Math 3C Spring 2015 Extra Credit Problem Set

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1 Instructions

You do NOT need to attempt all problems to receive partial extra credit. Full credit for attempting any given problem in this set will be given ONLY when your argument is written out neatly, legibly, logically and correctly. I reserve the right to award credit based on my assessment of how well you adhere to these criteria. I reserve the right to alter the content of the problem set below when necessary and at any time prior to the due date of **Thursday, May 14th**. You may work on these problems in groups, but the work you submit must be your own.

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2 Method of Images

In this problem set, we will compute various electric properties of a system consisting of a point particle of charge q held a distance a above an infinite, grounded, conducting plate.

First, choose a coordinate system where the charged particle sits at $(0, 0, a)$ along the z-axis and the conducting plane sits in the x-y plane. Since the charge is stationary with respect to the conducting plate, there are no magnetic fields generated and the equations of electrostatics apply:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \ , \ \nabla \times \boldsymbol{E} = \boldsymbol{0}$$

Here, ρ is the charge density, ϵ_0 is a constant called the permittivity of free space and $\boldsymbol{E}$ is the electric field, a vector field describing the electric force on a charged particle at any point in space above the x-y plane.

1. **(2 points)** We can assume that $\boldsymbol{E} = -\nabla V$ where V is a function of x, y and z. Why is this true? State any theorems you use in your argument.

2. **(1 point)** $V(x, y, z)$ is referred to as the electrostatic potential. Assume (x, y, z) is some point above the x-y plane and not equal to $(0, 0, a)$ where the charged particle sits. Since there is no charge at (x, y, z), we have $\rho(x, y, z) = 0$. Therefore, away from the conducting plate and from the point charge, $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} = 0$. Show that this implies $\nabla^2 V = 0$. This equation is known as Laplace's equation.

3. **(4 points)** Show that $\tilde{V}(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2+y^2+(z-a)^2}}$ satisfies $\nabla^2 \tilde{V} = 0$ away from $(0, 0, a)$. This is the electrostatic potential for the point charge alone.

4. **(4 points)** We need to take into account the conducting plate at $z = 0$. Because it is grounded, $V(x, y, 0) = 0$ by assumption. Show that

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{x^2+y^2+(z-a)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+a)^2}}\right) \tag{2.1}$$

satisfies $\nabla^2 V = 0$ for (x, y, z) away from $(0, 0, \pm a)$ and also $V(x, y, 0) = 0$. (The potential is constructed by replacing the difficult problem of a charge and a conducting plate with the easier problem of a charge and its mirror image; hence, the Method of Images.)

5. **(3 points)** Compute $-\nabla V$ to find the electric field above the conducting plate.

6. **(1 point)** The charge density σ on the conducting plate is defined to be $-\epsilon_0 \nabla V \cdot \boldsymbol{n}\big|_{z=0}$ where $\boldsymbol{n}$ is the upward pointing unit normal vector to the conducting plate. Compute $\sigma(x, y)$.

7. **(3 points)** Find the total charge on the conducting plate $Q = \int\int \sigma(x, y)dxdy$. Does the result surprise you?