Math 3F Homework 1 (Fall 2017) Instructions: Please select the best solution among the options presented. Once you have made your selection for ALL problems below, enter your answers into the corresponding Moodle quiz no later than 9:30AM Thursday, August 31st. Good luck!

The Fitzhugh-Nagumo model for the electrical impulse in a neuron states that, in the absence of relaxation effects, the electrical potential in a neuron $v(t)$ obeys the differential equation

$$\frac{dv}{dt} = -v \left[ v^2 - (1 + a)v + a \right]$$

where $a$ is a positive constant such that $0 < a < 1$.

**Problem 1:** What are the **equilibrium** solutions? Remember that equilibrium solutions correspond to solutions of the form $v(t) = \text{constant}$.

A All values in the interval $[-a, a]$.

B Just $v = 0$.

C There are no such solutions.

D $v = 1, \, v = a$.

E $v = 0, \, 1, \, a$.

**Problem 2:** Using the fact that this differential equation is separable, choose among the list below the implicit relationship between $v$ and time $t$ when $v$ is not an equilibrium solution. Here, $C$ is a constant and we have selected $a = \frac{1}{2}$.

A $\frac{v}{(1-2v)^2} = Ce^{-t}$.

B $\frac{v^2(1-v)^2}{(1-2v)^2} = Ce^{-t}$.

C $\frac{v(1-v^2)}{(1-2v)^2} = Ce^{-t}$.

D $\frac{v^2(1-v^2)}{(1-2v)^2} = Ce^{-2t}$.

**Problem 3:** Suppose that $v(0) = 2$. Use the quadratic formula to find $v(t)$ as an explicit function of time. Choose from among the list below the best possible answer.

A $v(t) = 2e^{-t}$.

B $v(t) = \frac{1 - \frac{\sqrt{5}}{2} e^{-t/2} - \sqrt{1 - \frac{\sqrt{5}}{2} e^{-t/2}}}{2(1 - \frac{\sqrt{5}}{2} e^{-t/2})}$. 

$C \ v(t) = 1 + e^{-t/2}.$  

$D \ v(t) = \frac{1 - \frac{g}{2} e^{-t/2} + \sqrt{1 - \frac{g}{2} e^{-t/2}}}{(1 - \frac{g}{2} e^{-t/2})}.$

According to Newton’s Law of Universal Gravitation, the gravitational force on an object of mass $m$ that has been projected vertically upward from the earth’s surface is

$$F = \frac{mgR^2}{(x + R)^2}$$

where $x(t)$ is the object’s distance above the surface at time $t$, $R$ is the earth’s radius, and $g$ is the acceleration due to gravity. Also, by Newton’s Second Law, $F = ma = m \frac{dv}{dt}$ and so

$$\frac{dv}{dt} = -\frac{gR^2}{(x + R)^2}$$

Suppose a rocket is fired vertically upward with an initial velocity $v_0$.

**Problem 4:** Using the Calculus II fact that

$$\frac{dv}{dx} = \frac{dv}{dt} \frac{dt}{dx},$$

choose the expression from the list below that best represents $\frac{dv}{dx}$.

- A $\frac{dv}{dx} = -\frac{v}{x}$
- B $\frac{dv}{dx} = -\frac{gR^2}{v(x + R)^2}$
- C $\frac{dv}{dx} = -\frac{2gR^2}{v^2(x + R)^2}$
- D $\frac{dv}{dx} = -\frac{g^2v}{(x + R)^2}$

**Problem 5:** The result of problem 1 is a separable differential equation for the speed of the projectile as a function of height about the surface of the Earth. Solve this differential equation using the initial condition $v(x = 0) = v_0$ and select the most appropriate solution from the list of functions below:
A \[ v(x) = \sqrt{\frac{2gR^2}{x+R} - 2gR + v_0^2} \]

B \[ v(x) = \sqrt{\frac{2gR^2}{x+R} + 2gR} \]

C \[ v(x) = \sqrt{\frac{2gR^2}{x+R} + gR - \frac{1}{2}v_0^2} \]

D \[ v(x) = -\sqrt{\frac{2gR^2}{x+R} - 2gR + v_0^2} \]

**Problem 6:** The escape velocity is the choice of \( v_0 \) so that \( v(\infty) = 0 \). Compute the escape velocity and select the answer from the list below

A \( v_0 = \sqrt{gR^2} \)

B \( v_0 = \sqrt{2gR} \)

C \( v_0 = gR \)

D \( v_0 = \sqrt{4gR} \)

Consider the chemical reaction whereby two reactant molecules \( A \) and \( B \) form the molecule \( C \). According to the law of mass action, the differential equation for \( x(t) = [C] \) (the concentration of \( C \)) is

\[
\frac{dx}{dt} = k_f(a - x)(b - x) - k_r x
\]

where \( k_f \) is the rate constant for the forward reaction and \( k_r \) is the rate constant for the reverse reaction. Both rate constants are positive. Here, the initial concentrations of \( A \) and \( B \) are \( a \) and \( b \) respectively and \( x(0) = 0 \). For simplicity, let’s assume that \( b = a = 1 \) and \( k_f = 1, k_r = 2 \).

**Problem 7:** What are the equilibrium solutions?

A Just \( 2 + \sqrt{3} \).

B There are no such solutions.

C \( x = 0, 2 \).

D \( x = 2 \pm \sqrt{3} \).

**Problem 8:** Find the solution to this equation with \( x(0) = 0 \).

A \( x(t) = 2 - \frac{1}{t+1} \).

B \( x(t) = 2 - \frac{1}{t+\frac{1}{2}} \).
C  \[ x(t) = \frac{e^{2 \sqrt{3} t} - 1}{(2 + \sqrt{3})e^{\sqrt{3} t} + \sqrt{3} - 2}. \]

D  \[ x(t) = \frac{e^{2 \sqrt{3} t} - 1}{(2 - \sqrt{3})e^{\sqrt{3} t} - \sqrt{3} - 2}. \]

**Problem 9:** Find the solution of \( y'(x) = xe^y \) that satisfies \( y(0) = 0 \).

A  \[ y(x) = -\ln(1 - \frac{1}{2} x^2). \]

B  \[ y(x) = \sin(1 - \frac{1}{2} x^2). \]

C  \[ y(x) = \ln(x^2 - 1). \]

D  \[ y(x) = \ln(x^2 + 1). \]

**Problem 10:** Find the solution of \( x \ln x = (1 + y^2)^{-1} y' \) that satisfies \( y(0) = 0 \) for \( x \geq 0 \).

A  \[ y(x) = \cot \left( \frac{x^2}{4} (2 \ln(x) - 1) \right). \]

B  \[ y(x) = \exp \left( \frac{x^2}{4} (2 \ln(x) - 1) \right). \]

C  \[ y(x) = \tan \left( \frac{x^2}{4} (2 \ln(x) - 1) \right). \]

D  \[ y(x) = \tan^{-1}(x). \]