403
1.41421356	2.665144138

of a **limit**. Limits will be at the center of everything we study in this course!

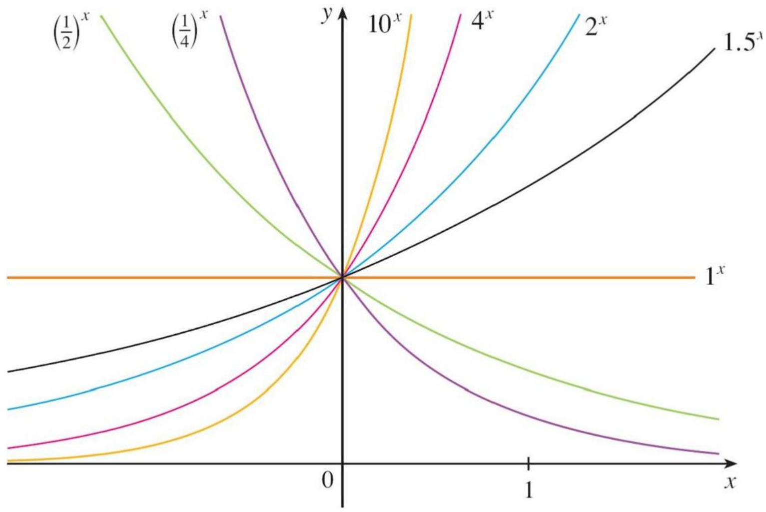
To summarize,

- We defined a^x for $x \in \mathbb{Z}$
- We extended this to a^x for $x \in \mathbb{Q}$ based on consistency of the laws of exponents.
- We then recognized that irrational numbers are approximated as well as we like by rational numbers
- We defined a^x for irrational numbers in terms of a limiting process.

Graphs of exponential functions look like
Notice

- All graphs intersect at the same point $(0, 1)$
- As the base a gets larger, the exponential grows more rapidly for $x > 0$.
- Given the way we define $f(x) = a^x$, the laws of exponents hold.

Of all the possible bases for an exponential function, there is one that is most useful not only in calculus, but in a wide range of mathematical applications. This base is e, sometimes referred to as **Napier's constant** (or Euler's constant, though there is another number which goes by that name).



Let us recall how the base e was introduced in precalculus since it touches on the idea of a **limiting process**.

In precalculus, we discussed the **Compound Interest Formula**:

Theorem: Suppose that a principal of P dollars is invested at an annual rate of r that is compounded annually. Then the amount A after t years is given by

$$A = P(1+r)^t$$

This formula is not too hard to prove:

At the beginning, we have P dollars (**the principal**) to invest. The money sits in the bank for one year and then, after that year is through, the bank gives us the principal back plus r times the principle. So, after one year,

$$A_1 = P + rP = P(1+r)$$

During the second year, A_1 becomes the new initial investment and, after the year is through, the bank gives us

$$A_2 = A_1 + rA_1 = A_1\,(1+r) = P\,(1+r)^2$$

Continuing this way, we see arrive at the compound interest formula.

Now suppose instead of compounding once a year, the bank compounds n times a year at an annual rate of r. The argument above gives us the following formula:

Theorem: Suppose that a principal of P dollars is invested at an annual rate r that is compounded n times a year. Then the amount A after t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

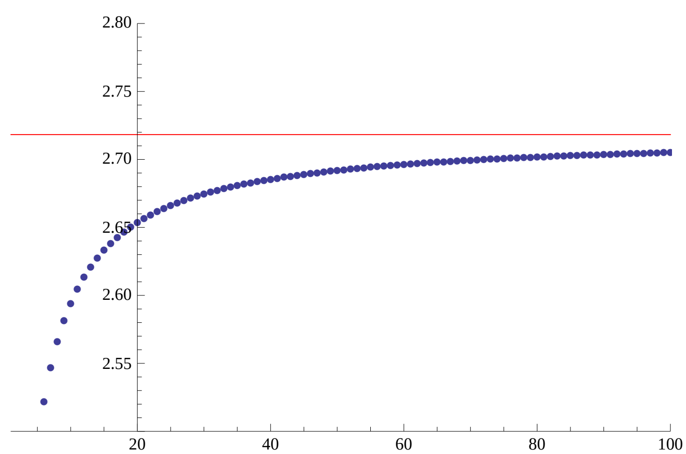
To see what this has to do with e, let's rewrite the formula as

$$A = P\left[\left(1 + \frac{1}{\frac{n}{r}}\right)^{\frac{n}{r}}\right]^{rt}$$

If we take n to be larger and larger, the formula describes **continuously compounded interest** better and better.

Let's take a look at $\left(1 + \frac{1}{\frac{n}{r}}\right)^{\frac{n}{r}}$ as n gets large. Let's assume $r = 1$ just for simplicity: it won't affect the end result.

The best way to see the result is via plotting $\left(1 + \frac{1}{n}\right)^n$ as a function of n:

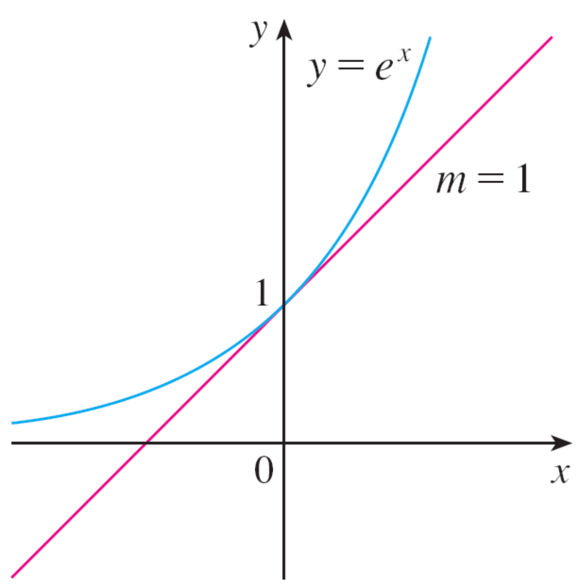


As we can see from the graph, $\left(1 + \frac{1}{n}\right)^n$ gets closer and closer to the red line which is the graph of $y = e \approx 2.71828\ldots$.

For continuously compounded interest, we therefore have the famous "Pert" formula: $A = Pe^{rt}$.

While the Pert formula is perhaps the most down-to-earth example of the appearance of e, this sort of formula appears all over the sciences and engineering. Examples include the growth of bacteria in a petrie dish and the decay of a radioactive isotope.

In calculus, we define e to the unique base such the the slope of the tangent line to $y = e^x$ at $x = 0$ is exactly 1. Since we don't know how to compute tangent lines yet, think of the Pert formula to define e for now.



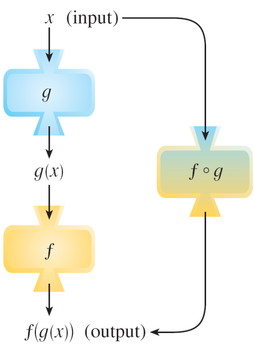
1.4 Inverse Functions and Logarithms: Stewart Sections 1.3 and 1.5

The exponential functions $f(x) = a^x$, $a > 0$ have natural counterparts called **logarithms**. Exponentials and logarithms "undo" each other in a sense we'll make precise. First, I must remind you of **function composition**.

Definition: Given two functions f and g, the **composite function** $f \circ g$ is defined by:

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ consists of all x where **both** $g(x)$ and $f(g(x))$ are defined.



Example: Define $f(x) = x^2 - 1$ and $g(x) = \sqrt{x}$. Compute $f \circ g$ and $g \circ f$ and find their respective domains.

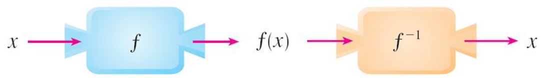
Example: Define $f(x) = x + \frac{1}{x}$ and $g(x) = \frac{x+1}{x+2}$. Compute $f \circ g$ and $g \circ f$ and find their respective domains.

Example: Express the function $R(x) = \sqrt{\sqrt{x} - 1}$ in the form $f \circ g \circ h$.

Decomposing functions this way will be extremely useful for us when we compute derivatives using the chain rule and when we compute certain integrals using u substitution.

Sometimes, it is possible to find a function $g(x)$ for a given function $f(x)$ so that $(f \circ g)(x) = (g \circ f)(x) = x$.

In this sense, f and g "undo" each other: whatever f transforms x into, g turns it back into x and vice versa.



When g exists, we say that f has an **inverse** and we usually represent g by f^{-1}.

Example: Show by direct substitution that $f(x) = 2x - 1$ has inverse $g(x) = \frac{1}{2}(x+1)$.

Example: Show by direct substitution that $f(x) = 1 + \sqrt{2+3x}$ has inverse $g(x) = \frac{1}{3}(x-1)^2 - \frac{2}{3}$ for $x \geq 1$.

How do we know a given function f has an inverse?

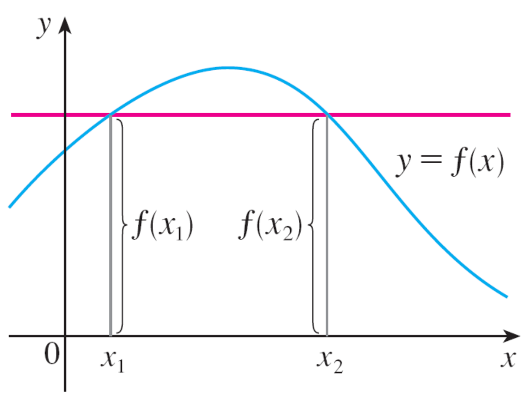
Only functions which are **one-to-one** can have an inverse function.

Definition: A function f is called a **one-to-one function** if it never takes on the same values twice; that is

$$f(x_1) = f(x_2) \text{ if and only if } x_1 = x_2.$$

Luckily, we can inspect the graph of f to determine if this holds:

Theorem (Horizontal Line Test): A function is one-to-one if and only if no horizontal line intersects its graph more than once.



This function is not one-to-one because $f(x_1) = f(x_2)$.

Example: Determine if $y = x^2$ and $y = x^3$ are one-to-one. If not, can you find a domain in which they are one-to-one?

Just to be clear, we have the following definition:

Definition: Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$\left(f \circ f^{-1}\right)(x) = \left(f^{-1} \circ f\right)(x) = x$$

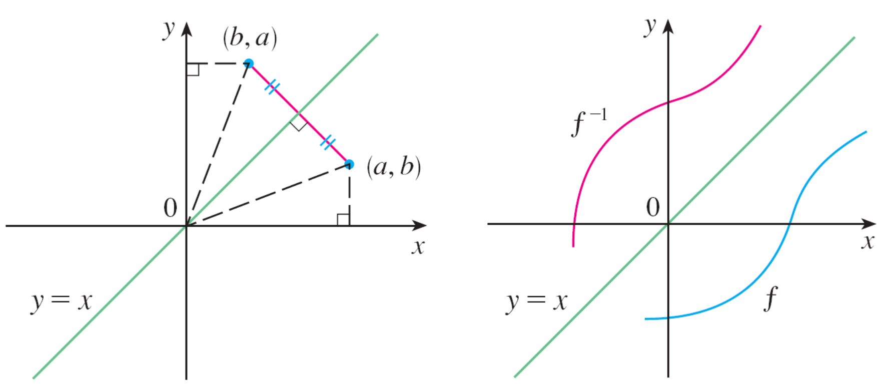
This definition of an inverse function provides us with a handy way of finding the formula for f^{-1} given a formula for f:

- Step 1: Write $y = f(x)$
- Step 2: Switch x and y : $x = f(y)$
- Step 3: Solve $x = f(y)$ for y.

Example: Find the inverse of $f(x) = 1 + \sqrt{2 + 3x}$.

Example: Find the inverse of $f(x) = \frac{4x-1}{2x+3}$.

It also follows from the definition of an inverse function that the graph of f^{-1} is the reflection of the graph of f over the line $y = x$:



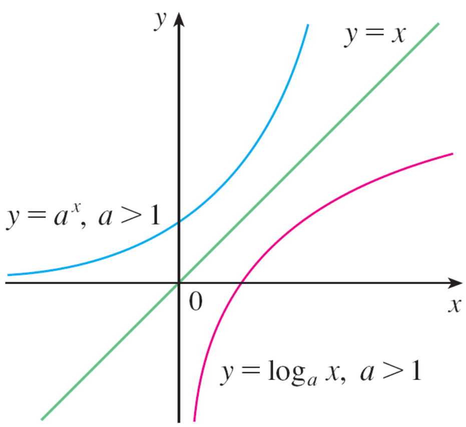
We may now speak of **logarithms**. If $a > 0$ and $a \neq 1$, the exponential functions $f(x) = a^x$ is either increasing or decreasing everywhere and so it is one-to-one by the horizontal line test.

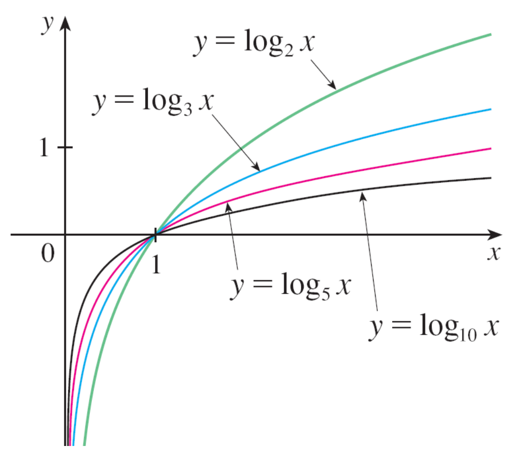
Therefore, an inverse exists which is called the **logarithmic function with base a** . We denote it by $\log_a$.

Since exponential and logarithm are inverse of each other, we have

$$a^{\log_a(x)} = x \ (x > 0) \ , \ \log_a(a^x) = x.$$

We also know what the graph of the logarithm must look like based on the concluding remark of the inverses section:





The following properties of logarithmic functions follow from the corresponding properties of exponential functions:

- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
- $\log_a(x^r) = r\log_a(x)$.

We already remarked that e is a special base. The base e logarithm has a special name, called the **natural logarithm** and is denoted by ln.

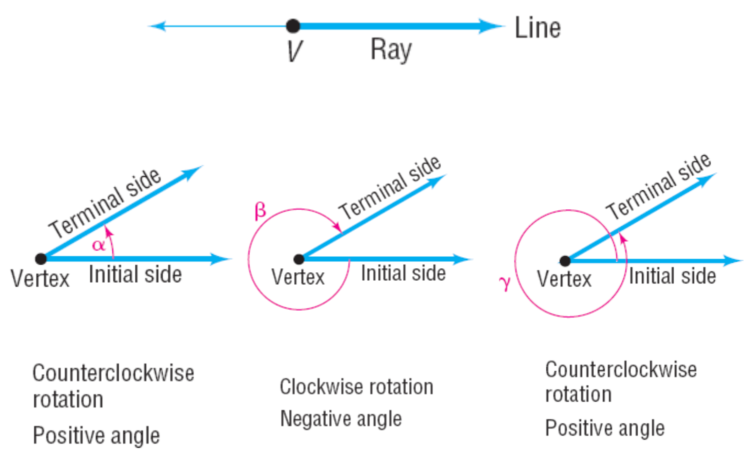
Since scientific calculators usually only compute ln, if we wish to compute $\log_a(x)$, we need the **change of base formula**:

$$\log_a(x) = \frac{\ln x}{\ln a} \tag{1.4.1}$$

1.5 Trigonometric Functions: Stewart Section 1.5 and Appendix D

The last family of functions we need to discuss before we can move into the calculus is the family of trigonometric functions. The trigonometric functions are usually introduced first by way of right triangle trigonometry.

Remember that an **angle** is formed by two rays (called **the sides**) sharing a common endpoint (called the **vertex**).

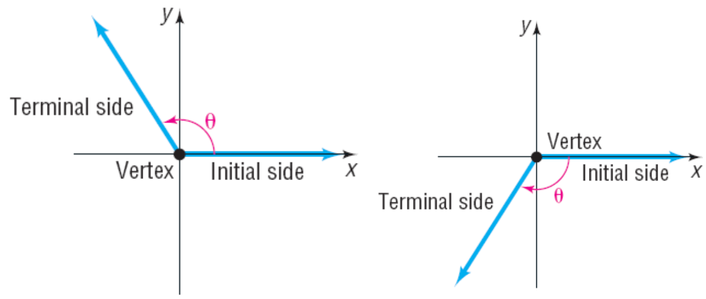


Angles can be measured in either radians or degrees. The conversion between the two units is

$$\pi \text{ rad} = 180^\circ.$$

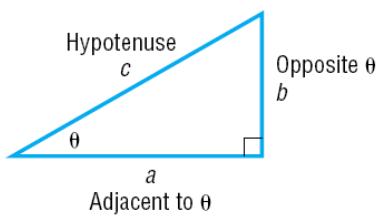
The **standard position** of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive x-axis. A **positive** angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Likewise, **negative** angles are obtained by clockwise rotation.

By forming a right triangle with one (**acute**) angle θ, we may define the trigonometric functions as follows:



(a) θ is in standard position; θ is positive

(b) θ is in standard position; θ is negative

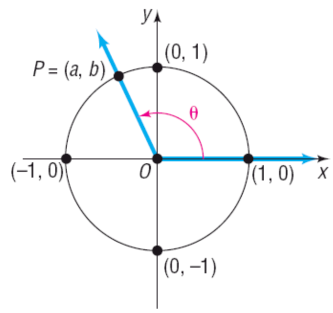


$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{b} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{a} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{a}{b}$$

To extend these definitions to arbitrary angles, we first place θ in standard position and then look at the point of intersection of the terminal side with the unit circle: The functions $\sin(x), \cos(x)$, and $\tan(x)$ are arguably more important than $\csc(x), \sec(x)$, and $\cot(x)$ for our purposes. When we have need of these later three functions, we will review their relevant properties. I refer you to Appendix D for more information. Based on the definition of sine, cosine, etc. on the unit circle, we can obtain the graphs of the trigonometric functions.

Based on the Horizontal Line Test, we see that $\sin(x), \cos(x)$ and $\tan(x)$ are not one-to-one functions. However, it is possible to restrict their respective domains so that the resulting function *is* one-to-one.

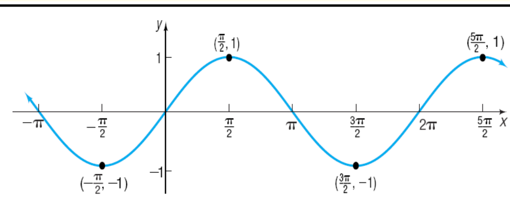


$\sin\theta = b$	$\cos\theta = a$	$\tan\theta = \dfrac{b}{a}, \quad a \neq 0$
$\csc\theta = \dfrac{1}{b}, \quad b \neq 0$	$\sec\theta = \dfrac{1}{a}, \quad a \neq 0$	$\cot\theta = \dfrac{a}{b}, \quad b \neq 0$

Properties of the Sine Function $y = \sin x$

1. The domain is the set of all real numbers.
2. The range consists of all real numbers from -1 to 1, inclusive.
3. The sine function is an odd function, as the symmetry of the graph with respect to the origin indicates.
4. The sine function is periodic, with period 2π.
5. The x-intercepts are $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$; the y-intercept is 0.
6. The maximum value is 1 and occurs at $x = \ldots, -\dfrac{3\pi}{2}, \dfrac{\pi}{2}, \dfrac{5\pi}{2}, \dfrac{9\pi}{2}, \ldots$;

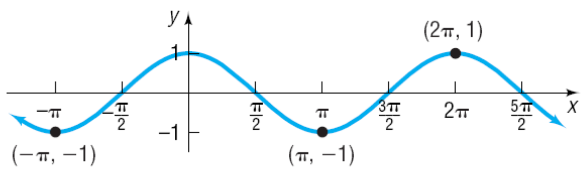
   the minimum value is -1 and occurs at $x = \ldots, -\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{7\pi}{2}, \dfrac{11\pi}{2}, \ldots$.

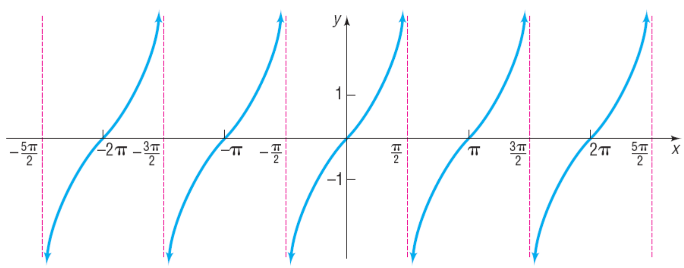


As Stewart mentions in section 1.5, the remaining inverse trigonometric functions are not used as frequently. See Section 1.5 if you are interested. Much time was spent in precalculus developing and proving trigonometric identities. While we may use of those identities in this class, it is not worth our time to review them all. When an identity is needed, I will remind you of it. You will not be required to memorize trigonometric identities in this class.

Properties of the Cosine Function

1. The domain is the set of all real numbers.
2. The range consists of all real numbers from -1 to 1, inclusive.
3. The cosine function is an even function, as the symmetry of the graph with respect to the y-axis indicates.
4. The cosine function is periodic, with period 2π.
5. The x-intercepts are $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$; the y-intercept is 1.
6. The maximum value is 1 and occurs at $x = \ldots, -2\pi, 0, 2\pi, 4\pi, 6\pi, \ldots$; the minimum value is -1 and occurs at $x = \ldots, -\pi, \pi, 3\pi, 5\pi, \ldots$.

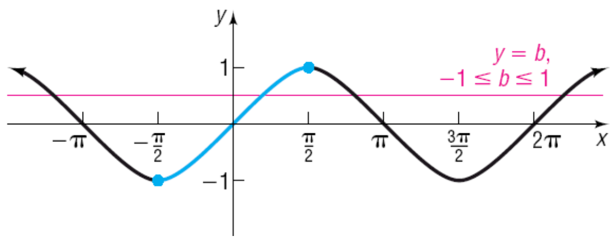




Properties of the Tangent Function

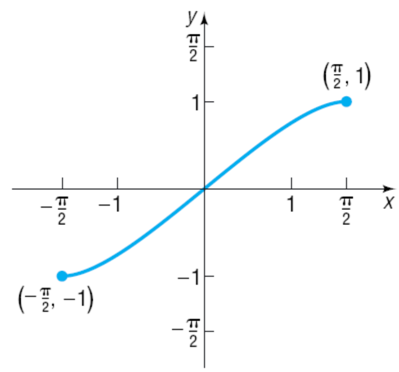
1. The domain is the set of all real numbers, except odd multiples of $\frac{\pi}{2}$.
2. The range is the set of all real numbers.
3. The tangent function is an odd function, as the symmetry of the graph with respect to the origin indicates.
4. The tangent function is periodic, with period π.
5. The x-intercepts are $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$; the y-intercept is 0.
6. Vertical asymptotes occur at $x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$.

The Inverse Sine Function



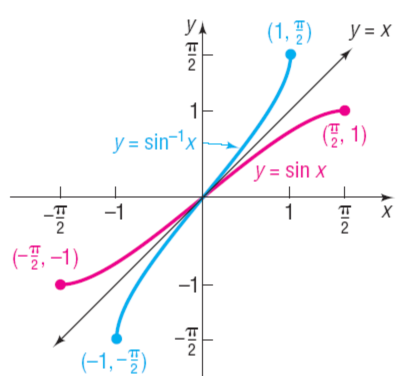
$y = \sin x, -\infty < x < \infty, -1 \leq y \leq 1$

$y = \sin x, -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}, -1 \leq y \leq 1$



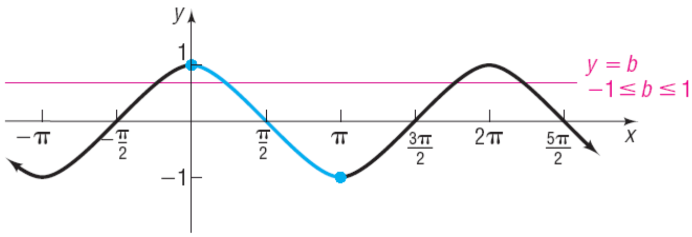
$$y = \sin^{-1} x \quad \text{means} \quad x = \sin y$$

$$\text{where} \quad -1 \leq x \leq 1 \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$



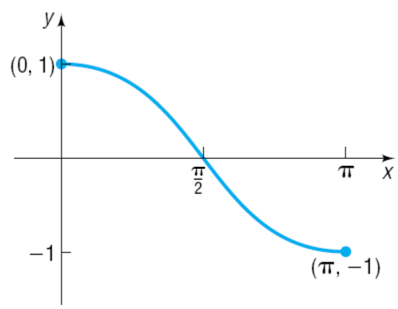
$y = \sin^{-1} x, -1 \leq x \leq 1, -\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$

The Inverse Cosine Function

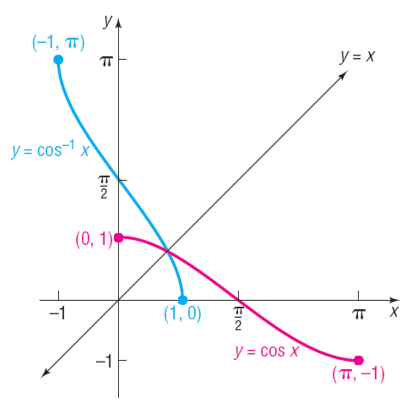


$y = \cos x,\ -\infty < x < \infty,$
$-1 \le y \le 1$

$y = \cos x,\ 0 \le x \le \pi,\ -1 \le y \le 1$

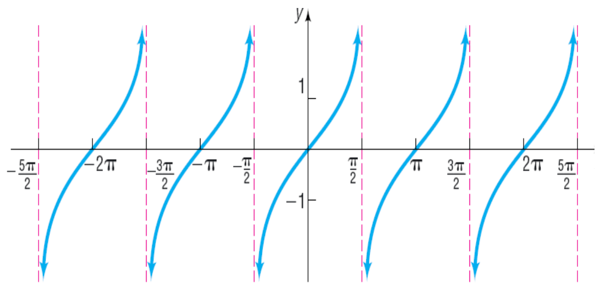


$$y = \cos^{-1} x \quad \text{means} \quad x = \cos y$$
$$\text{where} \quad -1 \le x \le 1 \quad \text{and} \quad 0 \le y \le \pi$$



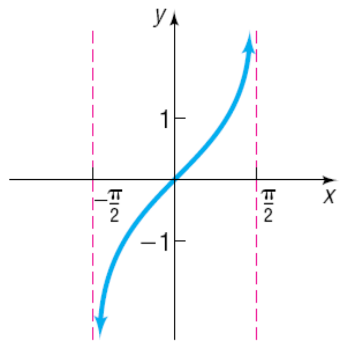
$y = \cos^{-1} x,\ -1 \le x \le 1,\ 0 \le y \le \pi$

The Inverse Tangent Function

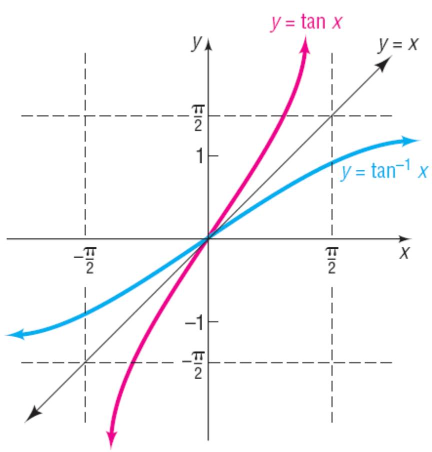


$y = \tan x$, $-\infty < x < \infty$, x not equal to odd multiples of $\frac{\pi}{2}$, $-\infty < y < \infty$

$y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $-\infty < y < \infty$



$$\boxed{\begin{array}{c} y = \tan^{-1} x \quad \text{means} \quad x = \tan y \\ \text{where} \quad -\infty < x < \infty \quad \text{and} \quad -\dfrac{\pi}{2} < y < \dfrac{\pi}{2} \end{array}}$$



$y = \tan^{-1} x$, $-\infty < x < \infty$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Chapter 2

Limits

2.1 Basic Concepts of Limits: Stewart Section 2.1 and 2.2

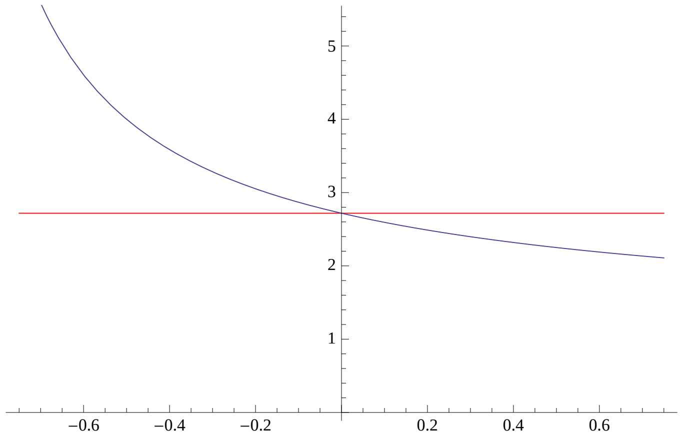
Calculus is the mathematics of **limits**. Let's get comfortable with limits before we learn how to calculate them. We'll start by thinking of limits in terms of graphs and then in terms of numbers with a few specific examples.

We've seen limits twice already:

- a^x for x irrational and $a > 0$ is defined as the **limit** of a^{r_n} for rational approximations r_n of x
- Napier's constant e is the **limit** of $\left(1+\frac{1}{x}\right)^x$ as x gets very large. Equivalently, it's possible to define e as the limit of $(1+x)^{\frac{1}{x}}$ as x becomes very small (or as x **approaches** 0).

We'll use this definition of e to start our discussion on limits. Here is a plot of $f(x) = (1+x)^{\frac{1}{x}}$. Notice how the point $x = 0$ is excluded from the graph since the exponent is $\frac{1}{x}$ and we can never divide by zero.

The red line indicates Napier's constant $e \approx 2.71828$. Even though the function is not defined at $x = 0$, we can trace with our pencils along the graph on either side and get closer and closer to e as we move our pencils closer and closer to 0.



In fact, we can make the values of $f(x)$ as close as we like to e by taking x sufficiently close to 0. Think of walking along the graph towards 0 from either side of the y-axis. Since there is a hole at $x = 0$ (the domain excludes this point), be careful not to fall in! Think of inching ever closer to the edge of a cliff: if you are brave enough, you can get as close to the edge as you like, but go to far and you fall.

On the graph in this example we can get very, very close to 0 without falling through the hole. If someone were to measure our altitude or height above the x-axis, they would report a number very, very close to e. Formally, we express this by saying "the limit of the function $f(x) = (1 + x)^{\frac{1}{x}}$ as x approaches 0 is equal to e."

Notation: The notation for this is

$$\lim_{x \to 0} f(x) = e.$$

Definition: Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

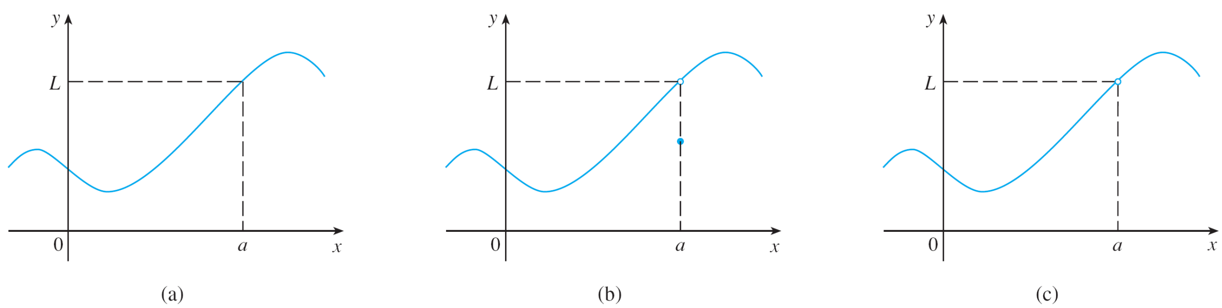
$$\lim_{x \to a} f(x) = L$$

and say " the limit of $f(x)$, as x approaches a, equals L" if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

In short, if $\lim_{x\to a} f(x) = L$, this means we can make f as close as we like to L by making $x \neq a$ sufficiently close to a.

Notice that we exclude $x = a$ in the general definition of the limit. The function $(1+x)^{\frac{1}{x}}$ is an example of why this is necessary and desirable: often times we are interested in the behavior of the function near a point where it is undefined. In this case, near $x = 0$ where $f(x)$ doesn't make sense, $f(x)$ approaches e.

The implication is that what happens *at* $x = a$ does not have an impact on the *limit* of $f(x)$ as x *approaches* a:



$\lim_{x\to a} f(x) = L$ in all three cases

Alternative Notation: Sometimes we'll use $f(x) \to L$ as $x \to a$ (read " $f(x)$ approaches L as x approaches a") to denote a limit.

Another way of thinking about limits (similar to how we defined a^x for $a > 0$ and $x =$ irrational) goes something like this:

At some point in your math education you probably entered the operation " one divided by three" into a standard handheld calculator. (If not, now is the time to try it!) You essentially found the result 0.333... with the number of threes depending on what calculator you use. Often times this **decimal representation** of $1/3$ is written as $0.\bar{3}$ where the bar indicates that the 3's repeat forever without terminating.

Let's think about what this actually means. When a positive number less than 1 is written as $0.a_1a_2a_3...$ where the numbers $a_n, n = 1, 2, \ldots$ are integers between 0 and 9 we really mean that

$$0.a_1a_2... = a_1 10^{-1} + a_2 10^{-2} + a_3 10^{-3} + \ldots.$$

For example, $0.5 = 5 \times 10^{-1} = \frac{5}{10} = \frac{1}{2}$ and $0.14 = 1 \times 10^{-1} + 4 \times 10^{-2} = \frac{1}{10} + \frac{4}{100} = \frac{14}{100} = \frac{7}{50}$. When our decimal representations terminate (*i.e.* don't go on forever) we can easily sum all of the terms together to find the rational form of the number. Even if the decimal representation is very long, it may take a long time to add all of the terms together, but it can be done as long as the decimal representation stops somewhere.

But what about $0.\bar{3}$? What does it mean to perform the sum to get back to the rational form which we known is $1/3$? Just because we know the answer is one third doesn't mean we fully understand the process of going back and forth between the decimal and rational forms. The problem I'm hinting at here has to do with the fact that, no matter how much time we have at our disposal, we can never *literally* add an infinite number of terms. Adding always takes some time and energy, even though it may not seem like it with today's computers. However, an infinite process like adding $3\times10^{-1}+3\times10^{-2}+\ldots$ term-by-term cannot be done!

Instead of doing the infinite sum, let's look at a **truncated** sum: we don't sum all of the terms, but we do go out to some place in the decimal representation and stop. The first few truncations are

$$0.3 = 3 \times 10^{-1} \ , \ 0.33 = 3 \times 10^{-1} + 3 \times 10^{-2} \ , \ 0.333 = 3 \times 10^{-1} + 3 \times 10^{-2} + 3 \times 10^{-3}$$

These sums can be done since they are finite.

It's a good idea to introduce some terminology here that will be used throughout our course. If you go on to higher calculus and calculus-based applications, you will certainly see this terminology again. First, a list of numbers $a_1, a_2, \ldots$ like that appearing in the expression for a decimal is referred to as a **sequence**. **Elements** of a sequence, or entries in the list, can be any real number, but the sequences in decimal representations are integers between 0 and 9.

A **partial sum** is defined by

$$\sum_{j=1}^{n} b_j = b_1 + b_2 + \cdots + b_n.$$

The Greek letter Σ stands for "sum" and indicates that you should add up all entries beginning with **index** $j = 1$ (at the bottom of Σ) and ending with **index** $j = n$ (at the top of Σ). For example,

$$0.3333 = \sum_{j=1}^{4} 3 \times 10^{-j}.$$

A useful formula that you will see derived in Calculus II (called the **geometric sum formula** if you are interested) tells us that

$$\sum_{j=0}^{n} r^j = \frac{1 - r^{n+1}}{1 - r}$$

where r is a real number. Try a few specific values of n and r to convince yourself that this is true.

We can use this formula to write any truncated sum as

$$0.\underbrace{3\ldots3}_{\text{n 3's}} = \sum_{j=1}^{n} 3 \times 10^{-j} = 3 \times \sum_{j=1}^{n} 10^{-j} = 3 \times \left(\frac{1 - \left(\frac{1}{10}\right)^{n+1}}{1 - \frac{1}{10}} - 1 \right)$$

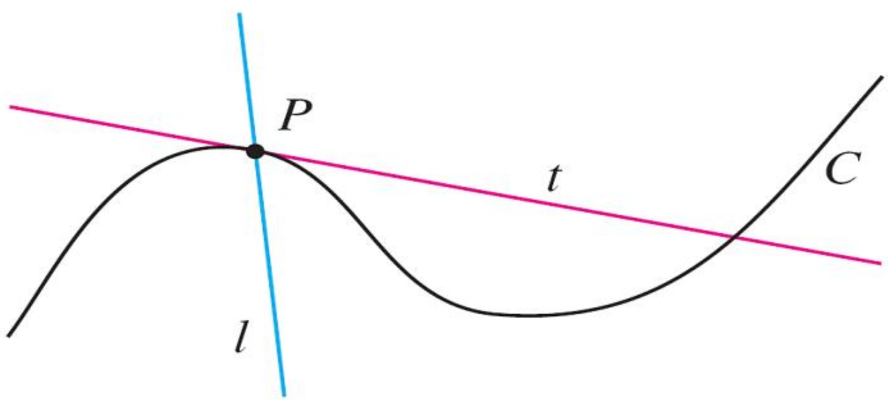
We can use this formula to figure out a **pattern** of our partial sums: notice that as n gets very, very large, the term $(1/10)^{n+1}$ becomes very, very small. So if we write the truncated sums in rational form, we notice a pattern that the rational forms get very, very close to $3 \times (1/(1 - 1/10) - 1) = 1/3$.

This is what we mean by $1/3 = 0.\bar{3}$! The pattern of the partial (truncated) sums as n gets very large (approaches ∞) is as follows: the partial sums become closer and closer to $1/3$. We can never do the infinite sum $0.\bar{3}$, but the pattern of the partial sums is to approach $1/3$. We say that the **limit** of the partial sums is $1/3$.

We went through the trouble of introducing limits in the precalculus review because we wanted to show how e arises naturally in continuously compounded interest (i.e. the Pert formula). I also tried to present limits intuitively from two different perspectives. How are limits useful? In other words, why should we care about limits?

This is a broad question, one which we couldn't fully answer in a single semester course! To start to answer this question, the book introduces in section 2.1 the tangent problem and the velocity problem (which is really just the tangent problem).

The Tangent Problem: In short, the tangent problem is this: suppose we are given a curve which for simplicity we assume to be the graph of a function $y = f(x)$. How can we find the equation of the line which just touches the graph at a point a? In order to find the

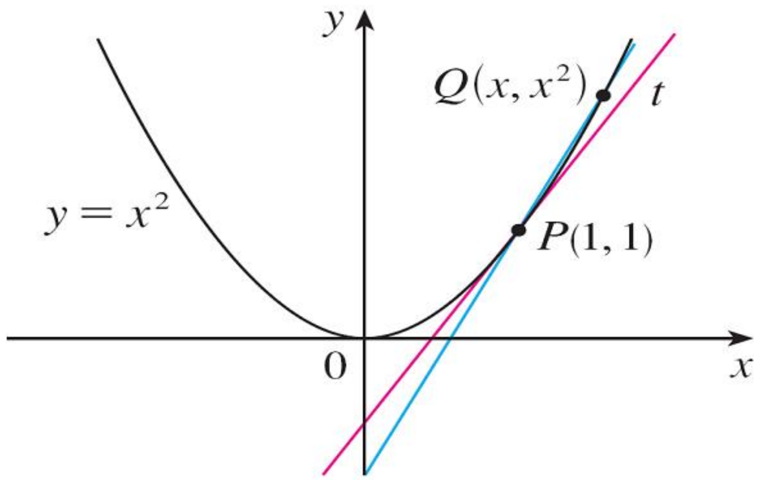


equation of a line, we need two points or one point and the slope. So far, we have only one point: the line must go through $(a, f(a))$ if it is to be a tangent line.

In precalculus, we briefly touched on the idea of a **secant line** to the graph of a function. The secant line was introduced as a rough measure of the *average* rate of change of a function over some interval $[a, a+h]$, $h > 0$.

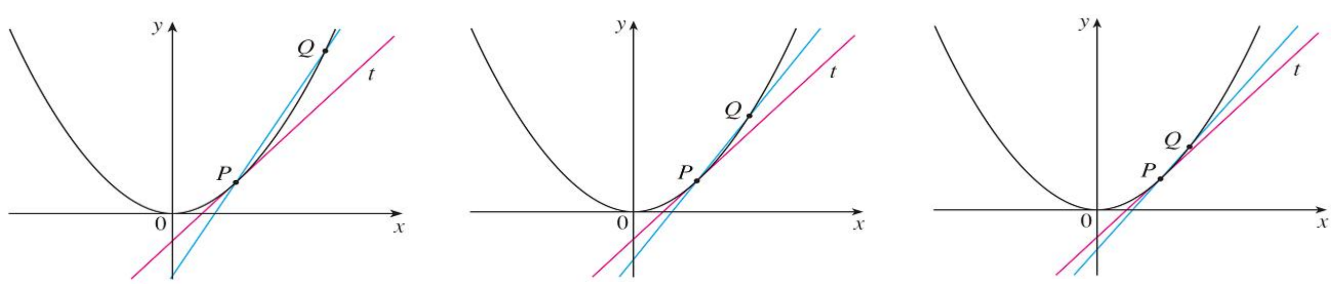
As we can see from this example figure where $f(x) = x^2$, the secant line is a reasonable approximation to the tangent line! Remember, the slope of the secant line on the interval $[a, a+h]$ is given by the **difference quotient**:

$$m_{sec} = \frac{f(a+h) - f(h)}{h}$$



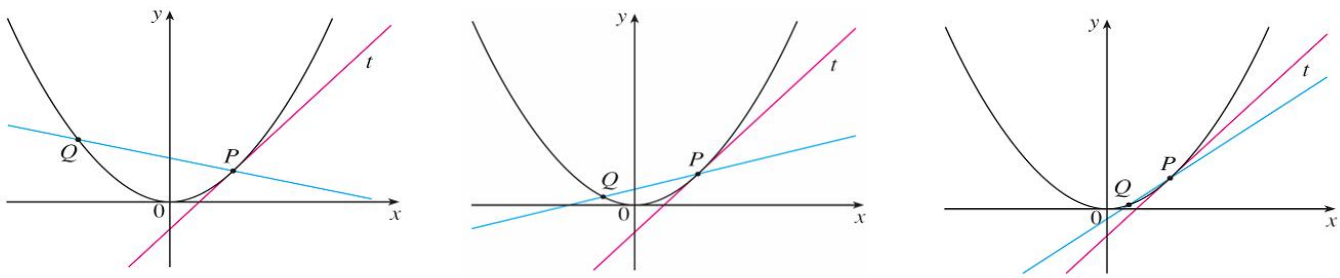
The reason, of course, is because the secant line goes through the two points $(a, f(a))$ and $(a+h, f(a+h))$.

If we take h to be smaller and smaller, we see that the secant line approaches the tangent line:



Q approaches *P* from the right

Of course, there is no reason that we had to look specifically at the interval $[a, a+h]$: the same argument holds if we compute the secant line on the interval $[a-h, a]$ ($h > 0$) for smaller and smaller h values:



Q approaches *P* from the left

This example suggests that the slope of the tangent line to the graph of $y = f(x)$ (when it exists) should be equal to the limit of the slopes of the secant lines as the interval defining the secant line becomes smaller and smaller:

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example: As an example, let's try and compute the equation for the tangent line to the curve $y = x^2$ at a fixed point a. Let's take $h > 0$. We first compute the slope of the secant line defined on the interval $[a, a+h]$.

The difference quotient is: $\frac{f(a+h)-f(a)}{h} = \frac{(a+h)^2-a^2}{h}$. By FOILing the square, we see that $(a+h)^2 = a^2 + 2ah + h^2$ and so

$$\frac{f(a+h) - f(a)}{h} = \frac{a^2 + 2ah + h^2 - a^2}{h} = \frac{2ah + h^2}{h} = 2a + h$$

Now suppose we compute the slope of the secant line defined on the interval $[a-h, a]$:

$$\frac{f(a) - f(a-h)}{h} = \frac{a^2 - (a-h)^2}{h}$$

We know that $(a-h)^2 = a^2 - 2ah + h^2$ and so

$$\frac{f(a) - f(a-h)}{h} = \frac{a^2 - a^2 + 2ah - h^2}{h} = 2a - h.$$

So in the first case (secant line on $[a, a+h]$), the slope is $2a + h$; in the second, (secant line on $[a-h, a]$, the slope is $2a - h$.

What happens when h gets smaller and smaller? Well, in both cases, the secant line slope approaches $2a$.

Therefore, the slope of the tangent line at $x = a$ must be $2a$! Does this make sense?

Well, choose $a > 0$. The slope of the tangent line is supposed to be $2a$. Look at the graph of $y = x^2$. The tangent lines to the right of the origin have positive slope and the slopes do increase as we move farther away from the origin!

Now choose $a < 0$. The slopes of the tangent lines to the left of the origin become steeper as you move away from the origin and they are negative.

So far so good. What about $a = 0$? There, the parabola reaches it's lowest point and so the tangent line must be horizontal. A horizontal line has slope equal to 0 which is consistent with our find.

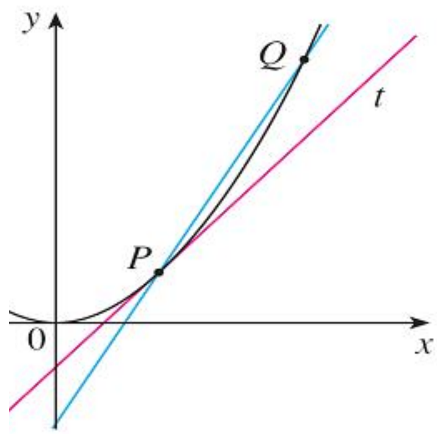
To answer the initial question, the equation of the tangent line for a given a is: $l(x) = 2ax - a^2$.

We'll have much more to say about tangent lines later when we know more about computing limits. Section 2.1 of Stewart also mentions the velocity problem as a key application of the idea of limits.

The Velocity Problem: In short, the velocity problem is this: given a description of the position of an object in space as a function of time, how can we determine the **instantaneous velocity** of the object.

Really, the velocity problem is a special case of the more general problem of finding the **instantaneous rate of change of some function**. In the case where we have a description of the position of an object along the real line (i.e. one-dimensional motion) $s(t)$, we want to know the velocity $v(t)$ as a function of time. If you imagine the object as a car, the velocity would essentially be the reading of the speedometer.

As I mentioned earlier, the slope of the secant line of a function f over an interval $[a, a+h]$ measures the **average rate of change** of f over that interval.



If you were to drop an object from a tall building, the distance of the object from the top of the building would be $s(t) = 4.9 \ (m/s^2)\, t^2$, so considering the parabola again for this example is a fine idea.

Suppose we again take the interval smaller and smaller: the average rate of change of the function over a very small interval is a very good approximation to the instantaneous rate of change.

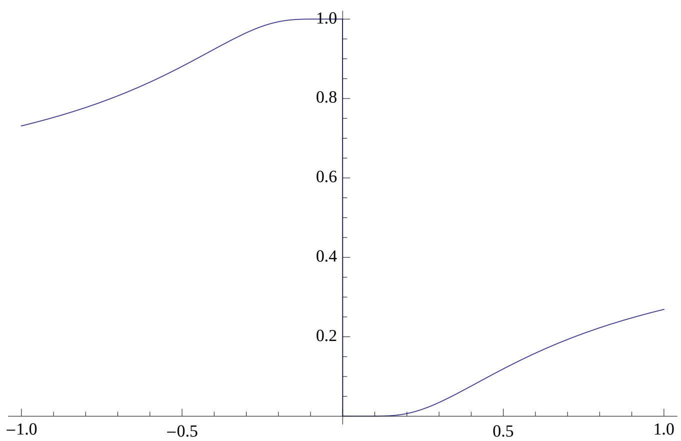
So we can define the instantaneous rate of change as the limit of the average rate of change as the size of the interval becomes smaller and smaller. But this means that the instantaneous rate of change of a function at a point $x = a$ is just the slope of the tangent line at $x = a$! Hence, the velocity problem (or, more generally, the instantaneous rate of change problem) is really just the tangent problem!

To be concrete, we have just established that $v(t) = \lim_{h\to 0} \frac{s(t+h)-s(t)}{h}$. As we computed in the last example, we find $v(t) = 9.8\ (m/s^2)\, t$.

Since we now have a sense of why limits are important, we'll spend some time coming up with ways of computing limits. As of right now, we don't have many tools to compute limits.

One option would be to use the graph of $f(x)$ if we have it.

Example: Find $\lim_{x\to 0} \frac{1}{1+e^{\frac{1}{x}}}$ if it exists using the graph provided.



Given a function $f(x)$, we could conceivably using a calculator or a computer to **guess** the value of the limit of f as x approaches a by plugging values of x close to a into $f(x)$. This is similar to what we did earlier to define the number e.

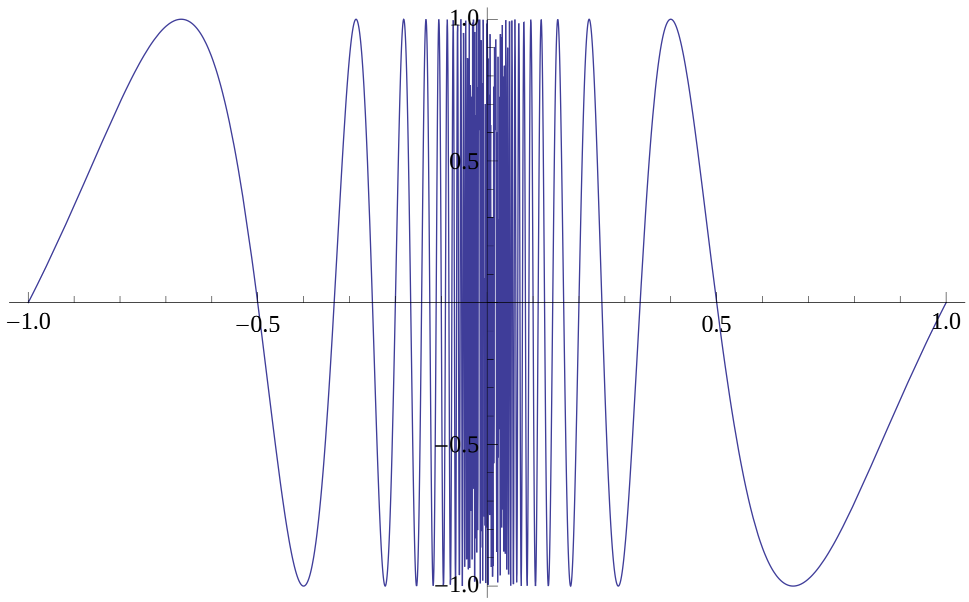
However, there are two major problems with this approach.

First, using a calculator to guess a limit is **not** acceptable as a mathematical proof. From here on, we expect a certain level of mathematical rigor.

Second, it is entirely possible that this computer-based search for a limit will give you **false** limits!

Example: Investigate $\lim_{x\to 0}\sin(\frac{\pi}{x})$.

ANS: As it turns out, this function has no limit as x approaches 0. First, let's look at the graph:



The graph oscillates so wildly near 0, it's really hard to tell what's going on.

Suppose you looked at $f(\frac{1}{n})$ using your computer/calculator and tried to guess the limit of f as $x \to 0$ by taking n to be really large. Since $f(\frac{1}{n}) = 0$, you would conclude that the limit is 0. Now your friend uses the same strategy but uses the points $x = \frac{2}{4n+1}$. Since $f(\frac{2}{4n+1}) = 1$, your friend would conclude that the limit is 1. Turns out, you're both wrong!

Example: If you use a computer/calculator to guess the limit $\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}$, you'll also arrive at a false result.

For $|t| \geq 0.01$, your computer will lead you to think the limit is $\frac{1}{6}$. It turns out this is correct. But, if you push to even smaller values, your calculator will lead you to suspect that the limit is actually 0, which is false.

When you plug in very small values of t (say, like $t = 0.0001$), your calculator will return 3.000 for $\sqrt{t^2+9}$ which is the calculator's value for $\sqrt{t^2+9}$ to as many digits as the calculator is capable of carrying.

To end this cautionary tale, you cannot always trust your calculator! Even when your calculator allows you to guess the correct limit, this is not a rigorous mathematical result!

Before we can move on to methods for computing limits, I have to define two more limit concepts: **one-sided limits** and **infinite limits**. We also need to spend some time understanding limits from a rigorous point of view but I'll postpone that discussion for now.

When we computed the tangent line for the graph of the function $f(x) = x^2$, we saw that the secant line on the interval $[a, a+h]$ had slope $2a+h$ and the slope of the secant line on the interval $[a-h, a]$ was $2a-h$.

In both cases, $h > 0$. When we took the limit of the slopes of the secant lines, we were secretly taking **one-sided** limits.

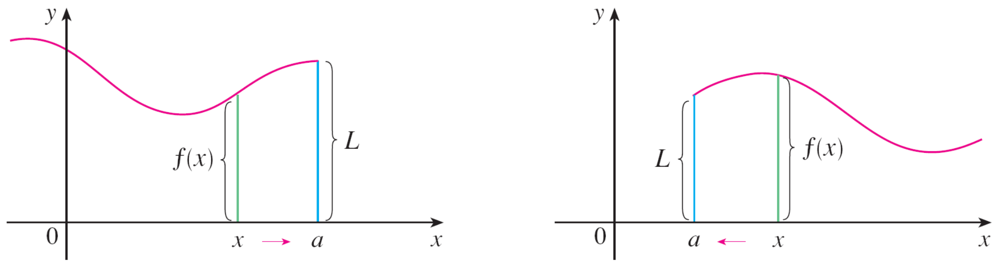
In the first case, we took the limit as $h \to 0$ while keeping $h > 0$. This amounted to keeping the left-hand point $x = a$ fixed while moving the right-hand point $x = a+h$ towards $x = a$. This is an example of a **right-hand limit**:

Definition: We write $\lim_{x \to a^+} f(x) = L$ and say the **right-hand limit** of $f(x)$ as x approaches a (or the limit of $f(x)$ as x approaches a from the right) is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and greater than a.

The second case amounted to keeping the right-hand point $x = a$ fixed while moving the left-hand point $x = a - h$ towards $x = a$. This is an example of a **left-hand limit**:

Definition: We write $\lim_{x \to a^-} f(x) = L$ and say the **left-hand limit** of $f(x)$ as x approaches a (or the limit of $f(x)$ as x approaches a from the left) is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and less than a.

A picture should help clarify the notion:



(a) $\lim_{x\to a^-} f(x) = L$ (b) $\lim_{x\to a^+} f(x) = L$

For the tangent line example, it turns out both the right-hand and left-hand limits exist and are equal.

In general, $\lim_{x\to a} f(x) = L$ **if and only if** $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.

Besides computing tangent lines, one-sided limits often arise in the context of piecewise-defined functions.

Example: Compute the one-sided limits of the Heaviside function as $t \to 0$. Does the limit exist?

Example: Compute the one-sided limits of

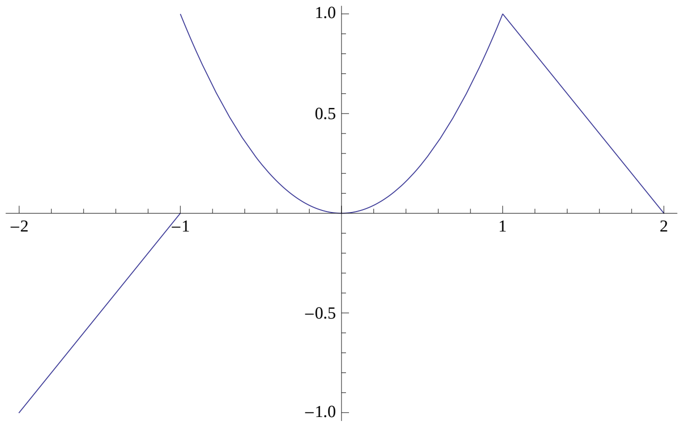
$$f(x) = \begin{cases} 1+x & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x \leq 1 \\ 2-x & \text{if } x \geq 1 \end{cases}$$

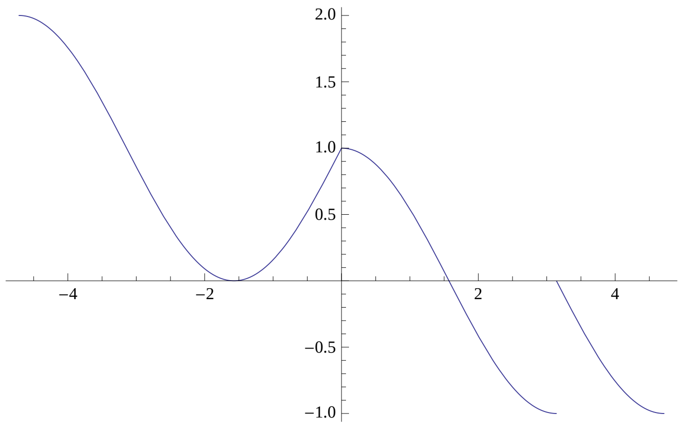
at $x = \pm 1$. Does f have a limit at $x = \pm 1$?

Example: Compute the one-sided limits of

$$f(x) = \begin{cases} 1+\sin(x) & \text{if } x < 0 \\ \cos(x) & \text{if } 0 \leq x \leq \pi \\ \sin(x) & \text{if } x > \pi \end{cases}$$

at $x = 0, \pi$. Does f have a limit at $x = 0, \pi$?





Finally, **infinite limits** are defined as follows:

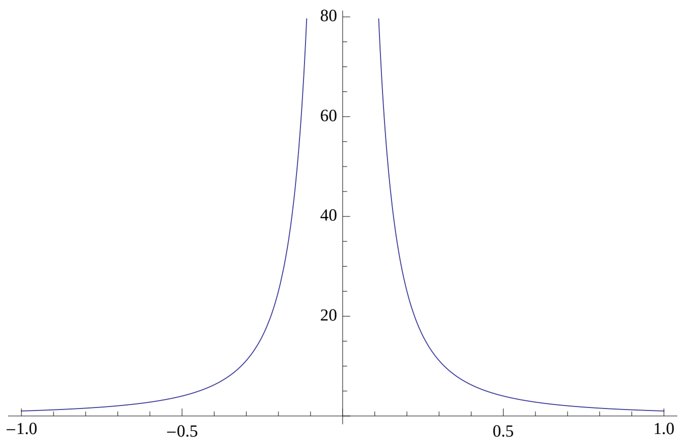
Definition: Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x to be sufficiently close to a, but not equal to a.

We don't actually think of ∞ as a number. Rather, this is just shorthand for saying that f "blows up" as $x \to a$. The quintessential example of this behavior is $f(x) = \frac{1}{x^2}$ near $x = 0$.

Whenever you take a small number and square it, it becomes an even smaller positive number. Divide 1 by this very small number and you get a very large number. Hence, the limit as $x \to 0$ for $f(x) = \frac{1}{x^2}$ is infinite.



A similar sort of limit, for functions that become large (in magnitude) and negative as x gets closer to a is defined by:

Definition: Let f be defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large and negative by taking x to be sufficiently close to a, but not equal to a.

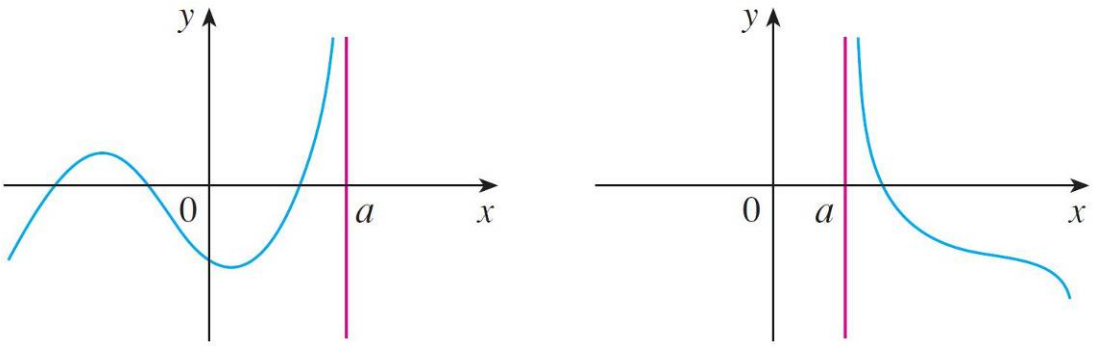
An example of this behavior is of course given by $f(x) = -\frac{1}{x^2}$.

Similar definitions can be given for **one-sided infinite limits**. Instead of rehashing the formal definitions, here's a picture:

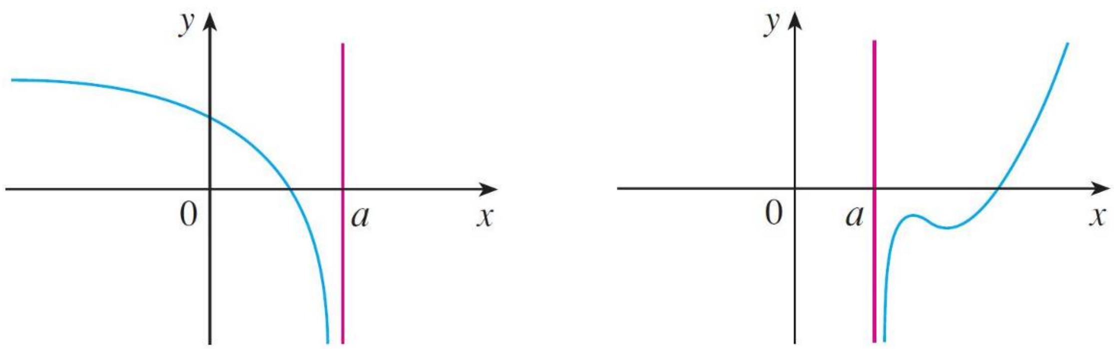
We mentioned **asymptotes** during our precalc review. Here's the formal definition of **vertical asymptote**:

We've already seen a few vertical asymptotes during our review of functions from precalculus.

- The graph of $f(x) = \frac{1}{x^2}$ has the y-axis for a vertical asymptote.
- The graph of $f(x) = \tan(x)$ has vertical asymptotes at $x = (2n+1)\frac{\pi}{2}$ when n is an integer.



(a) $\lim\limits_{x \to a^-} f(x) = \infty$ (b) $\lim\limits_{x \to a^+} f(x) = \infty$



6 Definition The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

- The logarithm function ln has the y-axis for a vertical asymptote.

In general, asymptotes give us information about the long-range behavior of functions. This will become more apparent when we finally define **horizontal** asymptotes.

Example: Determine the limit

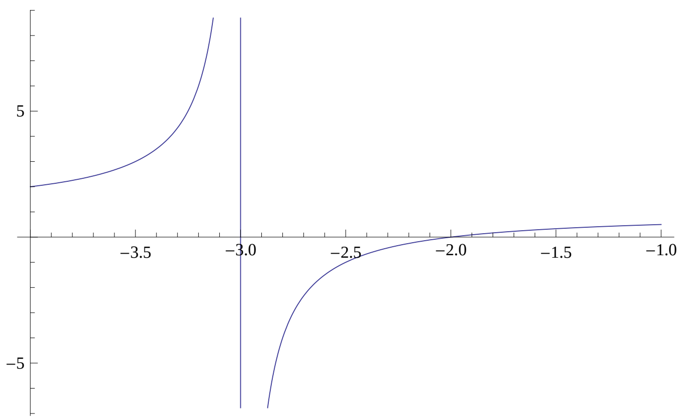
$$\lim_{x \to -3^+} \frac{x+2}{x+3}.$$

Does the graph have a vertical asymptote at $x = -3$?

ANS: If we had the graph of $\frac{x+2}{x+3}$, we could make some headway. We can obtain the graph by recognizing that

$$\frac{x+2}{x+3} = \frac{x+3-3+2}{x+3} = \frac{x+3}{x+3} + \frac{-1}{x+3} = 1 - \frac{1}{x+3}$$

Therefore, the graph of the function can be obtained by transforming the graph of $f(x) = \frac{1}{x}$.



Even without the graph, we can reason that the limit must be $-\infty$ just based on $f(x) = 1 - \frac{1}{x+3}$.

Example: Determine the limit

$$\lim_{x \to 3^+} \ln(x^2 - 9).$$

Does the graph have a vertical asymptote at $x = 3$?

Example: Determine the limit

$$\lim_{x \to 1} \frac{2-x}{(x-1)^2}.$$

2.2 The Precise Definition of a Limit and Limit Laws: Stewart 2.3 and 2.4

Our approach to limits up to this point relies mostly on intuition. Mathematics is an exact discipline, however, and so we need a much more precise definition of limit in order to put the subject of calculus on a firm foundation. I should emphasize that the following is rarely used in calculation of limits of interest. Rather, the Limit Laws and properties of limits (which are occasionally used to calculate limits but mostly used to prove limit theorems) all follow from painstaking proofs based on the precise definition of a limit.

Since the precise definition of a limit has limited utility for us (unless you plan on going into mathematical analysis in the future), we won't spend much time on it.

Definition: Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of f as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

In words, $\lim_{x \to a} f(x) = L$ means that the distance between $f(x)$ and L can be made arbitrarily small by taking the distance from x to a sufficiently small (but not 0).

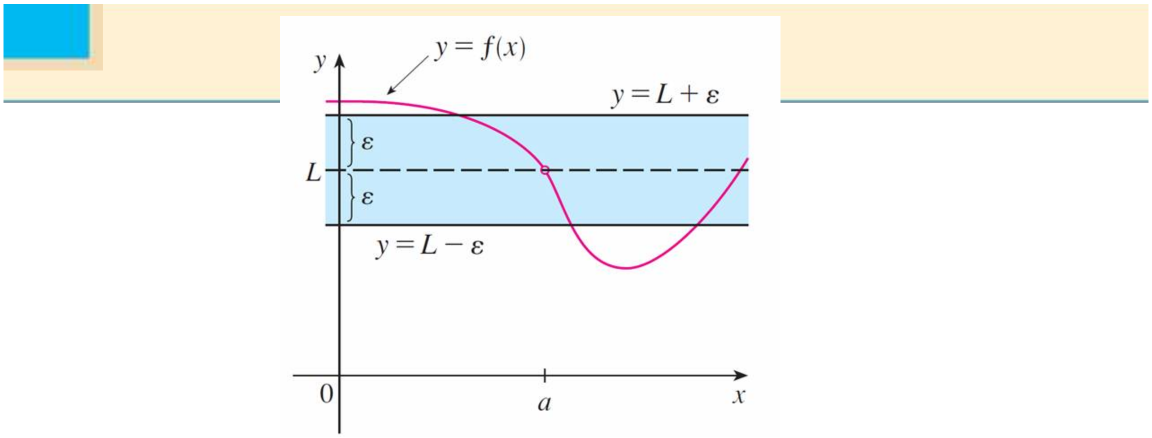
If we rephrase the definition, we can also visualize the precise definition of a limit:

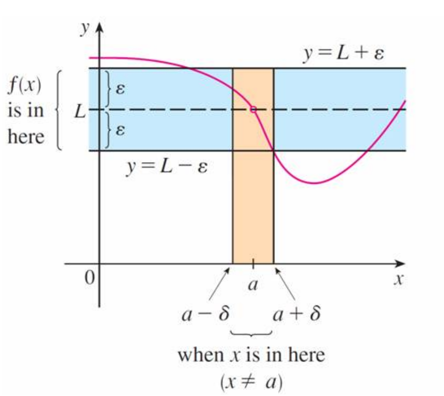
$\lim_{x \to a} f(x) = L$ means that for every $\epsilon > 0$ (no matter how small ϵ is) we can find $\delta > 0$ such that if x lies in the open interval $(a - \delta, a + \delta)$ and $x \neq a$, then $f(x)$ lies in the open interval $(L - \epsilon, L + \epsilon)$.

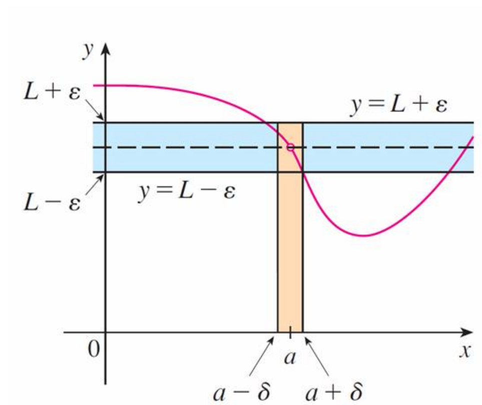
Example: Use the precise definition of a limit to show that $\lim_{x \to 2} (2x + 3) = 7$.

Example: Use the precise definition of a limit to show that $\lim_{x \to 2} x^2 = 4$.

ANS: Let's start with $|f(x) - L| = |x^2 - 4|$ and suppose we are given $\epsilon > 0$. We ask







the question "how small should we make $|x-2|$ so that $|f(x)-L|<\epsilon$.

First, factor x^2-4: $x^2-4=(x-2)(x+2)$ so that $|f(x)-L|=|x-2||x+2|$. If $\delta>0$ is just some number and it so happens that we choose x such that $|x-2|<\delta$, then it follows that

$$2-\delta<x<2+\delta$$

Therefore,

$$4-\delta<x+2<4+\delta$$

If we choose $\delta<4$, the left-hand quantity is positive and so we have

$$|x+2|<4+\delta$$

and

$$|f(x) - L| < (4 + \delta)\,\delta$$

So, given $\epsilon > 0$, we choose $\delta > 0$ so that $\delta < 4$ and $(4 + \delta)\delta < \epsilon$.

This completes the proof.

We're able to find δ explicitly for a given ϵ for only a small number of basic examples. Otherwise, we have to rely on numerical methods like graphing the function in order to find δ given ϵ. Of course, this rather defeats the purpose since relying on our calculator is NOT a mathematically rigorous way of proving a limit exists.

As I mentioned before, the importance of knowing the precise definition of a limit arises when we rigorously want to prove some general property of limits, like the Limit Laws.

The book proves the sum law; let's take a crack at proving the product law:

Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$$

To be concrete, suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = K$. For simplicity, let's also assume that both the limits are non-zero. We need to show that we can make

$$|f(x)g(x) - KL| < \epsilon$$

for $|x - a| < \delta$ for some δ where we are given $\epsilon > 0$.

Remember that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = K$. This means that there exist numbers $\delta_f > 0$ and $\delta_g > 0$ such that

$$|f(x) - L| < \frac{\epsilon}{2|K|}\ , \text{ for } |x - a| < \delta_f$$

and

$$|g(x) - K| < \frac{\epsilon}{4|L|} \text{ , for } |x - a| < \delta_f$$

The constants here may seem strange, but you'll see why I picked them in a just a few minutes.

$$|f(x)g(x) - LK| = |f(x)\,[g(x) - K] + K\,[f(x) - L]\,|$$

and by the triangle inequality $|x + y| \leq |x| + |y|$

$$|f(x)g(x) - LK| \leq |f(x)||g(x) - K| + |K||f(x) - L|$$

If $|x - a| < \delta_f$ and $|x - a| < \delta_g$, then

$$|f(x)g(x) - LK| < |f(x)|\frac{\epsilon}{4|L|} + |K|\frac{\epsilon}{2|K|}$$

From the other version of the triangle inequality $|x - y| \geq ||x| - |y||$, we have

$$|L| - \frac{\epsilon}{2|K|} < |f(x)| < |L| + \frac{\epsilon}{2|K|}$$

So as long as $\epsilon < 2|K||L|$, we have

$$0 < |f(x)| < 2|L|$$

and

$$|f(x)g(x) - LK| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, we must choose $\delta < \delta_f \ , \ \delta_g$.

For completeness, I present the precise definitions of **one-sided** and **infinite** limits:
A word about the definition of **infinite limits**: essentially, this definition means that we can make $f(x)$ as large as we like as long as we are sufficiently close to a.

Example: Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

3 Definition of Left-Hand Limit

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a \quad \text{then} \quad |f(x) - L| < \varepsilon$$

4 Definition of Right-Hand Limit

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a < x < a + \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

6 Definition Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) > M$$

Example: Prove that $\lim_{s \to 0^+} \ln(x) = -\infty$.

ANS: Let's fix M to be some negative number. We need to show that there is a δ so that if $0 < x < \delta$, then $\ln(x) < M$.

If we choose $\delta = \frac{1}{2}e^M$, say, then $0 < x < \delta = \frac{1}{2}e^M$ implies that $\ln(x) < M - \ln(2) < M$ since the logarithm is an increasing function.

Students often struggle with the so-called "ϵ-δ" definition of a limit. If you plan on pursuing mathematics beyond calculus you will have to become very comfortable with the definitions and proofs in this section. Fortunately, there is a nice equivalent way of thinking of limits which is sometimes a useful alternative to the "ϵ-δ" definition. The following definition of a limit harkens back to one of our original motivating examples: $0.\bar{3} = 1/3$. While like the "ϵ-δ" definition it is difficult to compute limits with this approach, students often find it easier to digest and understand.

Alternate Definition of a Limit: Let $f(x)$ be a real-valued function. Then $\lim_{x\to a} f(x) = L$ if and only if $\lim_{n\to\infty} f(p_n) = L$ whenever $\{p_n\}$ is a sequence of points such that $p_n \neq a$ and $\lim_{n\to\infty} p_n = a$.

To fully understand the above definition you need to remember:

Limit of Sequence of Points: A sequence of points (numbers) $\{p_n\}$ converges to L if and only if it is always possible to match the first M digits in the decimal representations of p_n and L for large enough n; *i.e.* it is possible to go far enough down the list of p_n's to match L to any arbitrary precision.

As I mentioned before, this is usually easier to understand than "ϵ-δ" but is not usually easier to work with since you have to somehow prove $\lim_{n\to\infty} f(p_n) = L$ for *any* possible sequence $\{p_n\}$. On the other hand, this definition is particularly suited to demonstrating that limits *don't* exist. Consider the example $\lim_{x\to 0} \sin(\frac{\pi}{x})$ we looked at above. By looking at two different sequences both converging to 0, we demonstrated that the limit could not possibly exist.

Enough definitions for the time being. So far, we either relied on graphs of functions, our intuition of functions based on their formulae or numerical guesses to compute limits. Occasionally Limit Laws can be useful for computing the actual numerical value of a limit. For completeness I include the Limit Laws below: If you are interested in the proofs of these laws, they are available in Appendix F and follow our "ϵ-δ" arguments above. Instead of focusing on these laws to compute limits I will introduce the concept of continuity which will lead us to a (relatively) simple process for computing all limits of interest in our course.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$

4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$

5. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer

7. $\lim_{x\to a} c = c$

8. $\lim_{x\to a} x = a$

9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer

10. $\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

    (If n is even, we assume that $a > 0$.)

11. $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer

    [If n is even, we assume that $\lim_{x\to a} f(x) > 0$.]

Root Law

2.3 Continuous Functions: Stewart 2.5

For functions like $f(x) = x^2$, $g(x) = \ln(x)$, $h(x) = e^x$, we can draw the graph on a coordinate plane without lifting our pencils. This intuitively means that the function is **continuous**.

In other words, there are no "breaks" in the curve of the graph.

Examples of **discontinuous** functions are $\frac{1}{x}, \frac{1}{x^2}$ and piecewise defined functions like

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x \geq 1 \end{cases}$$

Perhaps more subtle is the example $f(x) = \frac{x^2-1}{x-1}$: we can draw the graph of this function without lifting our pencil until we get to $(1, 2)$.

We can make this concept precise by defining **continuity** as follows: a function is continuous at a if

$$\lim_{x \to a} f(x) = f(a)$$

This definition implicitly requires three things if f is continuous at a:

- $f(a)$ is defined.
- $\lim_{x \to a} f(x)$ exists.
- $\lim_{x \to a} f(x) = f(a)$.

A continuous function f has the property that a small change in x produces only a small change in f.

Definition: If f is defined near a, we say that f is **discontinuous** at a if f is not continuous at a.

Example: For what values of b is the function

$$g(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if } x \neq 2 \\ b & \text{if } x = 2 \end{cases}$$

continuous at $x = 2$?

The function $f(x) = \frac{x^2-x-2}{x-2}$, like $\frac{x^2-1}{x-1}$, exhibits what is known as a **removable discontinuity**. A removable discontinuity of a function f at $x = a$ disappears with an appropriate definition of $f(a)$.

Example: Does the Heaviside function

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

have a removable discontinuity?

The Heaviside function exhibits what is known as a **jump discontinuity**: the function jumps from a value of 0 to a value of 1 as we pass through $x = 0$ from left to right.

Functions like $\ln(x)$ and $\frac{1}{x^2}$ exhibit **infinite discontinuities** since their graphs shoot off to $\pm\infty$. Points of infinite discontinuity are sometimes also called **singularities**.

Just as we defined one-sided limits, we also have a notion of **one-sided continuity**:

Definition: A function f is continuous from the right (left) at a number a if

$$\lim_{x \to a^+} f(x) = f(a) \quad \left(\lim_{x \to a^-} f(x) = f(a) \right) \tag{2.3.1}$$

The Heaviside function is a perfect example of being both continuous from the left and the right. Yet, it is not continuous at $x = 0$, though it is continuous everywhere else.

Definition: A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on side of an endpoint of the interval, we understand *continuous* at the endpoint to mean continuous from the right or continuous from the left.)

Knowing that a function is continuous at $x = a$ makes evaluating $\lim_{x \to a} f(x)$ trivial. Also, knowing which functions are continuous can help us determine where composite functions (functions made up of more basic ones) are continuous. For these reasons, the following theorems (which we won't prove) are invaluable:

Theorem: If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

$$f+g\ ,\ f-g\ ,\ cf\ ,\ fg\ ,\ \frac{f}{g}\ \text{ if } g(a)\neq 0$$

We can also prove

Theorem: If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x\to a} f(g(x)) = f\big(\lim_{x\to a} g(x)\big)$$

This theorem says that a limit symbol can be moved through a function symbol if the function is continuous and the limit exists. In other words, the order of these two symbols can be reversed.

Theorem: If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

Roughly, "a continuous function of a continuous function is a continuous function."

Theorem: The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions

- exponential functions
- logarithmic functions

Example: Show that $g(x) = 2\sqrt{3-x}$ is continuous on $(-\infty, 3]$.

Example: Where is the function

$$f(x) = \begin{cases} e^x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

continuous/discontinuous? What kind of discontinuity (if any) does it possess?

Example: Where is the function $f(x) = \frac{x^3-8}{x^2-4}$ continuous/discontinuous? What kind of discontinuity (if any) does it possess? Can we make this function continuous?

Example: Where is the function $f(x) = \frac{\ln(x)+\tan^{-1}(x)}{x^2-1}$ continuous?

Example: Explain why the function

$$f(x) = \begin{cases} \frac{x^2-x}{x^2-1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

is discontinuous. Can you make an adjustment so that the function is continuous?

Example: For what value of c is the function

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

continuous?

Important Summary: By constructing functions out of the precalculus functions using arithmetic operations and composition we can create sophisticated functions from basic building blocks that are continuous everywhere on their domains.

Example: Evaluate the limit

$$\lim_{x\to 4} \frac{5+\sqrt{x}}{\sqrt{5+x}}$$

Example: Evaluate the limit

$$\lim_{x\to \pi} \sin(x+\sin(x))$$

Now, we can tackle the examples we mentioned before. Additionally, consider

Example: Compute the limit

$$\lim_{x\to \frac{\pi}{2}} \ln(1+\cos(x))$$

Before I show you the general strategy for computing $\lim_{x\to a} f(x)$, I present a very important theorem: the **intermediate value theorem**. This theorem is used to prove a variety of calculus theorems and can also help in finding zeros of functions as we'll see when we discuss Newton's Method.

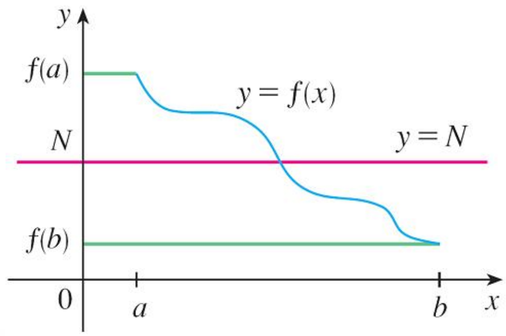
Theorem: Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

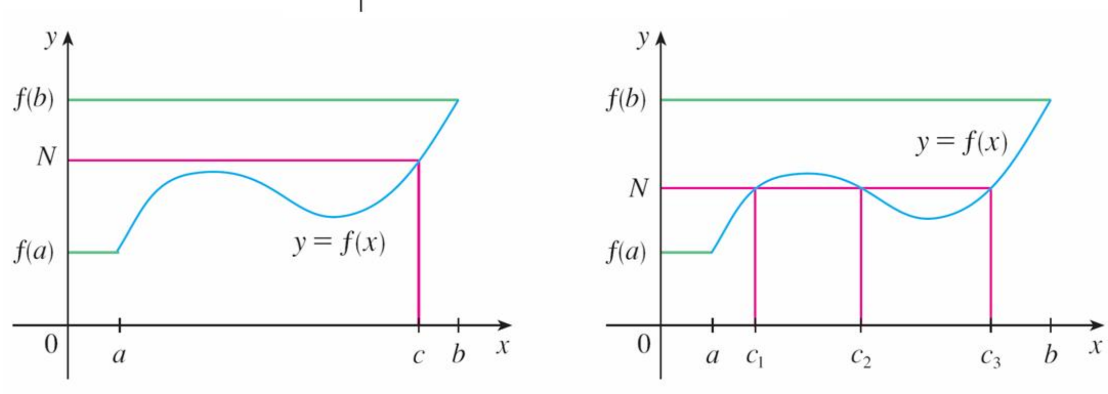
The Intermediate Value Theorem states that a continuous function takes on every intermediate value between $f(a)$ and $f(b)$. If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the Intermediate Value Theorem is true.

Example: Suppose f is continuous on $[1, 5]$ and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$, explain why $f(3) > 6$.

Example: Use the Intermediate Value Theorem to show that there is a root of the equation

$$x^4 + x - 3 = 0 \text{ in } (1, 2).$$





Example: Prove that the equation has at least one real root

$$\cos(x) = x^3.$$

The intermediate value theorem can also be exploited to develop a simple algorithm for approximating the roots whose existence we inferred above. The idea is to start with an interval known to contain the root and then halve it: another application of the IVT tells us which interval contains the root. Continuing in this way, we can achieve a fairly good approximation for the root which cannot be calculated otherwise in a relatively small number of iterations.

A few more comments regarding continuity:

1. The property of continuity allows us to prove a variety of useful theorems for functions that are continuous. But this does not necessarily make continuous

functions useful from a practical viewpoint. As it turns out many functions used to model physical reality are in fact continuous. Gravitational potential for instance is a continuous quantity in physical mechanics. It is possible to compute the gravitational attraction law both inside and outside the surface of a planet like the Earth and continuity demands that we match these solutions at the surface. Matching two functions to enforce continuity is a common problem in solving differential equations that model some physical aspect of our world.

2. Discontinuous functions also play a role. The Heaviside function comes up frequently in engineering applications. The simplest application of the Heaviside function is to model a switch in an electrical circuit.

3. Discontinuities also arise in fluid mechanics: sonic booms are shockwaves which in turn are discontinuities in the pressure, temperature and density associated with the atmosphere as a plane moves through it faster than the speed of sound.

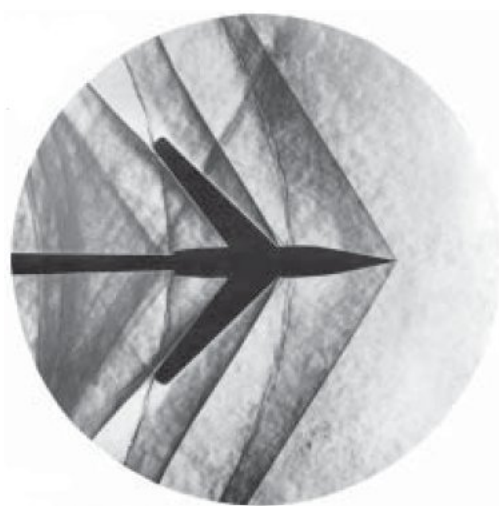


Figure 2.1: Photo courtesy of wikipedia.org

2.4 Direct Substitution and Evaluation of Limits

At long last we can state and use the Direct Substitution Strategy to compute many limits of interest. This direct substitution rule is equivalent to the collection of limit laws above and may be used as an alternative starting point for limit calculations. The direct substitution strategy is founded on the observation that a limit depends only what a function does near the limit point a and does not care what happens *at* a. We will use our algebra skills to construct a function $g(x)$ from $f(x)$ that is continuous at a and equal to $f(x)$ near a. Using continuity, we can compute $\lim_{x\to a} g(x)$ and therefore infer the limit of interest $\lim_{x\to a} f(x)$.

For instance, let's consider

$$\lim_{x\to 1} \frac{x^2-1}{x-1}$$

If we try direct substitution, we find $\frac{0}{0}$. This has no mathematical meaning since a is not in the domain of $f(x)$.

Clearly, 1 is not an element of the domain (look at the denominator). In cases like this, we remember that the limit $\lim_{x\to a} f(x)$ depends on the behavior of f *near* $x = a$ and doesn't care what happens to f at a.

This leads us to the very useful fact:

Fact: If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ provided the limits exist.

We can use this fact to evaluate the limit since

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1\ ,\ \ x \neq 1$$

Therefore,

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 2$$

Example: Find $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$.

Remember this example? We used this in our cautionary tale about using calculators to guess limits. The trick to evaluating this limit is to find a $g(t)$ which equals $\frac{\sqrt{t^2+9}-3}{t^2}$ for $t \neq 0$ and for which we can actually compute the limit.

Remember how we dealt with radicals in precalc: we can multiply and divide by the **conjugate** of the numerator which amounts to putting a minus sign in front of the radical.

$$\frac{\sqrt{t^2+9}-3}{t^2} = \frac{\sqrt{t^2+9}-3}{t^2} \times \frac{-\sqrt{t^2+9}-3}{-\sqrt{t^2+9}-3} = \frac{1}{\sqrt{t^2+9}+3}$$

Example: Evaluate the limit, if it exists

$$\lim_{h\to 0} \frac{(2+h)^3-8}{h}$$

Example: Evaluate the limit, if it exists

$$\lim_{t\to -3} \frac{(t^2-9)}{2t^2+7t+3}$$

HINT: $2t^2+7t+3=(3+t)(1+2t)$.

Example: Evaluate the limit, if it exists

$$\lim_{t\to 1} \frac{t^4-1}{t^3-1}$$

Example: Evaluate the limit, if it exists

$$\lim_{t\to 0} \frac{\sqrt{1+t}-\sqrt{1-t}}{t}$$

Example: Evaluate the limit, if it exists

$$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)$$

Example: Evaluate the limit, if it exists

$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)$$

Example: Evaluate the limit, if it exists

$$\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$

Sometimes, it helps to compute the left-hand and right-hand limits first. This is usually helpful when your function involves an absolute value or when the function is piecewise defined.

Remember, $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$.

Example: Prove that $\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

Example: For the function $g(x)$ given by

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

evaluate each of the following:

$\lim_{x\to 1^-} g(x)$, $\lim_{x\to 1} g(x)$, $\lim_{x\to 2^-} g(x)$, $\lim_{x\to 2^+} g(x)$, $\lim_{x\to 2} g(x)$
and sketch the graph of g.

To finish off this section, we add two more theorems to our arsenal:

2 Theorem If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

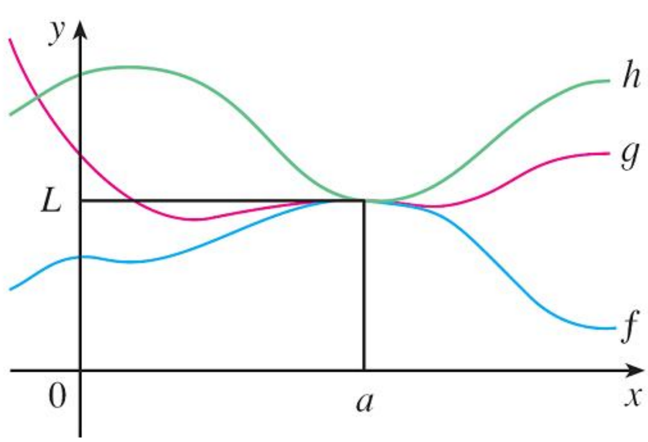
3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then $$\lim_{x \to a} g(x) = L$$

The **Squeeze theorem** is especially useful when we are dealing with the trigonometric functions sin and cos since they are both bounded between $\pm$.

To visualize the content of the theorem and get a sense of where it's name comes from, consider the plot:



Example: Prove that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ using the Squeeze Theorem.

HINT: For x very close to 0, but not equal to 0, we have $\cos(x) \leq \frac{\sin(x)}{x} \leq 1$. It would be challenging to prove this now, but you can convince yourself by graphing these functions.

Example: If

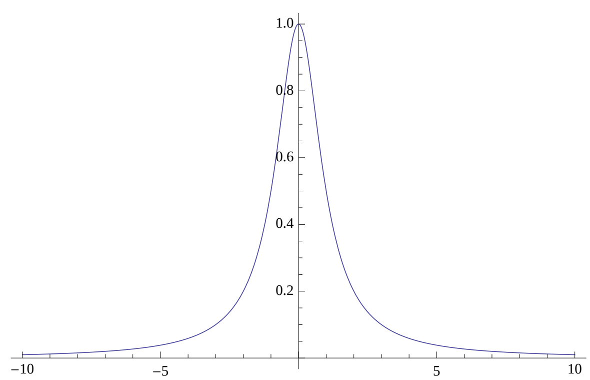
$$f(x) = \begin{cases} x^2 & \text{if x is rational} \\ 0 & \text{if x is irrational} \end{cases}$$

prove that $\lim_{x \to 0} f(x) = 0$.

2.5 Limits at Infinity: Stewart 2.6

So far, we've only considered limits at *finite* points a. To have a more complete understanding of how functions defined on intervals of the form $(-\infty, b]$, $[b, \infty)$, etc. behave, we need to understand how to take **limits at infinity**. In other words, we need to know what f does as x goes to either $\pm\infty$.

Let's first take a look at the function $f(x) = \frac{1}{1+x^2}$.



As we move farther and farther away from the origin in either direction, the graph becomes closer and closer to 0 (though it never dips below the y-axis).

This is easy to understand from the formula: as x becomes larger in magnitude, x^2 becomes much larger: 1 divided by a very large number is a very small number.

We can express this mathematically as

$$\lim_{x\to\infty} \frac{1}{1+x^2} = 0 \ , \ \lim_{x\to-\infty} \frac{1}{1+x^2} = 0$$

In general, we use the notation

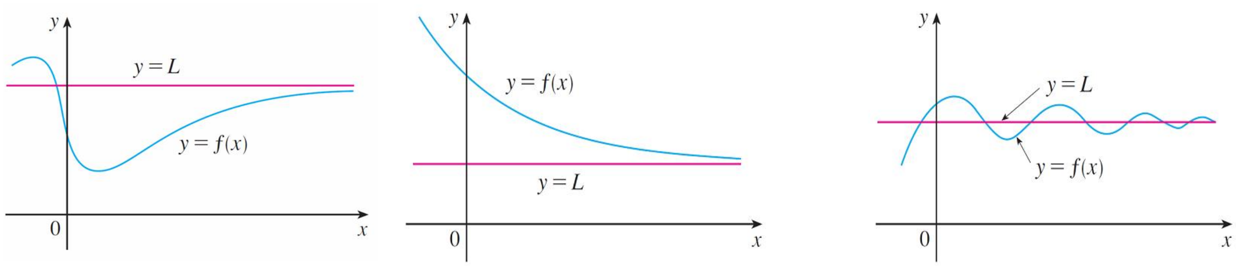
$$\lim_{x\to\infty} f(x) = L$$

to indicate that the values of $f(x)$ approach L as x becomes larger and larger.

If we let x decrease through negative values without bound, we use the notation

$$\lim_{x\to-\infty} f(x) = L$$

to indicate that the values of f approach L.



Definition: Let f be a function defined on some interval $(-\infty, a)$ (or (a, ∞)). Then

$$\lim_{x\to-\infty} f(x) = L \ , \ (\text{or } \lim_{x\to\infty} f(x) = L)$$

means that the values of f can be made arbitrarily close to L by taking x sufficiently large and negative (positive).

Just as we had vertical asymptotes, we can define **horizontal asymptotes**:

Definition: The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to-\infty} f(x) = L \text{ or } \lim_{x\to\infty} f(x) = L$$

As we've seen, the function $f(x) = \frac{1}{1+x^2}$ has a horizontal asymptote $y = 0$.

Example: $f(x) = \tan^{-1}(x)$ has two horizontal asymptotes, $y = \pm\frac{\pi}{2}$.

Example: $f(x) = e^x$ has $y = 0$ as a horizontal asymptote.

In general, we have the following theorem:

Theorem: If $r > 0$ is a rational number, then $\lim_{x\to\infty} \frac{1}{x^r} = 0$. If $r > 0$ is a rational number such that x^r is defined for all x, then $\lim_{x\to-\infty} \frac{1}{x^r} = 0$.

This basically tells us that if you make x large in either direction (assuming x^r is even defined), then x^r becomes large and the reciprocal becomes small.

Example: Evaluate $\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1}$.

Example: Find the vertical and horizontal asymptotes of $f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$.

Example: Find the limit or show that it doesn't exist

$$\lim_{x\to-\infty} \frac{4x^3+6x^2-2}{2x^3-4x+5}$$

Notice in these examples that we are essentially looking at the highest "power" of x occurring in the numerator and denominator. We then see which of these two powers dominates the other or if they are about as powerful.

This sort of viewpoint doesn't work when there are *differences* of functions which have about equal power as x tends towards infinity:

Example: Compute the limit or show that it doesn't exist

$$\lim_{x\to\infty} \sqrt{x^2+ax} - \sqrt{x^2+bx}$$

ANS: you may be tempted to argue that this must be 0 since the dominating power of both pieces is x and they cancel. This is incorrect as the lmit is $\frac{a-b}{2}$.

Example: Compute $\lim_{x\to\infty} \frac{\sin(x)}{x}$.
HINT: use the Squeeze Theorem.

Sometimes, a variable redefinition really helps in evaluating a limit:

Example: Compute $\lim_{x\to 0^+} \tan^{-1}(\ln(x))$.

Notation: The notation $\lim_{x\to\infty} f(x) = \infty$ is used to indicate that the values of $f(x)$ become larger and larger as x becomes larger and larger. Similar meanings are attached to the following symbols:

$$\lim_{x\to-\infty} f(x) = \infty \ , \ \lim_{x\to-\infty} f(x) = -\infty \ , \ \lim_{x\to\infty} f(x) = -\infty$$

We must be careful in defining **infinite limits at infinity** this way. We are not allowed to use the limit laws! For instance,

Example: Find $\lim_{x\to\infty}(x^2 - x)$.

Example: Find $\lim_{x\to\pm\infty} 2x^3 - x^4$.

I conclude this section just by mentioning that limits at infinity have a precise definition like their finite counterparts, as you would expect.

Definition: Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that for every $\epsilon > 0$ there is a corresponding number N such that if $x > N$ then $|f(x) - L| < \epsilon$.

A similar definition holds for limits of the form $\lim_{x\to-\infty} f(x)$. I refer you to the text for the precise definition of an infinite limit at infinity.

You may wonder what use limits at infinity have. There are many applications that you will encounter should you study higher mathematics and its applications. Here are a few:

1. Determining the long-range behavior of potentials in physics and chemistry. The long-range behavior will determine if your particle/system is bound or can "escape to infinity".

2. Determining the long-term behavior of systems depending on time like predator-prey models in biology and financial models.

3. Determining power law behavior like in the example of two oppositely charged particles separated by a fixed distance defining something called a dipole.

More examples (time permitting):

Compute the limit or show that it does not exist.

1.
$$\lim_{x\to -3} \frac{x^3+7x^2+15x+9}{x^2-9} = 0$$

2.
$$\lim_{x\to 1} \frac{2x^2-3x+1}{x^2+2x-3} = \frac{1}{4}$$

3.
$$\lim_{x\to 2} \frac{x^2-4}{x^3-8} = \frac{1}{3}$$

4.
$$\lim_{x\to 3} \frac{x^2-9}{x^2-2x-3} = \frac{3}{2}$$

5.
$$\lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2} = -\frac{5}{54}$$

6.
$$\lim_{x\to 2} \frac{\sqrt{4x+1}-3}{x-2} = \frac{2}{3}$$

7.
$$\lim_{t\to 0} \frac{\sqrt{4+t}-\sqrt{4-t}}{2t} = \frac{1}{4}$$

8.
$$\lim_{x\to 16} \frac{4-\sqrt{x}}{16x-x^2} = \frac{1}{128}$$

9.
$$\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{1}{2\sqrt{x}}$$

10.
$$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right) = 0$$

11.
$$\lim_{x\to 0^-} \left(\frac{1}{x} - \frac{1}{|x|}\right) = -\infty$$

12.
$$\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right) = 0$$

13.
$$\lim_{x\to 0^+} \sqrt{x}\; e^{\sin\left(\frac{\pi}{x}\right)} = 0$$

14.
$$\lim_{x\to -2} \frac{2-|x|}{2+x} = 1$$

15. Verify that a possible choice of δ for showing that $\lim_{x\to 3} x^2 = 9$ is $\delta = \min\{2, \frac{\epsilon}{8}\}$.

16. Find a number δ such that if $|x-2| < \delta$, then $|4x-8| < \epsilon$ where $\epsilon = 0.1, 0.01$.

Chapter 3

Derivatives

3.1 Tangent Problem Revisited: Stewart 2.7

When we discussed the tangent problem, we computed the slope of the tangent line to any point on the parabola $y = x^2$. The process we used was general:

Definition: The **tangent line** to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that the limit exists.

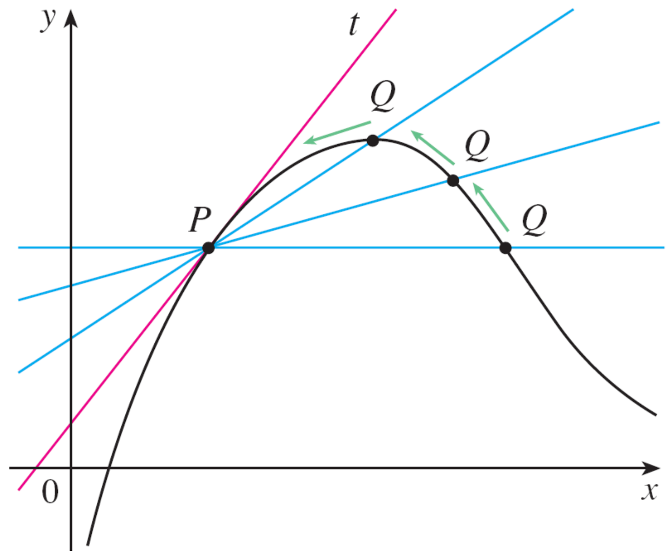
An alternative definition of the slope (which is precisely the one we used before) is

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{3.1.1}$$

Now that we know how to compute limits, let's work on a few examples where we compute the slope of the tangent line to a given $y = f(x)$.

Example: Compute the tangent line to the graph of $y = \frac{1}{x^2}$ at the point $(-1, 1)$.

Example: Compute the tangent line to the graph of $y = \frac{1}{\sqrt{x}}$ at the point $(1, 1)$.



Example: Compute the tangent line to the graph of $y = \frac{1-2x}{3+x}$ at the point $(0, \frac{1}{3})$.

The limit $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ occurs so frequently when we want to compute the **instantaneous rate of change** of a function that it is given its own special name.

Definition: The **derivative of a function f at a point a** , denoted by $f'(a)$ is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \tag{3.1.2}$$

if this limit exists.

Alternatively, when we want to be explicit about the independent variable, we also write the derivative as $\frac{df}{dx}$ or $\frac{dy}{dx}$ depending on the context. This notation is referred to as Leibniz notation after Gottfried Leibniz. The prime notation is due to Lagrange.

The Leibniz notation is useful to remember the meaning of the derivative: it measures the **instantaneous rate of change** of f at a given point. So if $x(t)$ represents the position of an object as it moves along one spatial dimension, the derivative $x'(a)$ represents the instantaneous rate of change at time $t = a$. In other words, if the object were a car, $x'(a)$

would be the reading of the speedometer at time $t = a$.

We usually denote average rates of change of a function over an interval $[a, b]$ as $\frac{\triangle y}{\triangle x}$ where $\triangle x = b - a$ and $\triangle y = f(b) - f(a)$. The "d" s appearing in the Leibniz notation of a derivative remind us that we are dealing with an instantaneous rate of change $\frac{dy}{dx}$ which can be thought of as $\lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x}$.

When we want to specify the point at which we are evaluating a derivative using Leibniz notation, we write

$$\frac{df}{dx}\Big|_{x=a} \quad \text{or} \quad \frac{dy}{dx}\Big|_{x=a} \tag{3.1.3}$$

Example: Compute the derivative of $f(x) = \sqrt{1 - 2x}$ at a where $a \leq \frac{1}{2}$

Example: Compute the derivative of $f(x) = 2x^3 - x$ at a

Note: The unit of a derivative is the unit of y divided by the unit of x. So if we are looking at the derivative of $x(t)$ with respect to time, the units will be length per time which is precisely the unit for velocity!

3.2 Derivatives and Differentiability: Stewart 2.8

In the previous section, we use the derivative to compute tangent lines to curves at specified points $(a, f(a))$. If we leave a arbitrary, the derivative $f'(a)$ becomes a function. If we allow a to be anything, we switch back to using x as the independent variable and write

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{3.2.1}$$

Given any x where the limit exists, we assign to x the number $f'(x)$. So we regard f' as a new function.

Properties of the graph of $y = f(x)$ can be obtained from the graph of $y = f'(x)$ as we illustrate with some examples:

Example: If $y = \sqrt{x}$, find the derivative and compare the graphs of f and f'.

Example: If $y = x^3$, find the derivative and compare the graphs of f and f'.

Just as we defined a function f to be continuous if $\lim_{x\to a} f(x) = f(a)$ for all a in the domain of f, we define a function to be **differentiable** at $x = a$ as follows:

Definition: A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a,b) if it is differentiable at every number in the interval. Here, a could be $-\infty$ and b could be ∞.

It turns out that not every function is differentiable in its domain. Consider the absolute value function $y = |x|$:

Example: Show that $y = |x|$ is not differentiable at $x = 0$.

So while $y = |x|$ is everywhere continuous, it is NOT everywhere differentiable. You may wonder if continuity and differentiability have anything to do with each other. The answer is: yes!

Theorem: If f is differentiable at a, then f is continuous at a.

Please note what this theorem does NOT say. It does not say that if a function is continuous at a, then it is differentiable at a. The absolute value function is a counterexample to this statement.

Intuitively, $|x|$ does not have a derivative at $x = 0$ because there is a sharp "corner" in the graph. Whenever there is a corner or a kink like in the graph of $y = x^{\frac{2}{3}}$, the derivative fails to exist.

Another way for a function to not have a derivative at $x = a$ is if f is not continuous at $x = a$. Remember the Heaviside step function!

Another possibility is that the graph of the function has a so-called vertical tangent. In other words, the function is continuous at a but $\lim_{x\to a} |f'(x)| = \infty$. This means the tangent becomes steeper and steeper as x approaches a. Remember $y = \sqrt{x}$ and the function describing the upper half of the unit circle $f(x) = \sqrt{1 - x^2}$.

Now that we are comfortable with thinking about f' as a function, it is possible that f' itself has a derivative. When the derivative of the derivative exists, we call it the **second derivative** and denote it by f''. If we continue to take more derivatives, we call the process **finding the higher order derivatives of f** .

Clearly, if we are looking for higher order derivatives, the prime notation is inconvenient. Sometimes we denote the n-th order derivative of f (if it exists) by $f^{(n)}$ where the n is supposed to stand for n primes. In this regard, Leibniz notation is more useful since the n-th order derivative is written as

$$\frac{d^n f}{dx^n} \tag{3.2.2}$$

Higher order derivatives are most often found in physics and engineering. If the velocity $v(t)$ is the derivative of $x(t)$, then the acceleration $a(t)$ is the second derivative of the position with respect to time.

When you get to Calculus II, you'll learn about Taylor series which are very useful in approximating functions. Taylor series are defined in terms of higher order derivatives of f.

Example: Compute all higher order derivatives of $x^2 + x + 1$.

Application: The height (in meters) of a projectile shot vertically upward from a point $2m$ above ground level with an initial velocity of 24.5 m/s is

$$h(t) = 2 + 24.5t - 4.9t^2$$

Find the velocity as a function of time. Find the acceleration as a function of time.

Application: The mass of the part of a metal rod that lies between its left end and a point x meters to the right is $3x^2$ kg. Find the linear density as a function of x.

Application: A spherical balloon is being inflated. Find the rate of increase of the surface area with respect to the radius.

Application: Consider populations of tundra wolves, given by $W(t)$, and caribou, given by $C(t)$, in northern Canada. The interaction between the two populations is modeled by

$$\frac{dC}{dt} = aC - bCW \ , \ \frac{dW}{dt} = -cW + dCW$$

where a, b, c, d are constants.

When is there an equilibrium between the two species? If the caribou go extinct, what happens to the wolves?

3.3 Derivatives of Polynomials and Exponentials: Stewart 3.1

We've spent a few sections defining and understanding the meaning of derivatives. Now it's time to start adding some tools to our toolbox: we want easy methods for computing derivatives.

From the examples we've seen, using the definition of a derivative to compute a derivative can be time consuming. Let's learn how to handle constant functions, power functions, polynomials and exponential functions.

First, the easiest function I can think of is $f(x) = c$. Using the definition of the derivative, it is very easy to see that $f' = 0$.

Now, how about power functions? First, let's ask what is the derivative of x^n when n is a positive integer?

For $n = 1$, the graph is a straight line through the origin of unit slope. So based on the graph, we conclude

$$\frac{d}{dx}(x) = 1$$

We can also use the exact definition of the derivative to prove this statement.

Unfortunately, the derivative of x^n cannot be calculated so easily. To compute this derivative, we need use of the Binomial Theorem (which also appears on the extra credit assignment, by the way!):

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \tag{3.3.1}$$

where

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \tag{3.3.2}$$

and $j! = j \times (j-1) \times (j-2) \ldots 2 \times 1$ is the factorial of j.

If we use the definition of the derivative to compute $\frac{d}{dx}x^n$, we see immediately why we need the Binomial theorem:

$$\frac{d}{dx}x^n = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}.$$

Using the Binomial Theorem, the only term that survives the limiting process is nx^{n-1}. Therefore, we have shown that

$$\frac{d}{dx}x^n = nx^{n-1}$$

for $n = 1, 2, 3, \ldots$. In the book, this result is referred to as the **Power Rule**.

We don't have the tools yet, but it is possible to show that the Power Rule extends to all real numbers:

$$\frac{d}{dx}x^a = ax^{a-1} \ , \ a \in \mathbb{R}.$$

The Power Rule enables us to find tangent lines without having to resort to the definition of a derivative.

Example: Compute the derivative of $g(x) = \frac{3}{4}x^8$ (assuming it exists)

Example: Compute the first and second derivatives of $g(x) = x^{\frac{2}{3}}$ wherever they exist.

When new functions are formed from old functions by addition, subtraction, or multiplication by a constant, their derivatives can be calculated in terms of derivatives of the old functions.

Theorem: If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x) \tag{3.3.3}$$

So, we can bring constants out of derivatives.

We can also show

Theorem (Sum Rule): If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \tag{3.3.4}$$

In words, *the derivative of a sum is the sum of the derivatives.*

Example: Compute the derivative of $g(x) = x^2(1 - 2x)$.

Example: Compute the derivative of $g(x) = \frac{\sqrt{x}+x}{x^2}$.

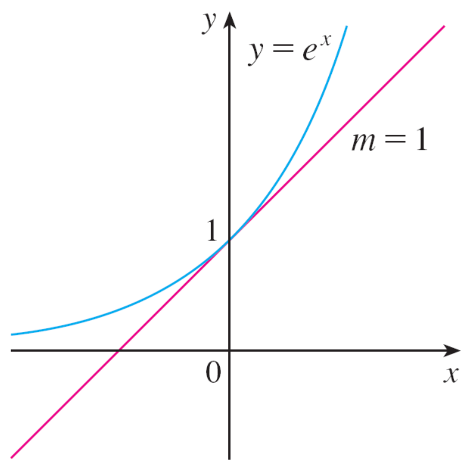
Example: Compute the derivative of $g(x) = (x - 2)(2x + 3)$.

So, combining the Power Rule and the Sum Rule, we are able to compute the derivatives of any polynomial.

How about exponential functions? First, we need to introduce a new definition for e. Remember, when we introduced e, I claimed that

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n. \tag{3.3.5}$$

That was perfectly good for our purposes. It turns out that e has an alternative definition: In calculus, we define e to the unique base such the the slope of the tangent line to $y = e^x$ at $x = 0$ is exactly 1.



So, in other words, e is the (unique) number such that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

From this, we can show (**show this!**) that

$$\frac{d}{dx}e^x = e^x \tag{3.3.6}$$

How about $f(x) = a^x$ for $a > 0$? We can use the fact that $a = e^{\ln(a)}$ to write $f(x) = e^{x\ln(a)}$. Therefore,

$$\frac{d}{dx}a^x = \ln(a)a^x \tag{3.3.7}$$

Example: Find the equation of the tangent line to the curve $y = x^4 + 2e^x$ at the point $(0, 2)$.

3.4 Product and Quotient Rules: Stewart 3.2

In the previous section, we established the sum rule which allowed us to compute derivatives of sums of power functions and exponentials. If only life were so simple!

There are functions which are written as products and quotients of simpler functions we can differentiate. How do we handle such functions?

The Product Rule: If f and g are differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Example: If $f(x) = xe^{-x}$, compute $f^{(n)}(x)$.

Example: Compute the derivative of $f(x) = (1 - e^x)(x + e^x)$.

Just as we have a rule for taking derivatives of a product of two functions, we also have a rule for differentiating quotients:

The Quotient Rule: If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Example: Compute the derivative of $f(x) = \frac{x^3}{1-x^2}$

Example: Compute the derivative of $f(x) = \frac{x}{(x-1)^2}$

Example: Compute the derivative of $f(x) = \frac{e^x}{x}$

Example: Show that $\frac{d}{dx}\left(x^{-n}\right) = -nx^{-n-1}$.

3.5 Derivatives of Trig Functions: Stewart 3.3

In this section, we will compute the derivatives of the trigonometric functions. To begin, we compute the derivatives of sine and cosine.

In order to calculate these derivatives, we will need the following limits:

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1 \ , \ \lim_{h\to 0} \frac{\cos(h) - 1}{h} = 0$$

The limit $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ can be obtained using the Squeeze Theorem (proved this already). The limit involving cosine can be obtained from the first by multiplying and dividing by $1 + \cos(h)$ and using the identity $\sin^2 + \cos^2 = 1$.

Here's how we may calculate the derivatives of sine and cosine:

If $f(x) = \sin(x)$, then

$$\frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin(x)}{h}$$

Recall that we may simplify $\sin(x+h)$ by using the Sum Formula $\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$:

$$\sin(x+h) = \sin(x)\cos(h) + \sin(h)\cos(x)$$

Hence,

$$\frac{\sin(x+h) - \sin(x)}{h} = \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

and we may conclude that $\frac{d}{dx}\sin(x) = \cos(x)$.
Using the same methods, we can deduce that $\frac{d}{dx}\cos(x) = -\sin(x)$. (Use $\cos(A+B) =$

$\cos(A)\cos(B) - \sin(A)\sin(B))$.

We may use the quotient rule to obtain $\frac{d}{dx}\tan(x) = \sec^2(x)$. In general, as long as x is measured in radians, we have

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Example: Compute the derivative of $f(x) = \frac{x}{2-\tan(x)}$.

Example: Compute the derivative of $f(x) = \cos(x)\sin(x)$.

3.6 The Chain Rule: Stewart 3.4

So far, we've considered derivatives of relatively simple functions. Often times we will be confronted with functions whose derivatives are not easily computed using the limit of the difference quotient. How can we handle complex derivatives?

The answer is to break the function up into more manageable pieces via function decomposition. Therefore, it is important to know how to compute the derivative of $f \circ g$ in terms of the derivatives of f and g.

Theorem (The Chain Rule):
If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x))g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Using Leibniz notation we can give a brief heuristic argument as to why the chain rule works:

$$\frac{dy}{dx} = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x} = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle u}\frac{\triangle u}{\triangle x} = \frac{dy}{du}\frac{du}{dx}.$$

Example: Compute the derivative of $f(x) = \sqrt{x^2 + 1}$.

Example: Compute the derivative of $f(t) = e^{-t}\cos(4t)$.

The chain rule is particularly useful when you have to take the derivative of a function like $[g(x)]^n$:

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$$

Another application of the chain rule is to compute the derivative of a^x for $a > 0$.

We know that $\frac{d}{dx}e^x = e^x$, so we can use this and the chain rule to prove that:

$$\frac{d}{dx}a^x = a^x \ln(a)$$

(Prove this using $a^x = e^{\ln(a)x}$.)

More examples (time permitting)

Compute the derivative using the Chain Rule.

1.
$$f(x) = \sin(x^2)$$

2.
$$f(x) = \left(x^4 + 3x^2 - 2\right)^5$$

3.
$$f(x) = (2x-3)^4(x^2+x+1)^5$$

4.
$$f(t) = (t+1)^{\frac{2}{3}}(2t^2-1)^3$$

5.
$$f(x) = \sqrt{1+2e^{3x}}$$

6.
$$f(x) = 10^{1-x^2}$$

7.
$$f(x) = x^2 e^{-\frac{1}{x}}$$

8.
$$f(x) = \sin(\sin(\sin(x)))$$

9.
$$f(x) = \sqrt{x+\sqrt{x+\sqrt{x}}}$$

10.
$$f(x) = \sin^2(e^{\sin^2(x)})$$

3.7 Implicit Differentiation and Derivatives of Logarithms: Stewart 3.5/3.6

So far, we've only considered derivatives of functions when they can be written as $y = f(x)$. Sometimes we are interested in computing tangent lines to curves which are defined **implicitly** by a relation between x and y.

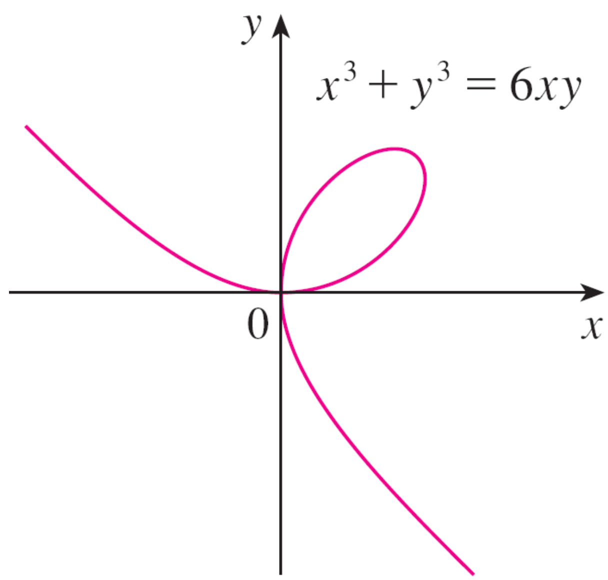
For instance, suppose we wish to compute the line tangent to the unit circle at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. The unit circle does not represent the graph of a function. However, we can use the vertical line test to amputate the lower half and solve the defining relation $x^2 + y^2 = 1$ to get

$f(x) = \sqrt{1-x^2}$. The usual machinery of calculus applies.

There are many cases where a curve is defined by an implicit relationship between the variables x and y which cannot be solved (even locally) for $y = f(x)$.

For example, consider the **folium of Descartes**:

$$x^3 + y^3 = 6xy \tag{3.7.1}$$



The folium of Descartes

The point $(3, 3)$ is on this curve: suppose we want the tangent line to the curve at this point. How can we obtain the slope of the tangent line if we can't even solve for y in terms of x near $(3, 3)$?

The answer is **implicit differentiation**. First, let's see how this works for the unit circle and then we'll apply the technique to folium of Descartes.

If we wish to compute the tangent line to the unit circle at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, we could write

$y = \sqrt{1-x^2}$ and crank out the derivative: $y' = \frac{-x}{\sqrt{1-x^2}}$. At $x = \frac{1}{\sqrt{2}}$ we have $y' = -1$.

Now let's try implicit differentiation. In this method, we don't solve y in terms of x. Rather, we start with the implicit defining relation. In this case, that relation is $x^2 + y^2 = 1$. Now, take the derivative of both sides of this equation:

$$\frac{d}{dx}\left(x^2 + y^2\right) = 2x + 2yy' = 0$$

If we plug in $x = y = \frac{1}{\sqrt{2}}$, we find that $y' = -1$ without the need to express y as some function of x!

This is even more useful for the folium:

$$\frac{d}{dx}\left(x^3 + y^3\right) = 3x^2 + 3y^2y' = \frac{d}{dx}(6xy) = 6\left(y + xy'\right)$$

If we plug in $x = y = 3$, we have $27 + 27y' = 18 + 18y'$ or $y' = -1$.

Example: Use implicit differentiation to find the equation of the tangent line to the curve at the given point:

$$y\sin(2x) = x\cos(2y)\ ,\ (\frac{\pi}{2}, \frac{\pi}{4})$$

Example: Use implicit differentiation to find the equation of the tangent line to the curve at the given point:

$$x^2 + 2xy - y^2 + x = 2\ ,\ (1, 2)$$

Example: Find y'' if

$$x^4 + y^4 = 16$$

Example: Find y'' at the point $x = 0$ by implicit differentiation.

$$xy + e^y = e$$

More importantly for our purposes, we can use implicit differentiation to obtain the derivatives of both the inverse trigonometric functions and the logarithmic functions.

The reason implicit differentiation is useful for computing the derivatives of these functions is simple: these functions are defined to be inverses of functions with derivatives we already know. Using $f \circ f^{-1} = x$, we can easily obtain the derivatives of the inverse functions.

For example, suppose we want to compute the derivative of $\sin^{-1}(x)$. Remember that $y = \sin^{-1}(x)$ means that $\sin(y) = x$. Therefore,

$$\frac{d}{dx}\sin(y) = \cos(y)\frac{dy}{dx} = \frac{d}{dx}(x) = 1$$

It follows that

$$\frac{dy}{dx} = \frac{1}{\cos(y)} \tag{3.7.2}$$

To complete the calculation, we need to express $\cos(y)$ in terms of x alone. Now, $\cos(y) \geq 0$ since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Therefore, $\cos(y) = \sqrt{1 - \sin^2(y)} = \sqrt{1 - x^2}$ by definition. We conclude that

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}$$

Example: Compute the derivative of $\tan^{-1}(x)$.

In general, we can establish the following:

Example: Suppose f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Use implicit differentiation to show that

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is nonzero.

This allows us to quickly establish the following:

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

We can use this in turn to establish

$$\frac{d}{dx}\ln(|x|) = \frac{1}{x}$$

which will be very useful for us when we come to integration.

Knowing the derivative of the natural logarithm allows us to introduce a new method for calculating the derivatives of complicated functions: **logarithmic differentiation**.

The idea is to take the logarithm of the complicated function and use the properties of logarithms and the chain rule to simplify desired derivative.

Example: Differentiate $y = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$.

Example: Differentiate $y = \frac{e^{-x}\cos^2(x)}{x^2+x+1}$.

Example: Differentiate $y = \sqrt{x}e^{x^2-x}(x+1)^{\frac{3}{2}}$.

The steps in logarithmic differentiation are thus:

1. Take the natural logarithm of both sides of an equation $y = f(x)$ and use the the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

We can use logarithmic differentiation to prove the **power rule**. (Prove if there is time.)

3.8 Related Rates: Section 3.9

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity using an equation that relates the two and the chain rule.

Example: A water tank has the shape of an inverted circular cone with base radius 2 m and height $4m$. If water is being pumped into the tank at a rate of 2 cubic meters per minute, find the rate at which the water level is rising when the water is 3 m deep.
ANS: 0.28 m / min.

Example: The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?

Example: A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?
ANS: $\frac{25}{3}$

Example: A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200ft of string have been let out?

Example: The top of a ladder slides down a vertical wall at a rate of 0.15 m/s. At the moment when the bottom of the ladder is 3m from the wall, it slides away from the wall at a rate of 0.2m/s. How long is the ladder?

Extra problems (time permitting)

1. A water tank has the shape of an inverted circular cone with base radius 2 m and height $4m$. If water is being pumped into the tank at a rate of 2 cubic meters per minute, find the rate at which the water level is rising when the water is 3 m deep.

ANS: 0.28 m / min.

2. The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?

3. A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole? ANS: $\frac{25}{3}$

4. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200ft of string have been let out?

5. The top of a ladder slides down a vertical wall at a rate of 0.15 m/s. At the moment when the bottom of the ladder is 3m from the wall, it slides away from the wall at a rate of 0.2m/s. How long is the ladder?

6. A man walks along a straight path at a speed of 4 feet per second. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

7. The length of a rectangle is increasing at a rate of 8 cm per second and its width is increasing at a rate of 3 cm per second. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

8. A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point $(4, 2)$, its x-coordinate increase at a rate of 3 cm/s. How fast is the distance from the particle to the origin changing at this instant?

3.9 Linear and Quadratic Approximations: Stewart 3.10

Very close to a point $x = a$, a differentiable function is well approximated by its tangent line:

$$f(x) \approx f(a) + f'(a)(x - a).$$

We can use this fact to approximate functional values

Example: Approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

Example: Approximate the numbers $e^{-0.005}$, $(1.999)^4$ and $(1.1)^{\frac{1}{3}}$.

While pocket calculators are just fine for such computations, the power in linear approximations comes from the realization that there is no need to *stop* at linear order. We can go on to define quadratic, cubic, etc. approximations for our function. In general, the higher the order of the approximation, the better the accuracy (as you'll see in Calculus II).

In the linear approximation, we match a line of the form $c_0 + c_1(x-a)$ to $f(x)$ near $x = a$ by matching the functional values and their derivatives at this point. Going to quadratic order, we do the same but with quadratic function of the form $c_0 + c_1(x-a) + c_2(x-a)^2$. It turns out that there is no need to reinvent the wheel by finding c_0 and c_1 again; they have the same values $f(a)$ and $f'(a)$ as before. By matching the second derivatives at $x = a$, we find that $\frac{1}{2}f''(a) = c_2$.

So, to second order, $f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$ (assuming some mild conditions). This observation will be crucial to our understanding of the second derivative test of extrema found in chapter 4.

Another application of this linear approximation idea is the concept of **differentials**. Differentials are essentially very small changes in f due to very small changes in x. According to the linear approximation idea, if dx represents and incredibly small change in x, then

$$dy = f'(x)dx$$

Differentials are primarily used in engineering and physics to determine experimental error. Typically, you have some understanding of how much error is present in one part of your experiment (say the voltage readout on your voltmeter or the current readout on your ammeter) and you want a bound on the error on some dependent output. Differential provide a way of relating the two error bounds.

Example: The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the relative error in using the value of the radius to compute the volume of the sphere.

ANS: relative error in radius measurement = .24%, relative error in volume measurement = .7%

Chapter 4

Applications of the Derivative

4.1 Maxima and Minima: Stewart Section 4.1

Some of the most important applications of differential calculus are optimization problems: given a function which relates two quantities of interest, find the maximum/minimum values of $y = f(x)$ if they exist.

For instance, if we could measure the acceleration of an aircraft as a function of time, we would want to know the maximum acceleration experienced by the passengers for design purposes.

Another example would be the amount of (desirable/undesirable) chemical byproduct as a function of some reactant concentration.

Before we describe how calculus helps us find solutions to optimization problems, let's define what we mean by **maxima** and **minima**:

Definition: A function f has an absolute maximum (or global maximum) at c if $f(c) \geq f(x)$ for all $x \in D$. The number $f(c)$ is called the maximum value of f on D. Similarly, f has an absolute minimum at c if $f(c) \leq f(x)$ for all $x \in D$ and the number $f(c)$ is called the minimum value of f on D. The maximum and minimum values of f are called extreme values of f.

Often times a function will have several **local** maxima/minima:

Definition: A function f has a local maximum (or relative maximum) at c if $f(c) \geq f(x)$ for all x near c. Similarly, f has an local minimum at c if $f(c) \leq f(x)$ for x near c.

From the definition, global maxima/minima are by default local maxima/minima.

Example: $f(x) = \cos(x)$ has infinitely many maxima/minima. The absolute max and min values are ± 1.

Example: $f(x) = x^2$ has absolute minimal value of 0. There are no local maxima of x^2.

Example: $f(x) = x^3$ has no local maxima/minima.

There is a nice theorem which guarantees that a continuous function on a closed interval has extreme values:

Theorem (Extreme Value Theorem): If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Example: $f(x) = \frac{1}{x}$ has no extrema on any interval containing 0.

Example: $f(x) = x^3$ does have extrema when restricted to $[-1, 1]$ for example.

The theorem tells us when to expect extreme values, but it doesn't tell us how to find them:

Theorem: If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$. (**Prove this!**)

Note what this theorem is not saying: we can't assume $f'(c) = 0$ means that f has a maximum/minimum at $x = c$. Think of $y = x^3$. Also, functions can can have extreme values without being differentiable. Think of $y = |x|$.

At the very least, we can use $f'(x) = 0$ to start to look for extreme values if they exist. The theorem narrows down the search to just a few possibilities.

Definition: A **critical number** of a function is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example: Find the critical numbers of $f(x) = x^{-2}\ln(x)$.

Example: Find the critical numbers of $f(x) = x^3 + 3x^2 - 24x$.

Example: Find the critical numbers of $f(x) = 4x - \tan(x)$.

With this definition in mind, we can say that if f has a local maximum or minimum at c, then c is a critical number of f.

All of this leads to the following useful three-step algorithm to find absolute maximum and minimum values of a continuous function on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest is the absolute minimum.

Example: Find the absolute maximum and minimum values of the function $f(x) = x^3 - 6x^2 + 9x + 2$ on the interval $[-1, 4]$.

Example: Find the absolute maximum and minimum values of the function $f(x) = \frac{x}{x^2+1}$ on the interval $[0, 2]$.

Example: Find the absolute maximum and minimum values of the function $f(x) = \ln(x^2 + x + 1)$ on the interval $[-1, 1]$.

4.2 The Mean Value Theorem: Stewart 4.2

Many of the results of Chapter 4 depend on the Mean Value Theorem. The MVT has utility in proving many important calculus theorems and in some practical applications as well. We'll touch on a few of these in this chapter. We'll build up to the Mean Value Theorem by starting with Rolle's Theorem:

Rolle's Theorem: Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.

Proof: There are three cases to consider: $f(x)$ is constant, $f(x) > f(a)$ for some $x \in (a, b)$, $f(x) < f(a)$ for some $x \in (a, b)$. In the first case, $f' = 0$ and so c can be taken to be any number in (a, b).

In the second case, the Extreme Value Theorem guarantees that f has a maximum value somewhere between in $[a, b]$. Since $f(a) = f(b)$, it must attain this maximum value at a number c in the open interval (a, b). Then f has a local maximum at c and so $f'(c) = 0$. A similar argument holds for case 3.

Example: If we imagine throwing an object straight up into the air, the position as a function of time t is differentiable and $x(0) = x(T)$ where T is the time it takes to go up and back. By Rolle's Theorem, there must be a time at which the velocity ($x'(t)$) is zero: this happens at the top of the trajectory.

Example: Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

$$f(x) = 5 - 12x + 3x^2, [1, 3]$$

Example: Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

$$f(x) = \sqrt{x} - \frac{1}{3}x, [0, 9]$$

The primary use of Rolle's Theorem is to prove the Mean Value Theorem:

Theorem: Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Basically, the theorem is telling us that there must be at least one tangent line which is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.

Proof: The proof involves applying Rolle's Theorem to

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

which is the difference between f and the secant line function. We can show that $h(x)$ satisfies the hypotheses of Rolle's Theorem.

Example: Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

$$f(x) = x^3 + x - 1, [0, 2]$$

Example: Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

$$f(x) = e^{-2x}, [0, 3]$$

Example: Let $f(x) = (x-3)^{-2}$. Show that there is no value of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4-1)$. Why does this not contradict the Mean Value Theorem?

Example: Let $f(x) = 2 - |2x-1|$. Show that there is no value of c such that $f(3) - f(0) = f'(c)(3-0)$. Why does this not contradict the Mean Value Theorem?

Example: Show that the equation $1 + 2x + x^3 + 4x^5 = 0$ has exactly one real root.

Example: Show that the equation $2x - 1 - \sin(x) = 0$ has exactly one real root.

Example: A number a is called a fixed point of a function if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

Two immediate applications of the mean value theorem:

Theorem: If $f'(x) = 0$ for all x in the interval (a, b), then f is constant on (a, b).

An immediate corollary to this theorem:

Corollary: If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b).

This corollary will be important later when we come to the section dealing with antidifferentiation. The constant appearing in the corollary above will be referred to as the constant of integration. More later!

4.3 Derivatives and Curve Shape/Sketching: Stewart 4.3/4.5

In this section, we're primarily interested in how the derivatives of f affect the shape of the graph of $y = f(x)$. Back in section 2.8, we discussed a very basic example of how this works when we considered $f'(x) = 3x^2$ and deduced some properties of $f(x) = x^3$.

Essentially, the derivative of $f(x)$ (if f is differentiable) tells us when the function increases or decreases. Remember that $f'(x)$ tells us the slope of the tangent line to $y = f(x)$ at x. We also know that the tangent line to $y = f(x)$ at x is a very good approximation to f near x. Therefore, if the slope is positive, we expect the function to increase; if the slope is negative, we expect the function to decrease:

Increasing/Decreasing Test:

1. If $f'(x) > 0$ on an interval, then f is increasing on that interval.
2. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The proof is quite simple: suppose $f' > 0$ on some interval. For any two x_1, x_2 in that interval with $x_1 < x_2$, it must be the case that $f(x_1) < f(x_2)$. This follows from the Mean Value Theorem: we know that there is a c between x_1 and x_2 such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. By assumption $f'(c)$ must be positive and, since $x_2 - x_1 > 0$, we must have $f(x_2) - f(x_1) > 0$.

Example: Find the intervals on which $f(x) = 2x^3 + 3x^2 - 36x$ is increasing or decreasing.

Example: Find the intervals on which $f(x) = 4x^3 + 3x^2 - 6x + 1$ is increasing or decreasing.

Example: Find the intervals on which $f(x) = \frac{x^2}{x^2+3}$ is increasing or decreasing.

Example: Find the intervals on which $f(x) = \sin(x) + \cos(x)$ is increasing or decreasing in $0 \le x \le 2\pi$.

Example: Find the intervals on which $f(x) = e^{2x} + e^{-x}$ is increasing or decreasing.

This observation gives us a useful test for determining if f has a maximum or a minimum at critical number c:

The First Derivative Test: Suppose that c is a critical number of a continuous function f.

1. If f' changes from positive to negative at c, then f has a local maximum at c.
2. If f' changes from negative to positive at c, then f has a local minimum at c.
3. If f' does not change sign at c, then f has no local maximum or minimum at c.

Examples: $y = |x|, x^3$.

Example: Find the local maximum and minimum values of $f(x) = 2x^3 + 3x^2 - 36x$.

Example: Find the local maximum and minimum values of $f(x) = 4x^3 + 3x^2 - 6x + 1$.

Example: Find the local maximum and minimum values of $f(x) = \frac{x^2}{x^2+3}$.

Example: Find the local maximum and minimum values of $f(x) = \sin(x) + \cos(x)$.

Example: Find the local maximum and minimum values of $f(x) = e^{2x} + e^{-x}$.

The question now is what does f'' say about f?

It turns out that f'' tells us about the **concavity** of $y = f(x)$.

Definition: If the graph of f lies above all of its tangent lines on an interval I, then it is called concave upward on I. If the graph of f lies below all of its tangents on I, it is called concave downward on I.

Examples: $y = x^2$, $y = x^3$.

We have the **Concavity Test:**

1. If $f''(x) > 0$ for all $x \in I$, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all $x \in I$, then the graph of f is concave downward on I.

The proof follows from the Mean Value Theorem: suppose f is a function with $f''(x) > 0$ for all $x \in I$. Choose $a \in I$. To show that f lies above the tangent line $f(a)+f'(a)(x-a)$ at $x = a$, we need only show that the function $h(x) = f(x) - f(a) - f'(a)(x-a)$ is increasing. Taking the derivative, this is equivalent to showing that $h'(x) = f'(x) - f'(a) > 0$. Assuming $f'' > 0$, we can demonstrate this using the mean value theorem.

When a function changes over from concave downward to concave upward or vice versa, we call the turnover point an inflection point:

Definition: A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or vice versa at P.

Example: $y = x^3$, $y = \tan^{-1}(x)$.

Example: Find the intervals of concavity and the inflection points of $f(x) = 2x^3+3x^2-36x$.

Example: Find the intervals of concavity and the inflection points of $f(x) = 4x^3 + 3x^2 - 6x + 1$.

Example: Find the intervals of concavity and the inflection points of $f(x) = \frac{x^2}{x^2+3}$.

Example: Find the intervals of concavity and the inflection points of $f(x) = \sin(x)+\cos(x)$.

Example: Find the intervals of concavity and the inflection points of $f(x) = e^{2x} + e^{-x}$.

The idea of concavity and its relation to the second derivative gives us a nice test for determining if a critical point is a local maximum or minimum:

The Second Derivative Test: Suppose f'' is continuous near c.

1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example: Find the local maximum and minimum values of $f(x) = x^5 - 5x + 3$ using both the First and Second Derivative Tests.

Example: Find the local maximum and minimum values of $f(x) = x + \sqrt{1-x}$ using both the First and Second Derivative Tests.

Example: Find the local maximum and minimum values of $f(x) = \frac{x}{x^2+4}$ using both the First and Second Derivative Tests.

To finish this section, we demonstrate how we may use calculus to sketch the graph of a given function.

The basic guidelines for curve sketching are:

1. **Domain:** Find the domain of $f(x)$.
2. **Intercepts:** Find all x and y intercepts by setting $y = 0$ and solving for x and also setting $x = 0$ in $y = f(x)$.
3. **Symmetry:** If $f(x) = f(-x)$ or $f(-x) = -f(x)$, we can save time by only sketching half the curve, then reflecting appropriately to find the other half.

4. **Asymptotes:** Find all vertical and horizontal asymptotes (if they exist).

5. **Intervals of Increase or Decrease:** Compute the derivative (if it exists) and find where it is positive and negative.

6. **Local Max/Min Values:** Find the critical points and use the first derivative test to find all local max and min values.

7. **Concavity and Points of Inflection:** Compute the second derivative (if it exists) and find the intervals where it is positive/negative and where it is 0.

Example: For $f(x) = \frac{x^2}{x^2-1}$,

Example: For $f(x) = \sqrt{x^2+1} - x$,

Example: For $f(x) = x\tan(x)$ on $-\frac{\pi}{2} < x < \frac{\pi}{2}$,

Example: For $f(x) = \ln(1 - \ln(x))$,

Example: For $f(x) = e^{\tan^{-1}(x)}$
(Please refer to included plots for solutions.)

4.4 Indeterminate forms: Stewart 4.4

We now come full circle and reexamine some limits of indeterminate forms. We saw many examples of such limits when we used the difference quotient to compute the derivative of a function. These examples were of the indeterminate form of type $\frac{0}{0}$. We now introduce a systematic method, known as l'Hospital's Rule, for evaluation of indeterminate forms.

l'Hosptal's Rule is also useful when we have limits which are indeterminate forms of type $\frac{\infty}{\infty}$:

l'Hospital's Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

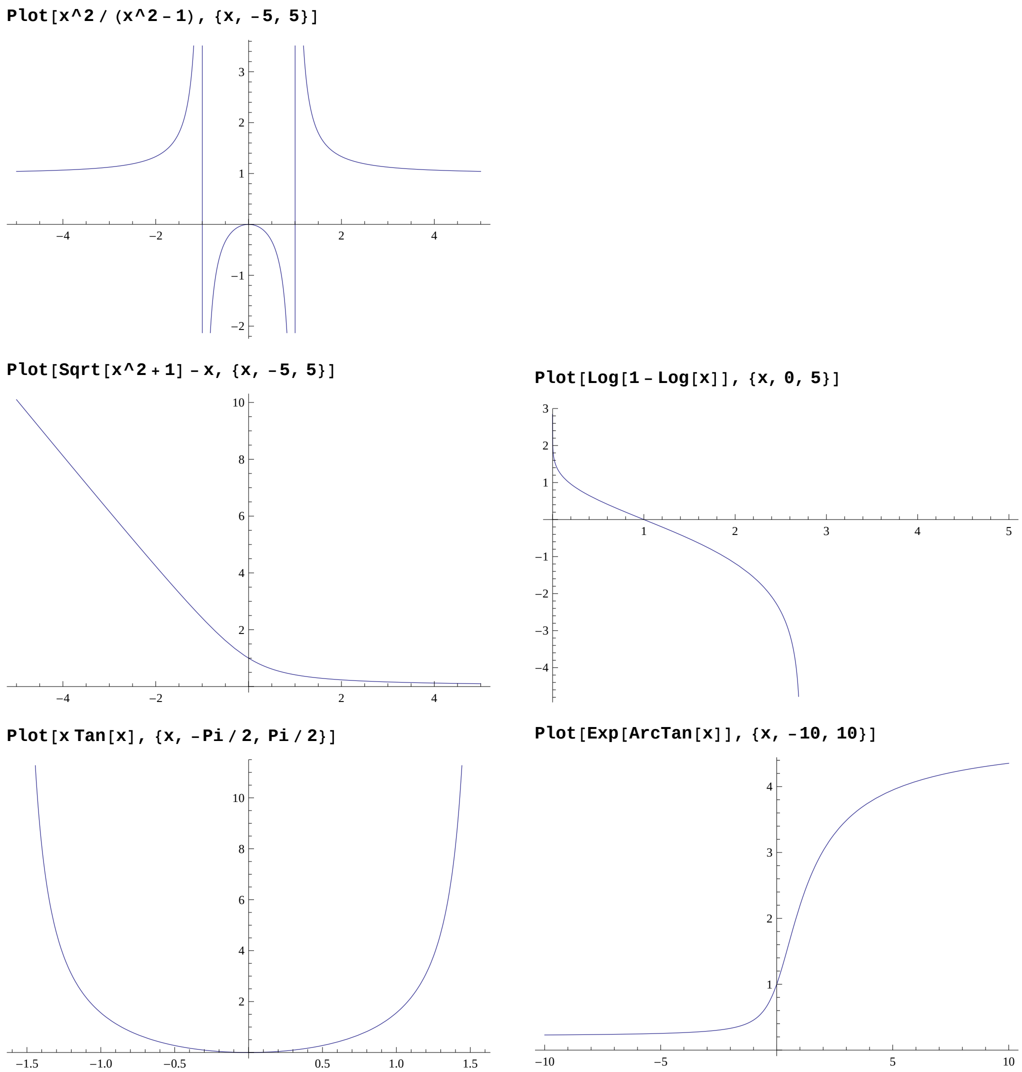


Figure 4.1: Solutions for Curve Sketching Examples.

or that $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$.

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ if the right side exists.

Note that l'Hospital's Rule is also valid for one-sided limits and for limits at infinity.

The general proof is hard to give (see Appendix F). However, we can give a heuristic argument for the indeterminate form $\frac{0}{0}$: let's keep things simple and assume that f and g are defined to be 0 at $x = a$. Then near $x = a$, $f(x) \approx f(a) + f'(a)(x-a)$ and $g(x) \approx g(a) + g'(a)(x-a)$. In other words, we are approximating f and g by the tangent lines at $x = a$.

Since $f(a) = g(a) = 0$,

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}$$

near $x = a$.

1. Compute
$$\lim_{x \to 0} \frac{\tan(x) - x}{x^3}$$
2. Compute
$$\lim_{x \to 1} \frac{\cos(x)}{1 - \sin(x)}$$
3. Compute
$$\lim_{x \to 0} \frac{\sin(4x)}{\tan(5x)}$$
4. Compute
$$\lim_{x \to 1} \frac{e^x - 1}{x^3}$$
5. Compute
$$\lim_{x \to \infty} \frac{\ln(\ln(x))}{x}$$
6. Compute
$$\lim_{x \to 1} \frac{\ln(x)}{\sin(\pi x)}$$

4.5 Optimization Problems: Stewart 4.7

In many practical situations, we can use calculus to optimize quantities of interest.

The procedure for approaching optimization problems has the following steps:

1. Understand the Problem.
2. Draw a Diagram.
3. Introduce Notation.
4. Find a relationship (or relationships) between given quantities and quantities of interest.
5. Use the methods of sections 4.1 and 4.3 to find absolute maxima and/or minima.

Example: A box with a square base and an open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.

Example: Find the point on the line $6x + y = 9$ that is closest to the point $(-3, 1)$.

Example: If a resistor of R ohms is connected across a battery of E volts with internal resistance r ohms, then the power (in watts) in the external resistor is

$$P = \frac{E^2 R}{(R + r)^2}$$

If E and r are fixed but R varies, what is the maximum value of the power?

4.6 Newton's Method: Section 4.8

In short, Newton's Method is a technique for finding a root of a differentiable function $f(x)$ using calculus and a pretty good initial guess.

Many real-world numerical algorithms for finding roots of functions $f(x)$ are based on the Newton Method.

Basically, the idea is to start with your guess x_0 and compute the tangent line to $f(x)$ at x_0. This gives us $y = f(x_0) + f'(x_0)(x - x_0)$.

We take this tangent line and find its root which becomes our next best guess for the root of $f(x)$: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. We repeat this process to find $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ and so on.

This gives a sequence of numbers x_n which is determined recursively:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Under the right circumstances, $\lim_{n\to\infty} x_n = r$ where $f(r) = 0$.

Go over power point slide demonstration

To finish this section, I review two very cool applications of Newton's method.

Application: Ancient Babylonians used the following recursive sequence to compute $\sqrt{a}$ given some initial guess:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

Derive this formula using Newton's Method.

Application: Ever wonder how a computer could compute reciprocals without dividing? The answer lies with Newton's Method. Define $f(x) = \frac{1}{x} - a$ and use Newton's Method to derive the following reciprocal algorithm:

$$x_{n+1} = 2x_n - ax_n^2$$

Chapter 5

Integrals

5.1 Indefinite Integrals/Antiderivatives: Stewart 4.9/5.2

We've seen some applications in which we have knowledge of the derivative of a function and wish to know the original function itself. For instance, suppose we are given the velocity as a function of time $v(t)$ for some object moving in one dimension.

We know that this quantity is the derivative of position: $v(t) = x'(t)$. We may be very interested to know what the position is as a function of time based on the velocity.

The problem then is to somehow "undo" the operation of differentiation much like the square root "undoes" the operation of squaring.

More precisely, our problem is to find a function F whose derivative is a known function f. If such a function F exists, it is known as an antiderivative:

Definition: A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Example: Suppose $f(x) = x^2$. We know that the power rule of differentiation: $\frac{d}{dx}x^n = nx^{n-1}$. Basically, differentiation of a power lowers the exponent by one. So, if we are interested in finding an antiderivative of $f(x) = x^2$, we should look for a function of the form $F(x) = Cx^3$. Then, $F'(x) = 3Cx^2$ and so, if we choose $C = \frac{1}{3}$, then all is well.

However, it turns out this is not the only possibility: $F(x) = \frac{1}{3}x^3 + c$ for any c will also work:

Theorem: If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + c$ where c is an arbitrary constant.

Example: Find the most general antiderivative of $y = \sin(x), e^x, \sec^2(x), \frac{1}{x}, x^n (n \neq 1)$

2 Table of Antidifferentiation Formulas

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sec^2 x$	$\tan x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec x \tan x$	$\sec x$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\dfrac{1}{x}$	$\ln \lvert x \rvert$	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$
e^x	e^x	$\cosh x$	$\sinh x$
$\cos x$	$\sin x$	$\sinh x$	$\cosh x$
$\sin x$	$-\cos x$		

Example: Find the most general antiderivatives of $f(x) = x - 3$, $\frac{1}{2}x^2 - 2x + 6$, $3\sqrt{x} - 2x^{\frac{1}{3}}$, $\frac{2+x^2}{1+x^2}$.

Example: Find f if $f'(x) = \frac{4}{\sqrt{1-x^2}}$, $f(\frac{1}{2}) = 1$.

Example: Find f if $f''(x) = 2 + \cos(x)$, $f(0) = -1$, $f'(\frac{\pi}{2}) = 0$.

Example: Find f if $f''(x) = \frac{1}{x^2}$, $x > 0$, $f(1) = 0$, $f(2) = 0$.

Example: Find f if $f'''(x) = \cos(x)$, $f(0) = 1$, $f'(0) = 2$, $f''(0) = 3$.

The process of finding an antiderivative of a function (when it exists) is referred to as finding the **indefinite integral** of the function:

$\int f(x)dx = F(x)$ means $F'(x) = f(x)$.

For example, $\int x^2 dx = \frac{1}{3}x^3 + C$ and $\int e^{-x}dx = -e^{-x} + C$. Later, when we introduce the Fundamental Theorem of Calculus, we'll see how indefinite integrals are connected to definite integrals which can be thought of as areas of regions bounded by a curve $y = f(x)$.

To bring this section to a close, we return to our original problem of finding the position as a function of time when either the velocity or the acceleration of the object is given:

Example: A particle is moving with $v(t) = \sin(t) - \cos(t)$. If the initial position is 0, find the position as a function of time.

Example: A particle is moving with $a(t) = t^2 - 4t + 6$. If the initial position is 0 and after 1 second the particle is found at 20, find the position as a function of time.

5.2 Areas and Distances: Stewart 5.1

In the last section, we focused on finding antiderivatives of functions (when they exist). We viewed this process as "undoing" the operations of differentiation.

This process is intimately related to the process of finding **integrals** which will occupy our time for the remainder of the course. The relationship between antidifferentiation and integration will be made clear when we reach the Fundamental Theorem of Calculus.

When we introduced derivatives, we did so in the context of the so-called tangent problem. We'll motivate integration with a similar problem.

The Area Problem: Find the area of the region that lies under the curve $y = f(x)$ from a to b where $f \geq 0$ and continuous. (Draw this! Figure 1)

First, we need to clarify what area means. For a rectangle, we know that the area is the product of the width and the length. For a triangle, the area is $\frac{1}{2}bh$. However, it is difficult to find the area of a region with a curved side. This is precisely the area problem.

When we solved the tangent problem, we approximated the tangent line at P by a secant line through P and Q and then took the limit as $Q \to P$. Similarly, we can chop up the area beneath a curve $y = f(x)$ and approximate the area by rectangles. The more rectangles we have, the better the approximation (**Figure 4, page 366**).

Let's start with a basic example: $y = x^2$ between $x = 0$ and $x = 1$.

We have two options when approximating the area of this region: we can pick rectangles which overshoot the curve or we can pick rectangles which are all below the curve (**Figure 4b, Figure 5** pages 366-367).

To keep things simple at first, let's use four rectangles with left-hand vertices at $x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$. If we take the heights of these rectangles to be at $f(\frac{1}{4}), f(\frac{1}{2}), f(\frac{3}{4}), f(1)$, (i.e. the righthand endpoints) then the rectangles will overshoot the curve. If we take the heights of these rectangles to be at $f(0), f(\frac{1}{4}), f(\frac{1}{2}), f(\frac{3}{4})$, (i.e. the lefthand endpoints) then the rectangles will be below the curve.

Let's call the approximation using the righthand-endpoint rectangles R_4 and call the approximation using the lefthand-endpoint rectangles L_4.

Then $R_4 = \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}\left(1\right)^2 = \frac{15}{32} = 0.46875$ and

$L_4 = \frac{1}{4}\left(0\right)^2 + \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 = \frac{7}{32} = 0.21875$

So, we know that $0.21875 < A < 0.46875$. We can do better with more rectangles:

$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2 = \frac{1}{n^3}\sum_{j=1}^{n} j^2.$$

Also,

$$L_n = \frac{1}{n}\left(\frac{0}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n-1}{n}\right)^2 = \frac{1}{n^3}\sum_{j=0}^{n-1} j^2.$$

At this point, we need the important formula (see Extra Credit assignment!):

$$\sum_{j=0}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

From this, we may show that $\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \frac{1}{3}$. So, as we add more rectangles, the left and right approximations approach each other. Since the area of the region is always bounded between these two quantities, the area must be $\frac{1}{3}$.

We now generalize this idea to regions bounded by vertical lines $x = a$ and $x = b$ and the curve $y = f(x)$ where $f(x) \geq 0$.

First, let's break up the interval $[a, b]$ into n equal pieces of length $\triangle x = \frac{b-a}{n}$. These strips divide the interval $[a, b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3] \ldots [x_{n-1}, x_n]$ where $x_0 = a$, $x_n = b$ and

$$x_1 = a + \triangle x \ , \ x_2 = a + 2\triangle x \ , \ x_3 a + 3\triangle x \ , \ \ldots$$

Let's approximate the i-th strip S_i by a rectangle with width $\triangle x$ and height $f(x_i)$, which is the value of f at the right endpoints (**Figure 11** pg 370). Then the area of the i-th rectangle is $f(x_i)\triangle x$.

Then, $R_n = \triangle x \sum_{i=1}^{n} f(x_i)$. Similarly, we have $L_n = \triangle x \sum_{i=0}^{n-1} f(x_i)$, in which we are evaluating the function at the left endpoints of each interval.

Assuming the function f is continuous, the following definition makes sense:

Definition: The **area** A of the region S bounded by the graph of the continuous function f and the vertical lines $x = a$ and $x = b$ is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$$

In fact, it isn't necessary to use just the left or right endpoints: we can use any point x_i^* in the interval $[x_{i-1}, x_i]$ to evaluate f. The starred numbers are called **sample points**.

Sometimes, we'll find it convenient to choose x_i^* to be either the minimum or maximum value of f on $[x_{i-1}, x_i]$: the sums defining the area are then known as **lower** and **upper** sums respectively.

To conclude this section, we outline another analogy between the area problem and the tangent problem. We found that the tangent problem solves another problem, namely, the velocity problem. It turns out that solving the area problem also solves the so-called distance problem: Find the distance traveled by an object during a certain time period if the velocity of the object is known at all times.

As we'll see in coming sections, finding the area bounded by the curve $v(t)$ and lines $t = a$ and $t = b$ is equivalent to finding the total distance traveled by the object between those times.

Unlike the example $y = x^2$, it is generally not true that we'll be able to find the area bounded by a curve $y = f(x)$ directly using the ideas of this section.

Example: Find the expression for the area under the graph of $f(x) = \sqrt{\sin(x)}$, $0 \leq x \leq \pi$ as a limit. Do not evaluate the limit.

Example: Find the expression for the area under the graph of $f(x) = \frac{2x}{x^2+1}$, $1 \leq x \leq 3$ as a limit. Do not evaluate the limit.

Example: Determine a region whose area is equal to the given limit. Do not evaluate the limit.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \left(5 + \frac{2i}{n}\right)^{10}$$

Example: Determine a region whose area is equal to the given limit. Do not evaluate the limit.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{4n}\tan\left(\frac{\pi}{4n}\right)$$

Example: Let A_n be the area of the polygon of n equal sides inscribed in a circle of radius r. By dividing the polygon into n congruent triangles with central angle $\frac{2\pi}{n}$, show that

$$A_n = \frac{1}{2}nr^2\sin\left(\frac{2\pi}{n}\right)$$

Show that $A_n \to \pi r^2$.

5.3 Definite Integrals: Stewart 5.2

We now formalize the discussion of the previous section:

Defintion: If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\triangle x = \frac{b-a}{n}$. We let $x_0 = a, x_1, x_2, \ldots, x_n = b$ be the endpoints of these subintervals and we let $x_1^*, \ldots, x_n^*$ be any sample points in these subintervals, so $x_i^* \in [x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\triangle x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Notes:

1. We do not require $f \geq 0$
2. We do not require f to be continuous.
3. The symbol $\int$ is an elongated S which stands for "sum." The function inside the integral is called the **integrand** and a and

b are the **limits of integration**. The lower limit is a and the upper limit is b.

4. The dx has no independent meaning. $\int_a^b f(x)dx$ is all one symbol.

5. The definite integral is a number and does not depend on x. In fact, we can use any letter in place of x without changing the value of the integral.

6. When $f \geq 0$, we can interpret the integral as the area bounded by the curve, the x axis and the lines $x = a$ and $x = b$. When f can be negative, we interpret the integral as

$$\int_a^b f(x)dx = A_+ - A_-$$

where A_+ is the area of the region above the x-axis and A_- is the area below. In other words, $\int_a^b f(x)dx$ is the net area.

Not all functions are integrable. Consider $f(x) = \frac{1}{x}$ defined on $(0, 1]$. The integral of this function does not exist since there is infinite area beneath the curve.

We have a theorem which guarantees that most commonly encountered functions are integrable:

Theorem: If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)dx$ exists.

Much of the remainder of the course will be devoted to computing integrals and using them in applications.

Unless we are very lucky, the Riemann sums rarely allow us to compute the definite integral of a function directly. Exceptions are positive powers of x. The Extra Credit problem set let's us establish, for instance, that

$$\int_0^a x^n dx = \frac{1}{n+1}a^{n+1}$$

However, we can sometimes evaluate a definite integral by interpreting the integral as an area of a region:

Example: Compute $\int_0^a \sqrt{a^2 - x^2}dx$ and $\int_0^3 (x-1)\, dx$.

In some numerical applications, it is convenient to approximate the definite integral by a sum. Recall that the sample points may be chosen anywhere within the subintervals defining the Riemann sum:

Midpoint Rule:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\triangle x_i$$

with $\triangle x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$.

Midpoint rule estimates improve as n increases.

To go beyond basic approximations of the definite integral, we need to establish some of its properties.

Properties of the Integral:

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b c dx = c\,(b-a)$
4. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
5. $\int_a^b [cf(x)]dx = c\int_a^b f(x)dx$
6. $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

All of these properties (except the last one) can be proven using the definition of the Riemann integral in terms of the Riemann sum and using the properties of summation. (Prove one if there is time).

There are also several comparison properties which are useful from time to time:

1. If $f(x) \geq 0$ for all $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$.
2. If $f(x) \geq g(x)$ for all $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
3. If $m \leq f(x) \leq M$ for all $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

These properties are useful if we are only interested in estimating the value of an integral.

Example: Estimate $\int_1^4 \sqrt{x}dx$ and $\int_0^2 xe^{-x}dx$ and $\int_\pi^{2\pi} (x - \sin(x))\, dx$.

Clearly, we are not going to make much headway with just these properties and the definition of the definite integral in terms of Riemann sums.

We need something more: the Fundamental Theorem of Calculus.

5.4 Fundamental Theorem of Calculus: Stewart 5.3

The Fundamental Theorem of Calculus relates the two processes of calculus: differentiation and integration.

Through the Fundamental Theorem, we see that differentiation and integration are in a sense inverse operations.

The first part of the Fundamental Theorem deals with functions defined by an equation of the form

$$g(x) = \int_a^x f(t)dt$$

where f is a continuous function on $[a, b]$ and x varies between a and b.

$g(x)$ is a function: it takes a number x and outputs another number. If $f \geq 0$, then $g(x)$ can be interpreted as the area bounded by f, the x-axis and the lines a and x. As x changes, the right hand side of the region slides back and forth and the area changes.

The first part of the Fundamental Theorem tells us that the derivative of g is equal to f: in this way, we see that differentiating the function with a variable limit of integration returns the integrand, thus “undoing” the operation of integration.

The Fundamental Theorem of Calculus, Part 1: If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

In Leibniz notation, we can state the last condition as

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

when f is continuous.

Examples: Compute the derivative of $\int_1^x \frac{1}{t^3+1}dt$, $\int_x^\pi \sqrt{1+\sec(t)}dt$, $\int_1^{e^x} \ln(t)dt$, $\int_{\sin(x)}^{x^2} \sqrt{1+t^2}dt$

The second part of the Fundamental Theorem allows us to compute many integrals easily.

The Fundamental Theorem of Calculus, Part 2: If f is continuous on $[a, b]$ and differentiable on (a, b), then

$$\int_a^b f'(x)dx = f(b) - f(a)$$

The proof can be easily obtained using the Mean Value Theorem: start with $f(b) - f(a)$ and break up the interval $[a, b]$ into n subintervals: $x_i = a + i\triangle x$ where $\triangle x = \frac{b-a}{n}$. Then

$$f(b) - f(a) = \sum_{j=0}^{n-1} [f(x_{j+1}) - f(x_j)]$$

By the Mean Value Theorem, we can find $c_i \in (x_{i+1}, x_i)$ for every i so that $f'(c_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$ or

$$f'(c_i)(x_{i+1} - x_i) = f(x_{i+1}) - f(x_i) = f'(c_i)\triangle x$$

Therefore,

$$f(b) - f(a) = \sum_{j=0}^{n-1} f'(c_i)\triangle x$$

In the limit as $n \to \infty$, the sum on the right hand side becomes $\int_a^b f'(x)dx$ and so

$$f(b) - f(a) = \int_a^b f'(x)dx$$

This is a very powerful result: if we can identify an antiderivative of the integrand, then the definite integral is simply the difference of the antiderivative at the endpoints.

Examples: Evaluate the integral $\int_{-1}^{3} (x^3 - 2x)\, dx$, $\int_0^1 \left(1 + \frac{1}{2}u^4 - \frac{2}{3}u^9\right) du$, $\int_0^1 e^x dx$.

A basic consequence of the Fundamental Theorem is the Net Change Theorem:

The integral of a rate of change is the net change: $\int_a^b f'(x)dx = f(b) - f(a)$. This principle can be applied to all of the rates of change in the natural and social sciences. For example:

1. If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$ and so $\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$. In other words, the net change in the position (displacement) is the integral of the velocity.

2. If we want the total distance traveled by an object moving in one dimension, the answer is $\int_{t_1}^{t_2} |v(t)|dt$

3. Power is the rate of change of the energy: $P(t) = E'(t)$. So the net change in energy is the integral of the power.

4. The change in concentration of a chemical reactant is the integral of the reactant rate.

Example: The current in a wire is defined as the derivative of the charge: $I(t) = Q'(t)$. What does $\int_{t_1}^{t_2} I(t)dt$ represent?

Example: The acceleration function (in m/s^2) for an object is $a(t) = t + 4$ for $t \in [0, 10]$. If the initial velocity is 5 (in m/s), what is the total distance traveled and the displacement over the given time interval?

5.5 The Substitution Rule: Stewart 5.5

While the Fundamental Theorem is a very powerful result which helps us evaluate many integrals, it relies on our ability to correctly identify an antiderivative of the integrand.

In many situations, the following rule is very useful:

The Substitution Rule: If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f'(g(x))g'(x)dx = \int f(u)du$$

This rule can be justified using the Chain Rule: $f'(g(x))g'(x) = \frac{d}{dx}(f \circ g)(x)$ and so $\int \frac{d}{dx}(f \circ g)(x)dx = f(g(x)) + C$.

Viewed this way, $u = g(x)$ implies $du = g'(x)dx$ and so $\int f(u)du = \int f(g(x))(g'(x))\,dx$. So, we can think of the du and dx appearing here as differentials inside of the integral.

Examples: Evaluate the integral $\int x\sin(x^2)dx$, $\int x^2e^{x^3}dx$, $\int \frac{1}{5-3x}dx$, $\int \frac{(\ln(x))^2}{x}dx$, $\int sec^2(\theta)\tan^3(\theta)d\theta$,

$\int \sqrt{x}\sin(1+x^{\frac{3}{2}})dx$, $\int 5^t \sin(5^t)dt$.

When evaluating a definite integral by substitution, two methods are possible. One method is to evaluate the indefinite integral first and then use the Fundamental Theorem.

Example: Evaluate $\int_0^4 \sqrt{2x+1}dx$.

Another method, which is usually preferable, is to change the limits of integration when the variable is changed:

The Substitution Rule for Definite Integrals: If g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Observe that when using this substitution rule we do not return to the variable x after integration.

Examples: Evaluate $\int_0^3 \frac{1}{5x+1}dx$, $\int_0^1 xe^{-x^2}dx$, $\int_0^a x\sqrt{x^2+a^2}dx$, $\int_e^{e^4} \frac{1}{x\sqrt{\ln(x)}}dx$, $\int_0^1 \frac{1}{(1+\sqrt{x})^4}dx$

Sometimes, when our function possesses symmetry, it is possible to exploit this fact to simplify integrals.

Integrals of Symmetric Functions Suppose f is continuous on $[-a,a]$.

1. If f is even, then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$
2. If f is odd, then $\int_{-a}^a f(x)dx = 0$.

Example: Evaluate $\int_{-1}^1 \frac{\tan(x)}{1+x^2+x^4}dx$.