HW 9

Due at the beginning of lecture on Thursday, April 30th.

In order to receive full credit, you must attempt all problems and write out all steps leading to your answers neatly and legibly. You cannot simply write the correct answer to demonstrate your mathematical understanding.

You must include your name, the course title and section number on the first page. All homework sets must be stapled. No late homework will be accepted without my express permission. You may receive no credit if these guidelines are not followed.

Good luck!

1. Find the general solution of the system of equations $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$ for any set of initial conditions where
$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}.$$

   Classify the equilibrium point at $(0, 0)$.

   ANS: $\boldsymbol{x}(t) = e^{tA}\boldsymbol{x}(0)$ for $A = \begin{bmatrix} e^t(\cos 2t + \sin 2t) & -e^t \sin 2t \\ 2e^t \sin 2t & e^t(\cos 2t - \sin 2t) \end{bmatrix}$. The equilibrium $(0, 0)$ is unstable.

2. Find the general solution of the system of equations $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$ for any set of initial conditions where
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}.$$

   Classify the equilibrium point at $(0, 0, 0)$.

   ANS: $\boldsymbol{x}(t) = e^{tA}\boldsymbol{x}(0)$ for $A = \begin{bmatrix} e^t & 0 & 0 \\ \frac{1}{2}e^t(-3 + 3\cos(2t) + 2\sin(2t)) & e^t\cos(2t) & -e^t\sin(2t) \\ -\frac{1}{2}e^t(-2 + 2\cos(2t) - 3\sin(2t)) & e^t\sin(2t) & e^t\cos(2t) \end{bmatrix}$.
   The equilibrium $(0, 0, 0)$ is unstable.

3. In this problem, we wish to solve the differential equation $\frac{d^3y}{dt^3} - y = 0$ for initial conditions $y(0) = -1, y'(0) = 0$ and $y''(0) = 1$. Start by defining $x_1(t) = y(t)$, $x_2(t) = y'(t)$, $x_3(t) = y''(t)$ and obtain a linear system of first order differential equations $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$. Solve through the usual diagonalization procedure.

   ANS: $y(t) = -\frac{1}{3}e^{-\frac{t}{2}}\left(3\cos\frac{\sqrt{3}t}{2} + \sqrt{3}\sin\frac{\sqrt{3}t}{2}\right)$

4. Find and analyze the stability of the equilibrium points of the **nonlinear** system $\frac{dx_1}{dt} = x_1 + x_2^2$ and $\frac{dx_2}{dt} = -x_2$.

ANS: The unique equilibrium point is $x_1 = 0, x_2 = 0$. Near the equilibrium point, $x_1 = 0 + \delta x_1(t)$, $x_2 = 0 + \delta x_2(t)$ where δx_i is small. The nonlinear system is approximately

$$\frac{d\delta x_1}{dt} = \delta x_1 \ , \ \frac{d\delta x_2}{dt} = -\delta x_2$$

which has solutions

$$\delta x_1(t) = \delta x_1(0) e^t \ , \ \delta x_2(t) = \delta x_2(0) e^{-t}.$$

Therefore, the equilibrium point $(0, 0)$ is a saddle.